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Statistical Bootstrap Models for Build-Up and Decay of Isoscalar and Isovector Fireballs.

J. ENGELS, K. FABRICIUS and K. SCHILLING

Department of Theoretical Physics, University of Bielefeld - Bielefeld

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« Und wir Jünglinge sangen,
Und empfanden, wie Hagedorn. »
F. G. KLOPSTOCK, 1750

Summary. — The level density and the decay of isoscalar and isovector fireballs (with input pions) are studied in the framework of the statistical bootstrap model. We find that different versions of this model lead to the same total multiplicity distribution for large fireball mass M. Asymptotic expressions for the integrated multiplicity correlation of charged particles are derived. The correlations are calculated numerically in the range $1\,\text{GeV} \leqslant M \leqslant 8\,\text{GeV}$.

1. – Introduction.

There is increasing evidence that the production of particles in high-energy hadron collisions proceeds via the production of clusters, *i.e.* subsystems of hadronic matter that decay isotropically in their rest frames. Though the nature of these clusters is not at all understood from present data (1) (mainly because of the ignorance of theoreticians about the production mechanism), it is tempting to identify them with the fireballs of Hagedorn's thermodynamical bootstrap model (2), which are mainly characterized by their limited temperature for energy going to infinity. On the other hand there is the intriguing pos-

(1) See, *e.g.*, A. BIAŁAS: invited talk given at the *IV International Symposium on Multiparticle Hadrodynamics, Pavia, 1973*. CERN preprint TH 1745.
(2) R. HAGEDORN: CERN preprint 71-12; *Suppl. Nuovo Cimento*, **3**, 147 (1965).

sibility that individual fireballs might be discovered and studied in e^+e^- annihilation into hadrons: it has been speculated that virtual timelike photons behave like fireballs in their decay, and there is in fact some evidence for this speculation from the first measurements at SPEAR.

It seems therefore appropriate to investigate fireball decay in some detail. The inclusive spectra from this decay can be calculated at finite fireball mass in Frautschi's (3) and Yellin's (4) version of the bootstrap (henceforth called statistical bootstrap) once it is accepted that fireballs decay in the way they are built up. Various versions of the statistical bootstrap (which implies energy-momentum conservation) have been investigated previously to study the predictive power of the model at finite fireball masses. Most of these applications neglected isospin in the bootstrap. It is suggestive to use Cerulus coefficients (5) on the solutions of isospinless bootstrap to calculate prong cross-sections (6). ILGENFRITZ and KRIPFGANZ (7) succeeded in solving the bootstrap equations with isospin conservation for fireballs of unrestricted isospin which would seem somewhat unphysical. In this paper we formulate and solve the bootstrap problem with isospin and G-parity conservation for a system of nonexotic fireballs composed of π-mesons and investigate their decay, with particular emphasis on nonasymptotic features (*).

2. – Build-up of fireballs: statistical bootstrap.

This Section is mainly devoted to the implication of isospin and G-parity conservation on the statistical bootstrap of mesonic fireballs. In order to establish our notations, let us first review quickly the conventional bootstrap scheme without quantum number conservation.

2·1. *Bootstrap without quantum number conservation.* – The simplest, since explicitly solvable version of statistical bootstrap is the linear bootstrap (9,10), where a fireball is composed of another fireball plus a pion. The bootstrap equation for the density of states of a fireball of mass $M=\sqrt{Q^2}$, $\tau(Q^2)$, reads in

(3) S. FRAUTSCHI: *Phys. Rev. D*, **3**, 2821 (1971).
(4) I. YELLIN: *Nucl. Phys.*, **52** B, 583 (1973).
(5) F. CERULUS: *Nuovo Cimento*, **19**, 528 (1960).
(6) J. ENGELS, H. SATZ and K. SCHILLING: *Nuovo Cimento*, **17** A, 535 (1973).
(7) E.-M. ILGENFRITZ and J. KRIPFGANZ: *Nucl. Phys.*, **62** B, 141 (1973).
(*) After completion of this work we received a preprint of KRIPFGANZ and ILGENFRITZ which treats some aspects of nonexoticity requirements in the linear-bootstrap case (8).
(8) J. KRIPFGANZ and E.-M. ILGENFRITZ: Leipzig University preprint KMU-HEP-7401, December 1973.
(9) R. HAGEDORN and I. MONTVAY: *Nucl. Phys.*, **59** B, 45 (1973).
(10) C. B. CHIU: *Nucl. Phys.*, **54** B, 170 (1973).

this case

$$(2.1)\qquad \begin{cases} B\tau(Q^2) = B\delta_0(Q^2-\mu^2) + B^2\int\prod_{i=1}^{2} \mathrm{d}^4k_i\,\tau(k_i^2)\delta_0(k_2^2-\mu^2)\delta^4(Q-k_1-k_2)\,, \\ \delta_0(x^2-\mu^2) = \theta(x_0)\delta(x^2-\mu^2)\,, \end{cases}$$

where μ denotes the pion mass and B stands for a coupling constant («volume»). Equation (2.1) is solved by means of the covariant Laplace transform

$$(2.2)\qquad \mathscr{L}(\tau) := Z(\beta) = B\int \mathrm{d}^4Q\,\tau(Q^2)\exp[-\beta_\mu Q^\mu]\,.$$

We choose β_μ purely timelike in the c.m.s.

$$(2.3)\qquad \beta_\mu = (\beta, 0, 0, 0)$$

with $\beta > 0$.

The bootstrap equation then reads very simply

$$(2.4)\qquad Z(\beta) = t(\beta) + t(\beta)Z(\beta)$$

with the solution

$$(2.5)\qquad Z(\beta) = \frac{t(\beta)}{1-t(\beta)} = \sum_{N=1}^{\infty} t^N(\beta)\,.$$

Here $t(\beta)$ stands for the Laplace transform of the pion inhomogeneity

$$(2.6)\qquad t(\beta) = B\int \mathrm{d}^4Q\,\delta_0(Q^2-\mu^2)\exp[-\beta_\mu Q^\mu] = \frac{2\pi\mu}{\beta}\,BK_1(\mu\beta)\,.$$

K_1 is a modified Hankel function. Inverse Laplace transformation of eq. (2.5) then yields the solution to eq. (2.1)

$$(2.7)\qquad \tau(Q^2) = \sum_{N=1}^{\infty} B^{N-1}\Omega_N(Q^2)$$

in terms of invariant N-particle phase space

$$(2.8)\qquad \Omega_N(Q^2) = \int\prod_{i=1}^{N} \mathrm{d}^4k_i\,\delta_0(k_i^2-\mu^2)\,\delta^4\left(\sum_{j=1}^{N} k_j - Q\right).$$

It is clear from the definition of $Z(\beta)$ that the singularity of $Z(\beta)$ at β_0, defined through

$$(2.9)\qquad t(\beta_0) = \bar{t} = 1\,,$$

induces an exponential increase of $\tau(Q^2)$ as $Q^2 \to \infty$, namely

$$\tau(Q^2) \sim M^{-a} \exp[\beta_0 M] . \tag{2.10}$$

The value of a follows immediately from the fact that the singularity of Z as a function of t is a pole, and is $a = \frac{3}{2}$ (9).

In the full bootstrap scheme (3,4,11) a fireball is composed of any number of fireballs:

$$B\tau(Q^2) = B\delta_0(Q^2 - \mu^2) + \sum_{N=2}^{\infty} \frac{B^N}{N!} \int \prod_{i=1}^{N} \mathrm{d}^4 k_i\, \tau(k_i^2)\, \delta^4\left(\sum_{j=1}^{N} k_j - Q\right) . \tag{2.11}$$

The Laplace transform of this equation

$$2Z = t + \exp[Z] - 1 \tag{2.12}$$

is solved by

$$Z = \sum_{N=1}^{\infty} g_N t^N , \tag{2.13}$$

where the g_N obey the recursion formula (6,12)

$$\begin{cases} g_1 = 1 , \\ g_{N+1} = \dfrac{-1}{N+1}\left[N g_N - 2 \sum_{k=1}^{N} k g_k \cdot g_{N+1-k}\right] . \end{cases} \tag{2.14}$$

The radius of convergence of expansion—eq. (2.13)—is determined through

$$\frac{\mathrm{d}t}{\mathrm{d}Z} = 0 , \tag{2.15}$$

and follows from eq. (2.12) to be $\tilde{t} = 2 \ln 2 - 1$. $Z(t)$ has a square-root branch point at $t = \tilde{t}$, since

$$\frac{\mathrm{d}^2 t}{\mathrm{d}Z^2} \neq 0 .$$

The equation

$$t(\beta_0) = \tilde{t} = 0.386 \tag{2.16}$$

is the analogue to eq. (2.9) in the linear-bootstrap case and allows one to calculate B from β_0 and *vice versa*. The solution to the full bootstrap equation reads

$$\tau(Q^2) = \sum_{N=1}^{\infty} g_N B^{N-1} \Omega_N(Q^2) , \tag{2.17}$$

(11) E.-M. Ilgenfritz and J. Kripfganz: *Nucl. Phys.*, **56** B, 241 (1973).
(12) K. Fabricius and U. Wambach: *Nucl. Phys.*, **62** B, 212 (1973).

the asymptotic behaviour of which is given in eq. (2.10) with $a=3$ [4,13].

We note that, intermediary to the full bootstrap and the linear bootstrap, one defines any sort of bootstrap model by changing the upper limit to the summation in N. All these versions produce a square-root branch-point singularity of $Z(t)$ and thus $a=3$.

2'2. *Bootstrap with isospin and G-parity conservation.* – In the following we want to introduce isospin into the bootstrap equations, or rather their Laplace transforms, eq. (2.4) and eq. (2.12). This will be done with the physical idea in mind that fireballs represent the continuation of the observed resonance spectrum towards higher masses. Therefore we will allow only nonexotic fireballs to appear in the scheme. Mesonic fireballs in particular should only have isospin zero or one.

2'2.1. Linear bootstrap. Let us start with the introduction of isospin and leave aside G-parity for the moment. If we denote the Laplace transforms of the isoscalar and isovector densities $B\tau^0(Q^2)$ and $B\tau^1_m(Q^2)$ by $S(\beta)$ and $Z_m(\beta)$, respectively, where m stands for the 3-component of isospin, the bootstrap equations for these quantities take the form

$$(2.18)\qquad \left|\begin{aligned} Z_m(\beta) &= t_m + \sum_{\substack{m_1,m_2\\ m_1+m_2=m}} C(1,1,1;\,m_1,m_2)^2 Z_{m_1} t_{m_2} + S t_m\,,\\ S(\beta) &= \sum_{m_1} C(1,1,0;\,m_1,-m_1)^2 Z_{m_1} t_{-m_1}\,.\end{aligned}\right.$$

$C(j_1, j_2, j;\, m_1, m_2)$ are standard Clebsch-Gordan coefficients.

We have added an index to the inhomogeneous term indicating the pion charge. Since $t_+ = t_0 = t_- \equiv t > 0$, it follows directly that $Z_+ = Z_- = Z_0 \equiv Z$. Thus the linear bootstrap predicts the isotriplet densities $\tau^1_m(Q^2)$ to be identical to each other, as one would require from isospin symmetry of the fireball spectrum

$$(2.19)\qquad \tau^1_+(Q^2) = \tau^1_-(Q^2) = \tau^1_0(Q^2)\,.$$

Therefore the bootstrap equations finally look as simple as this

$$(2.20)\qquad \left\{\begin{aligned} Z &= t + (Z+S)t\,,\\ S &= Zt\,,\end{aligned}\right.$$

and have the solution

$$(2.21)\qquad Z = \frac{-t}{(t-t_1)(t-t_2)}$$

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[13] W. NAHM: *Nucl. Phys.*, **45** B, 525 (1972).

with two simple poles at

$$t_{1,2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}\,. \tag{2.22}$$

Consequently the convergence radius $\bar{t}$ of the series expansion

$$Z = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} \binom{n}{\nu} t^{n+\nu+1} \tag{2.23}$$

is given by

$$\bar{t} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} = 0.618\,. \tag{2.24}$$

Thus, the isovector and isoscalar fireball densities from the linear bootstrap have the form

$$\begin{cases} \tau^1_+(Q^2) = \tau^1_-(Q^2) = \tau^1_0(Q^2) = \sum\limits_{\substack{N \geqslant 0 \\ \nu \leqslant N/2}} \binom{N-\nu}{\nu} B^N \Omega_{N+1}(Q^2)\,, \\ \tau^0(Q^2) = \sum\limits_{\substack{N \geqslant 0 \\ \nu \leqslant N/2}} \binom{N-\nu}{\nu} B^{N+1} \Omega_{N+2}(Q^2)\,, \end{cases} \tag{2.25}$$

the asymptotic behaviour of which is given by eq. (2.10) with β_0 this time determined from

$$t(\beta_0) = \bar{t} = 0.618\,. \tag{2.26}$$

It is interesting to see what happens to the linear bootstrap if only isovector fireballs are considered: one sees immediately from eqs. (2.20) that this *simplified linear bootstrap* amounts to the linear bootstrap without isospin as given in eq. (2.4).

Before closing this Subsection, we want to turn to the question of G-parity conservation. In order to include this quantum number into the scheme, we have to assign G-parity to the fireballs and to double the number of bootstrap equations. It is proved in Appendix A that the linear bootstrap with only pions as input particles yields the solution, which one might have naively anticipated for the even and odd G-parity fireball level densities $\tau^{(e)}$, $\tau^{(o)}$, namely

$$\tau_+^{1\binom{e}{o}}(Q^2) = \frac{1}{2} \sum_{\substack{N \geqslant 1 \\ \nu \leqslant N/2}} (1 \mp (-1)^N) \binom{N-\nu}{\nu} B^N \Omega_{N+1}(Q^2)\,, \tag{2.27}$$

and similarly for $\tau^{0\binom{e}{o}}$, *i.e.* one just has to drop in the previous solution eqs. (2.25) the terms with the wrong G-parity!

2'2.2. Full boostrap. We require from the beginning that the outcome of the bootstrap should be isospin symmetric, *i.e.* we postulate

$$Z_m = Z \quad \text{for} \quad m = +, -, 0 . \tag{2.28}$$

The task is then to generalize eq. (2.12) to a coupled system of bootstrap equations for S and Z. As a first step we find

$$\begin{cases} 2Z = t + \sum\limits_{\substack{N_0 \geqslant 0 \\ N_1 \geqslant 1}} \dfrac{\varrho^{(1)}_{N_1}}{(N_0+N_1)!} \dbinom{N_0+N_1}{N_0} S^{N_0} Z^{N_1} , \\ 2S = \sum\limits_{N_0=1}^{\infty} \dfrac{1}{N_0!} S^{N_0} + \sum\limits_{\substack{N_0 \geqslant 0 \\ N_1 \geqslant 2}} \dfrac{\varrho^{(1)}_{N_1}}{(N_0+N_1)!} \dbinom{N_0+N_1}{N_0} S^{N_0} Z^{N_1} . \end{cases} \tag{2.29}$$

The binomial coefficients account for the combinatorial weight of the term characterized through (N_0, N_1) in the sum. $\varrho^{(I)}_{N_1}$ denotes the isospin phase-space volume, which in this case equals the number of ways that N_1 isovector representations can be coupled to the total isospin I. The lower limits of the summation are readily deduced from isospin arguments.

The N_0-summations can easily be performed with the result

$$\begin{cases} 2Z = t + \exp[S] \sum\limits_{N=1}^{\infty} \dfrac{\varrho^{(1)}_N}{N!} Z^N , \\ 2S = \exp[S] - 1 + \exp[S] \sum\limits_{N=2}^{\infty} \dfrac{\varrho^{(0)}_N}{N!} Z^N . \end{cases} \tag{2.30}$$

The solution to eq. (2.30) proceeds again by series expansion

$$Z(t) = \sum_{N=1}^{\infty} c_N t^N , \qquad S(t) = \sum_{N=2}^{\infty} b_N t^N . \tag{2.31}$$

The recursion relations for the coefficients c_N, b_N are this time somewhat long and therefore relegated to Appendix B. The numerical values for c_N, b_N resulting from these recursion relations—eqs. (B.3) and (B.4)—are given together with the previous coefficients g_N and the isospin phase-space volumes $\varrho^{(0)}_N$, $\varrho^{(1)}_N$ in Table I.

The asymptotic behaviour of the fireball densities

$$\begin{cases} \tau^1(Q^2) = \sum\limits_{N=1}^{\infty} c_N B^{N-1} \Omega_N(Q^2) , \\ \tau^0(Q^2) = \sum\limits_{N=2}^{\infty} b_N B^{N-1} \Omega_N(Q^2) \end{cases} \tag{2.32}$$

TABLE I.

N	$\varrho_N^{l=0}$	$\varrho_N^{l=1}$	g_N	b_N	c_N
1	0	1	1	0	1
2	1	1	0.5	0.5	0.5
3	1	3	0.666 666 7	0.666 666 7	1.5
4	3	6	1.083 333 3	2.375	3.791 666 7
5	6	15	1.966 666 7	6.716 666 7	1.220 833 3 $\cdot 10^{1}$
6	15	36	3.822 222 2	2.310 416 7 $\cdot 10^{1}$	3.989 166 7 $\cdot 10^{1}$
7	36	91	7.779 365 1	7.905 714 3 $\cdot 10^{1}$	1.393 180 6 $\cdot 10^{2}$
8	91	232	1.636 984 1 $\cdot 10^{1}$	2.853 265 6 $\cdot 10^{2}$	4.989 016 1 $\cdot 10^{2}$
9	232	603	3.532 548 5 $\cdot 10^{1}$	1.047 674 9 $\cdot 10^{3}$	1.837 877 0 $\cdot 10^{3}$
10	603	1 585	7.774 962 1 $\cdot 10^{1}$	3.935 670 2 $\cdot 10^{3}$	6.896 280 7 $\cdot 10^{3}$
11	1 585	4 213	1.738 590 7 $\cdot 10^{2}$	1.500 231 2 $\cdot 10^{4}$	2.630 271 2 $\cdot 10^{4}$
12	4 213	11 298	3.938 738 0 $\cdot 10^{2}$	5.796 455 6 $\cdot 10^{4}$	1.016 132 7 $\cdot 10^{5}$
13	11 298	30 537	9.020 925 4 $\cdot 10^{2}$	2.263 675 5 $\cdot 10^{5}$	3.968 797 7 $\cdot 10^{5}$
14	30 537	83 097	2.085 274 9 $\cdot 10^{3}$	8.923 052 4 $\cdot 10^{5}$	1.564 465 0 $\cdot 10^{6}$
15	83 097	227 475	4.858 798 3 $\cdot 10^{3}$	3.545 281 9 $\cdot 10^{6}$	6.216 213 8 $\cdot 10^{6}$
16	227 475	625 992	1.139 972 1 $\cdot 10^{4}$	1.418 374 9 $\cdot 10^{7}$	2.487 021 7 $\cdot 10^{7}$
17	625 992	1 730 787	2.690 851 5 $\cdot 10^{4}$	5.708 972 1 $\cdot 10^{7}$	1.001 061 5 $\cdot 10^{8}$
18	1 730 787	4 805 595	6.385 723 9 $\cdot 10^{4}$	2.310 198 2 $\cdot 10^{8}$	4.051 013 5 $\cdot 10^{8}$
19	4 805 595	13 393 689	1.522 651 2 $\cdot 10^{5}$	9.393 103 1 $\cdot 10^{8}$	1.647 154 4 $\cdot 10^{9}$
20	13 393 689	37 458 330	3.646 241 0 $\cdot 10^{5}$	3.835 507 9 $\cdot 10^{9}$	6.726 011 2 $\cdot 10^{9}$
21	37 458 330	1.050 892 3 $\cdot 10^{8}$	8.765 197 3 $\cdot 10^{5}$	1.572 198 1 $\cdot 10^{10}$	2.757 087 8 $\cdot 10^{10}$
22	1.050 892 3 $\cdot 10^{8}$	2.956 739 9 $\cdot 10^{8}$	2.114 423 6 $\cdot 10^{6}$	6.467 032 5 $\cdot 10^{10}$	1.134 112 6 $\cdot 10^{11}$
23	2.956 739 9 $\cdot 10^{8}$	8.340 864 2 $\cdot 10^{8}$	5.116 819 6 $\cdot 10^{6}$	2.668 579 2 $\cdot 10^{11}$	4.679 918 7 $\cdot 10^{11}$
24	8.340 864 2 $\cdot 10^{8}$	2.358 641 4 $\cdot 10^{9}$	1.241 845 2 $\cdot 10^{7}$	1.104 368 3 $\cdot 10^{12}$	1.936 772 6 $\cdot 10^{12}$
25	2.358 641 4 $\cdot 10^{9}$	6.684 761 1 $\cdot 10^{9}$	3.021 970 1 $\cdot 10^{7}$	4.582 502 8 $\cdot 10^{12}$	8.036 619 3 $\cdot 10^{12}$
26	6.684 761 1 $\cdot 10^{9}$	1.898 505 7 $\cdot 10^{10}$	7.371 853 2 $\cdot 10^{7}$	1.906 140 9 $\cdot 10^{13}$	3.342 959 6 $\cdot 10^{13}$
27	1.898 505 7 $\cdot 10^{10}$	5.402 271 5 $\cdot 10^{10}$	1.802 378 1 $\cdot 10^{8}$	7.946 752 2 $\cdot 10^{13}$	1.393 705 1 $\cdot 10^{14}$
28	5.402 271 5 $\cdot 10^{10}$	1.540 005 6 $\cdot 10^{11}$	4.415 966 1 $\cdot 10^{8}$	3.319 973 9 $\cdot 10^{14}$	5.822 648 4 $\cdot 10^{14}$
29	1.540 005 6 $\cdot 10^{11}$	4.397 422 2 $\cdot 10^{11}$	1.084 055 8 $\cdot 10^{9}$	1.389 714 3 $\cdot 10^{15}$	2.437 338 2 $\cdot 10^{15}$
30	4.397 422 2 $\cdot 10^{11}$	1.257 643 2 $\cdot 10^{12}$	2.666 034 7 $\cdot 10^{9}$	5.827 796 8 $\cdot 10^{15}$	1.022 112 6 $\cdot 10^{16}$

is governed by the singularity of the series eqs. (2.31). In Appendix C we prove the following

Statement. There exist solutions to the bootstrap equations, eqs. (2.30), which are regular in $|t|<\tilde{t}$, where $\tilde{t}$ is a square-root branch point on the real axis, *i.e.*

$$Z(t)=g(t)+h(t)\sqrt{\tilde{t}-t}$$

with g, h regular in $|t|<\tilde{t}$ and $t=\tilde{t}$, and similarly for S.

Since the closest singularity to the origin with $\operatorname{Re}\tilde{t}>0$ controls the asymptotic behaviour of $\tau(Q^2)$, the situation corresponds very closely to the result of the full bootstrap obtained previously without isospin.

The functions $Z(t)$, $S(t)$ are plotted in Fig. 1. The value of the convergence radius $\tilde{t}$ was calculated on the computer and is $\tilde{t}=0.2265$. Thus the relation between B and β_0 reads now

$$t(\beta_0)=\tilde{t}=0.2265\,, \tag{2.33}$$

to be compared to eq. (2.16) in the case of no isospin. G-parity conservation, if imposed on the bootstrap equation eq. (2.30), leads again to the most natural

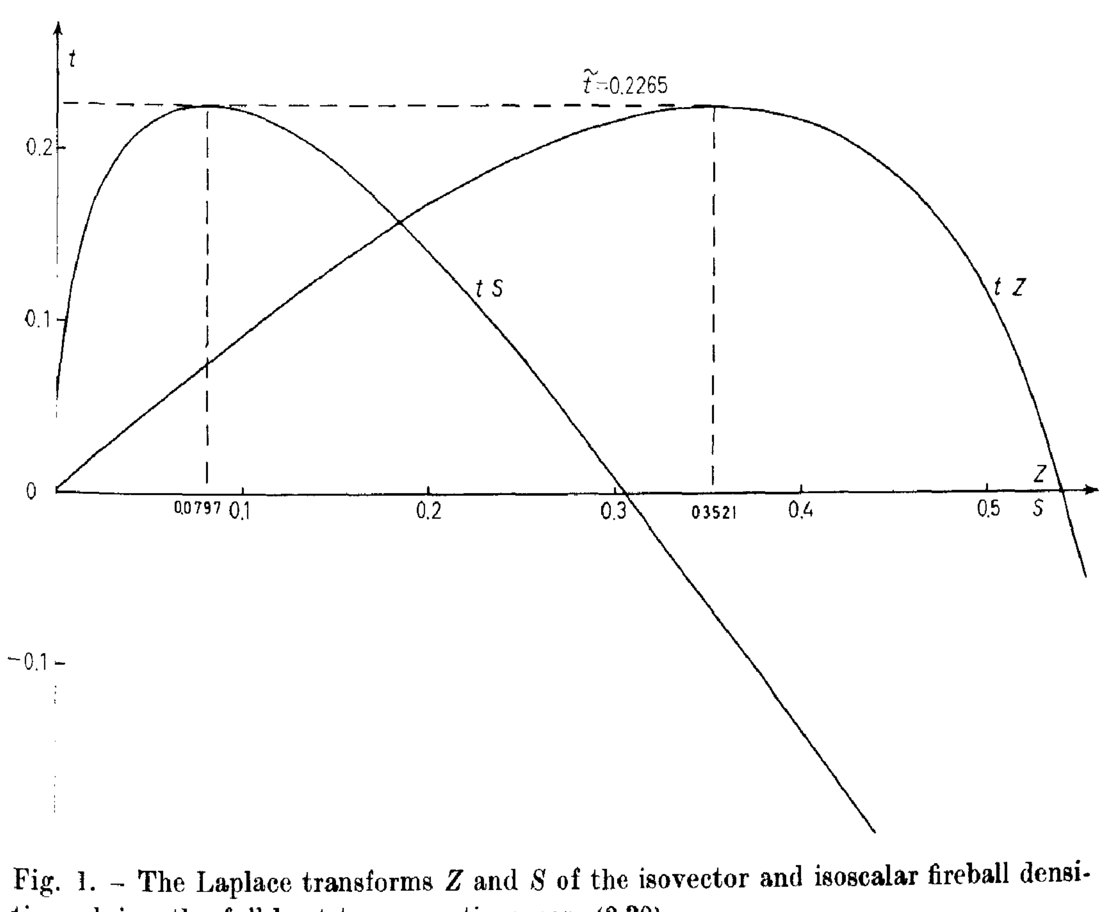

Fig. 1. – The Laplace transforms Z and S of the isovector and isoscalar fireball densities solving the full-bootstrap equations eqs. (2.30).

modification of the solution

$$(2.34)\qquad Z^{(\mathrm{e})}_{(\mathrm{o})} = \tfrac{1}{2}\sum_{N=1}^{\infty}(1 \pm (-1)^N)c_N t^N, \qquad S^{(\mathrm{e})}_{(\mathrm{o})} = \tfrac{1}{2}\sum_{N=2}^{\infty}(1 \pm (-1)^N)b_N t^N.$$

This statement is proved in Appendix A.

We can summarize this Section by stating that isospin conservation in conjunction with G-parity conservation and nonexoticity conditions do not change the asymptotic behaviour of the bootstrap solutions. The only changes with respect to the previous bootstrap occur in the relations between B and β_0 and in the nonasymptotic behaviour of $\tau(Q^2)$. We now want to investigate the features that appear in fireball decay as a consequence of quantum number constraints on the bootstrap.

3. – Fireball decay.

3·1. *Basic philosophy.* – In order to describe fireball decay, we want to make the very natural assumption that these objects decay as they are built up [6,14]. In other words, we set the partial decay width for the decay of a fireball into N pions, Γ_N, to be proportional to the N-th term in the expansion of the fireball level density in terms of phase space integrals

$$\tau(Q^2) = \sum_{N=1}^{\infty} d_N B^{N-1}\Omega_N(Q^2),$$

namely

$$(3.1)\qquad \Gamma_N \sim d_N B^N \Omega_N(Q^2),$$

where d_N is the appropriate coefficient deduced from the bootstrap chosen. Equation (3.1) holds up to a normalization that depends on Q^2, but not on N. If we introduce the (unknown) total width Γ_{tot}, eq. (3.1) can be rewritten as an equality

$$(3.2)\qquad \Gamma_N = \frac{d_N B^N \Omega_N}{\sum_{n=2}^{\infty} d_n B^n \Omega_n}\,\Gamma_{\mathrm{tot}}.$$

Correspondingly the inclusive one-particle distribution reads

$$(3.3)\qquad 2k_0\frac{\mathrm{d}^3\sigma}{\mathrm{d}k^3} = \frac{\sum_{N=2}^{\infty} N d_N B^N \Omega_{N-1}((Q-k)^2)}{\sum_{n=2}^{\infty} d_n B^n \Omega_n(Q^2)}\,\sigma_{\mathrm{tot}},$$

(14) I. MONTVAY: *Nucl. Phys.*, **53** B, 521 (1973).

which is obtained by just dropping (symmetrically) the integration over one particle in the phase-space integral of eq. (3.1).

In the subsequent considerations, where we compare the predictions of various bootstrap versions, we shall once and for all assume a limiting temperature

$$T_0 = \frac{1}{\beta_0} = 160\ \text{MeV}\,,$$

which is a reasonable value from experiment. This requirement fixes the « volume » parameter B within each of these versions through eqs. (2.9), (2.16), (2.26), (2.33), respectively. This philosophy is not in the spirit of ref. (9), which argues that B should equal the pion Compton wavelength.

3·2. *No charge detected: asymptotic equivalence.* – We are now in a position to extract predictions of the various bootstrap versions on all sorts of inclusive measurements as long as the charge of the final particles is not detected. For the integrated correlation functions $f_N(Q^2)$, which are given by

$$(3.4)\qquad \begin{cases} f_1 = \langle N\rangle\,, \\ f_2 = \langle N(N-1)\rangle - \langle N\rangle^2\,, \\ f_3 = \langle N(N-1)(N-2)\rangle - 3\langle N\rangle\langle N(N-1)\rangle + 2\langle N\rangle^3\,, \end{cases}$$

and for general N through (15)

$$(3.5)\qquad f_N = B^N \frac{\mathrm{d}^N}{\mathrm{d}B^N} \ln\left(B\tau(Q^2)\right)\,,$$

we prove the following

Statement. To leading order in the fireball mass M the linear and full bootstrap models with or without isospin predict identical correlation functions, if T_0 is chosen universal.

Proof. From eq. (3.5), which we derive for the convenience of the reader in Appendix D, and from the asymptotic form

$$(3.6)\qquad \tau(Q^2) \underset{M\to\infty}{=} f(B)\,M^{-a}\exp[\beta_0 M]\left\{1 + \frac{1}{M}h(B) + \ldots\right\},$$

the leading term of f_N is derived to be (7)

$$(3.7)\qquad f_N \simeq M B^N \frac{\mathrm{d}^N}{\mathrm{d}B^N}\beta_0(B)\,.$$

(15) A. H. MUELLER: *Phys. Rev. D*, **4**, 150 (1971); L. S. BROWN: *Phys. Rev. D*, **5**, 748 (1972); K. J. BIEBL and J. WOLF: *Nucl. Phys.*, **44** B, 301 (1972).

Here β_0 stands for the function that solves the singularity equation of the respective bootstrap. Since all of these relations—eqs. (2.9), (2.16), (2.26), (2.33)—are of the type

$$\frac{\tilde{t}}{B} = \frac{2\pi\mu}{\beta_0} K_1(\mu\beta_0) := \varphi(\beta_0) \tag{3.8}$$

with properly chosen convergence radius $\tilde{t}$, $\beta_0(B)$ has the form

$$\beta_0(B) = \varphi^{-1}\left(\frac{\tilde{t}}{B}\right) := g\left(\frac{B}{\tilde{t}}\right), \tag{3.9}$$

and consequently

$$f_N \simeq M\varkappa^N \frac{\mathrm{d}^N}{\mathrm{d}\varkappa^N} g(\varkappa)\bigg|_{\varkappa=(\tilde{t}/B)^{-1}}. \tag{3.10}$$

Since g is a universal function in all bootstrap versions and the value of $\varkappa$ is the same in all of them (see eq. (3.8)), the proof is completed.

3'3. *Pion charge detected.* – In order to obtain predictions on charge-dependent quantities such as $f_{N_+N_0N_-}$, we have to carefully keep track of the charges of the pion inhomogeneities, *i.e.* we have to expand the solution to the bootstrap equations in terms of powers of t_+, t_-, t_0:

$$Z_m = \sum_{\substack{N_+,N_-,N_0\\ N_+-N_-=m}} c_{N_+N_0N_-} t_+^{N_+} t_0^{N_0} t_-^{N_-}. \tag{3.11}$$

We want to go in detail through this calculation of the coefficients c in the case of the linear bootstrap, which has been formulated in terms of t_+, t_0, t_- through eq. (2.18).

3'3.1. Linear bootstrap. If we discriminate formally between the pion charges the solution to the linear bootstrap reads

$$\begin{cases} Z_{+,0,-} = \dfrac{\mathcal{N}_{+,0,-}}{\Delta}, \\ S = \frac{1}{3}\{Z_+t_- + Z_-t_+ + Z_0t_0\}, \end{cases} \tag{3.12}$$

where

$$\begin{cases} \mathcal{N}_\pm = t_\pm\left(1+\dfrac{t_0}{2}\right), \\ \mathcal{N}_0 = t_0 - \dfrac{t_0^2}{2} + t_+t_-, \\ \Delta = 1 - \frac{3}{2}t_0 - \frac{7}{6}t_+t_- - \frac{1}{3}t_0^2 - \frac{2}{3}t_+t_-t_0 + \frac{1}{6}t_0^3. \end{cases} \tag{3.13}$$

Explicit expressions for the coefficients $c_{N_+N_0N_-}$ are given in Appendix E.

From the explicit form of the solution, eq. (3.12), one can prove the following

Statement. In the linear-bootstrap scheme the natural moments of the decay pion multiplicity distribution of a neutral fireball, $\langle N_+^{l_+} N_0^{l_0} N_-^{l_-}\rangle$, do not depend, to leading order in M, on the individual l_i, but only on their sum:

$$\langle N_+^{l_+} N_0^{l_0} N_-^{l_-}\rangle \simeq \left(\frac{1}{3}\frac{M}{\mu}\frac{K_1(\mu\beta_0)}{K_2(\mu\beta_0)}\right)^{l_++l_0+l_-}. \tag{3.14}$$

The proof of this statement is given in Appendix D.

As a corollary to the last statement we find, not unexpectedly,

$$\langle N_+\rangle = \langle N_-\rangle \simeq \langle N_0\rangle . \tag{3.15}$$

More important are the charge correlations $f_{N_+N_0N_-}$ (for the definition see Appendix D). They have been calculated numerically for fireball masses $1\ \text{GeV} \leqslant M \leqslant 8\ \text{GeV}$ by using the explicit bootstrap solution and will be presented in Sect. **4**.

The asymptotic expression for the charge correlations of a decaying fireball of charge m reads

$$\underset{m=N_+-N_-}{f^m_{N_+N_0N_-}} = MB^N \frac{\mathrm{d}^N}{\mathrm{d}B_+^{N_+}\,\mathrm{d}B_0^{N_0}\,\mathrm{d}B_-^{N_-}}\,\beta_0(B_+, B_0, B_-)\Bigg|_{\substack{B_+=B_0=B_-=B\\ N=N_++N_0+N_-}}, \tag{3.16}$$

which is the analogue to eq. (3.7), where the B's, like the t's, carry a charge index. Note that this leading term depends neither on the charge nor on the G-parity of the decaying fireball. In contrast to the asymptotic behaviour of f_N, eq. (3.5), the asymptotic expressions for the $f_{N_+N_0N_-}$ depend strongly on the bootstrap version. This is demonstrated in Appendix D.

3'3.2. Formulation of the problem for the full bootstrap. Having solved the problem of charge correlations in the linear-bootstrap case, we turn now to the full bootstrap, eq. (2.30).

Evidently, we lost some information, namely that on charge, by suppressing the charge indices in this equation. For the purpose of the bootstrap, this was perfectly all right and in fact enabled us to find the solution with reasonable effort. Being now interested in charge correlations among secondaries, however, we have to differentiate between the various Z's, thus ending up with rather horrible bootstrap equations. Although we are not motivated enough to solve these equations, we want to write them down in the following for reasons of completeness.

To formulate the problem one has to make an assumption about the branching ratios between the different charge distributions of fireballs from direct fireball decay. In accord with the philosophy underlying eq. (2.30), we assume conservation of isospin and charge as well as equal probability of each possible isospin state among the immediate daughter fireballs emerging from a fireball decay. This corresponds to the assumptions of CERULUS ([5]), except that we apply them to a state of secondary fireballs, not pions. As a consequence, we introduce Cerulus coefficients $^*P^{(I)}_{N_+N_0N_-}$ (in the notation of ref. ([5])) to weight the charge distributions and obtain the bootstrap equations ([*])

$$(3.17)\quad \begin{cases} 2Z_m = t_m + \exp[S] \displaystyle\sum_{\substack{N_+,N_0,N_- \\ \sum_i N_i = N \geqslant 1,\, N_+ - N_- = m}} \frac{1}{N!}\, {}^*P^{(1)}_{N_+N_0N_-} Z_+^{N_+} Z_0^{N_0} Z_-^{N_-}\,, \\ 2S = \exp[S] - 1 + \exp[S] \displaystyle\sum_{\substack{N_+,N_0,N_- \\ \sum_i N_i = N \geqslant 2,\, N_+ = N_-}} \frac{1}{N!}\, {}^*P^{(0)}_{N_+N_0N_-} Z_+^{N_+} Z_0^{N_0} Z_-^{N_-}\,. \end{cases}$$

4. – Numerical results and discussion.

In order to exhibit the impact of isospin conservation on the bootstrap solution and the decay characteristics of a fireball at finite M, we have calculated average multiplicities and integrated correlations in the range $1\ \text{GeV} \leqslant M \leqslant 8\ \text{GeV}$, using pions as stable input particles and choosing $T_0 = 160\ \text{MeV}$ for all bootstrap models. The phase space integrals were calculated with a program of KAJANTIE and KARIMÄKI ([17]).

A characteristic sample of results is given in Fig. 2 to 6 for the decay of neutral fireballs with the quantum numbers $(I, G) = (1, +)$ and $(0, -)$, respectively (which are the quantum numbers of the photon).

In Fig. 2 we show the predictions of the full-bootstrap model (eq. (2.29)) and the linear-bootstrap model (eq. (2.18)) for the f_n, if no charge is detected. Notice, that asymptotia sets in fairly early. As was already found in ref. ([6]), the average multiplicity f_1 is practically proportional to M for $M \geqslant 1\ \text{GeV}$. The same is true for f_2 above 2 GeV and f_3 above 3 GeV. Comparing the two parts of Fig. 2, we realize that the quantum numbers (I, G) have no influence on the correlations in the asymptotic region, *i.e.*

$$(4.1)\qquad f_n^{(I,G)} = d_n M + c_n\,.$$

([*]) There is of course no unique way of injecting isospin into the full bootstrap. WAMBACH ([16]), *e.g.*, uses a special ansatz for the isospin coupling, motivated by the Chan-Paton method for the construction of dual amplitudes with isospin.

([16]) U. WAMBACH: University of Bielefeld preprint Bi-74/01, Jan. 1974.

([17]) K. KAJANTIE and V. KARIMÄKI: *Computer Phys. Comm.*, **2**, 207 (1971).

While d_n is model independent, the intercepts c_n are seen to differ between the full and the linear bootstrap models.

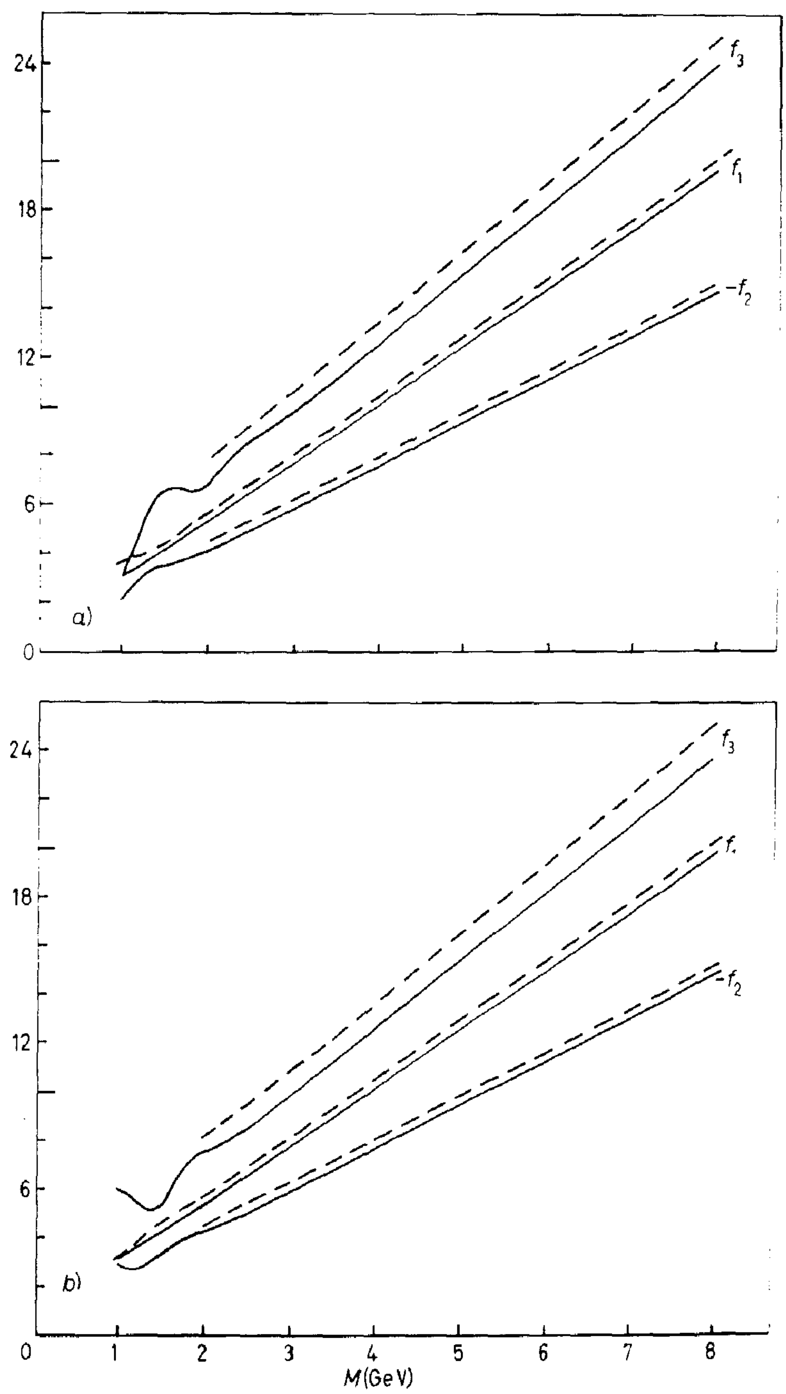

Fig. 2. – Correlation functions f_1, f_2, f_3, eq. (3.4), for the decay of «photonlike» fireballs of mass M: full lines: full-bootstrap model (eq. (2.30)); broken lines: linear-bootstrap model (eq. (2.20)). *a*) $G = +$, $I = 1$; *b*) $G = -$, $I = 0$.

Turning now to the case when charges are detected, we show the results of the linear bootstrap (eq. (2.18)) for average multiplicities and independent charge correlations in Fig. 3 and 4. The remaining f's can be calculated from

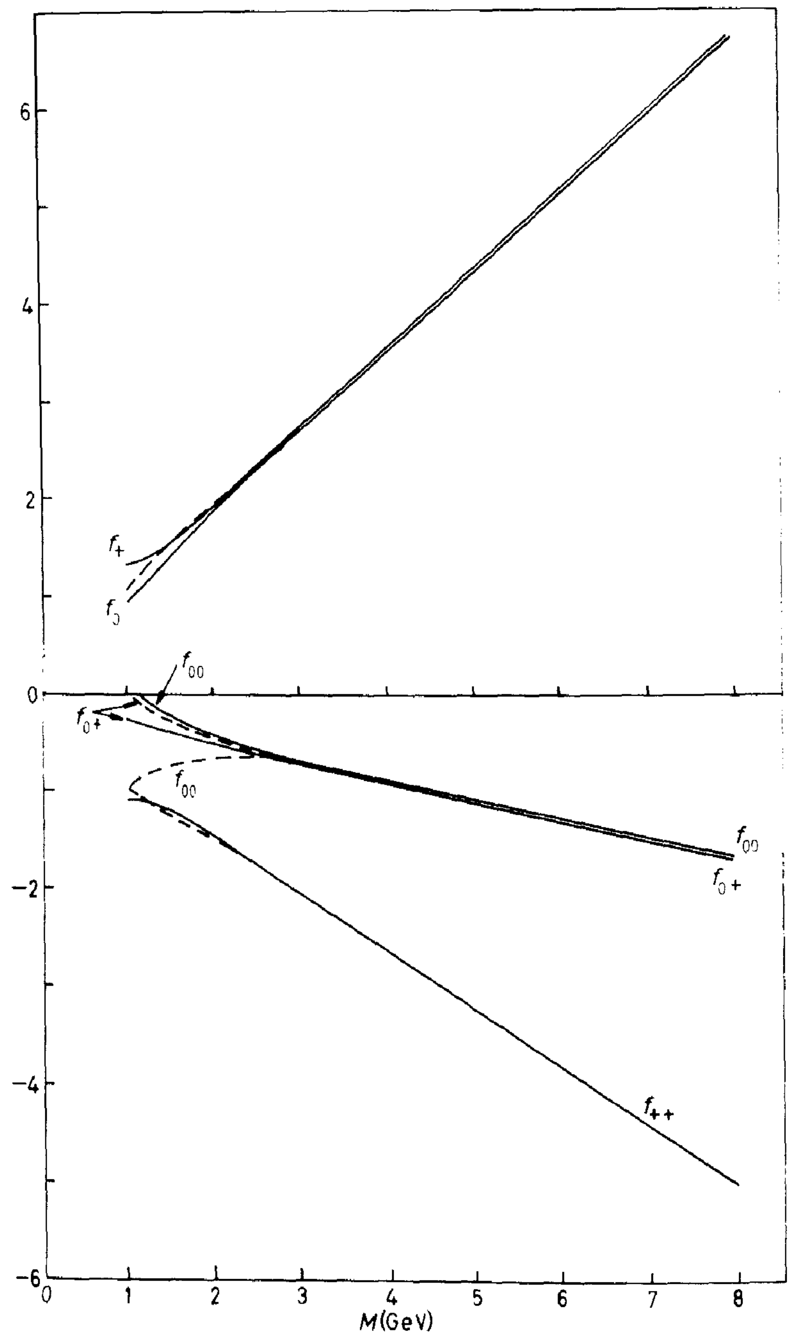

Fig. 3. – The charge correlation functions f_q and $f_{q_1q_2}$ predicted by the linear-bootstrap model for isoscalar and isovector fireballs (eq. (2.20)). The quantum numbers of the decaying fireballs are $G=+$, $I=1$, $I_3=0$ (full lines), $G=-$, $I=I_3=0$ (broken lines).

the equations

$$(4.2)\quad \left\{\begin{aligned} f_- &= f_+ , & f_{0--} &= f_{0++} ,\\ f_{0-} &= f_{0+} , & f_{0+-} &= f_{0++} + f_{0+} ,\\ f_{--} &= f_{++} , & f_{++-} &= f_{+--} = f_{+++} + 2f_{++} ,\\ f_{+-} &= f_{-+} = f_{++} + f_+ , & f_{---} &= f_{+++} .\\ f_{00-} &= f_{00+} , \end{aligned}\right.$$

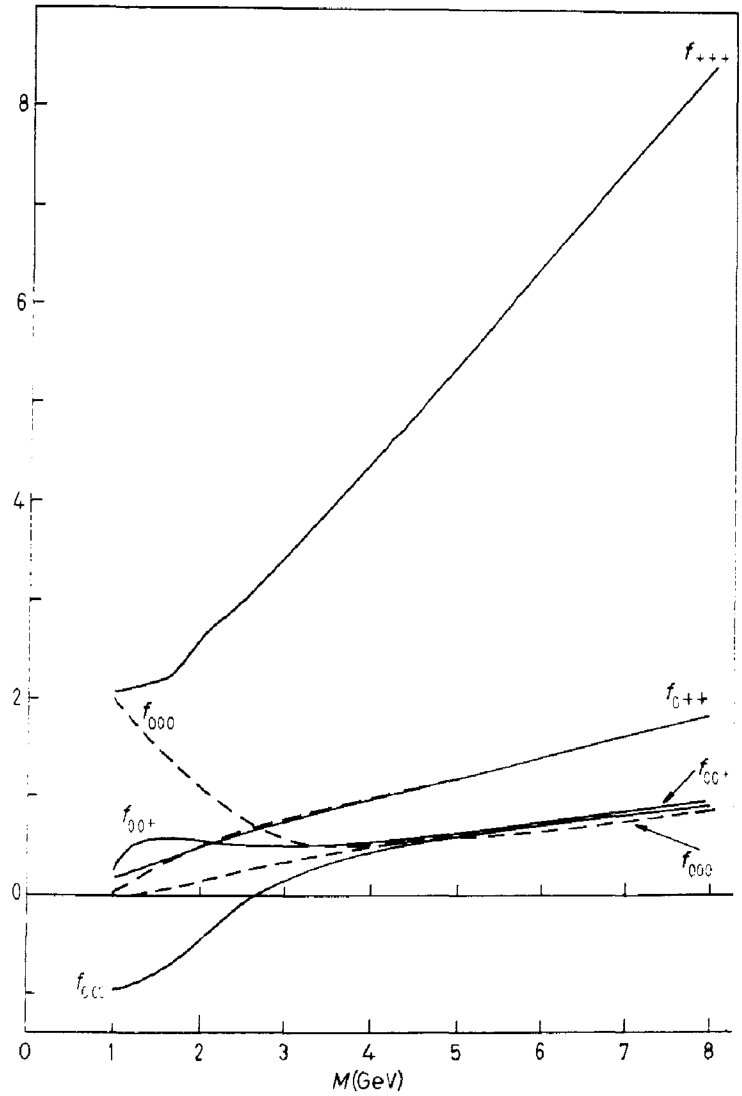

Fig. 4. – The charge correlation functions $f_{q_1q_2q_3}$ predicted by the linear-bootstrap model for isoscalar and isovector fireballs. Same notation as in Fig. 3.

As to the onset of asymptotics the picture remains roughly unchanged compared to Fig. 2; again it is found that isospin and G-parity of the decaying fireball are of no importance in the asymptotic regime. In the nonasymptotic region, however, there is a strong (I, G)-dependence, which increases with the number of neutral particles involved in the measurement. We note that the only positive two-particle correlation is f_{+-}:

$$f_{+-} \approx 0.28 f_{++} . \tag{4.3}$$

The coincidence of f_{00} and f_{0+} is fortuitous, as can be seen from a comparison of the linear-bootstrap result given in Fig. 3 with the prediction of the simplified linear bootstrap (eq. (2.4)) contained in Fig. 5. A comparison of the asymptotic predictions is given in Fig. 6. This demonstrates that the asymptotic behaviour of the charge correlations depends sensitively on the isospin quantum numbers

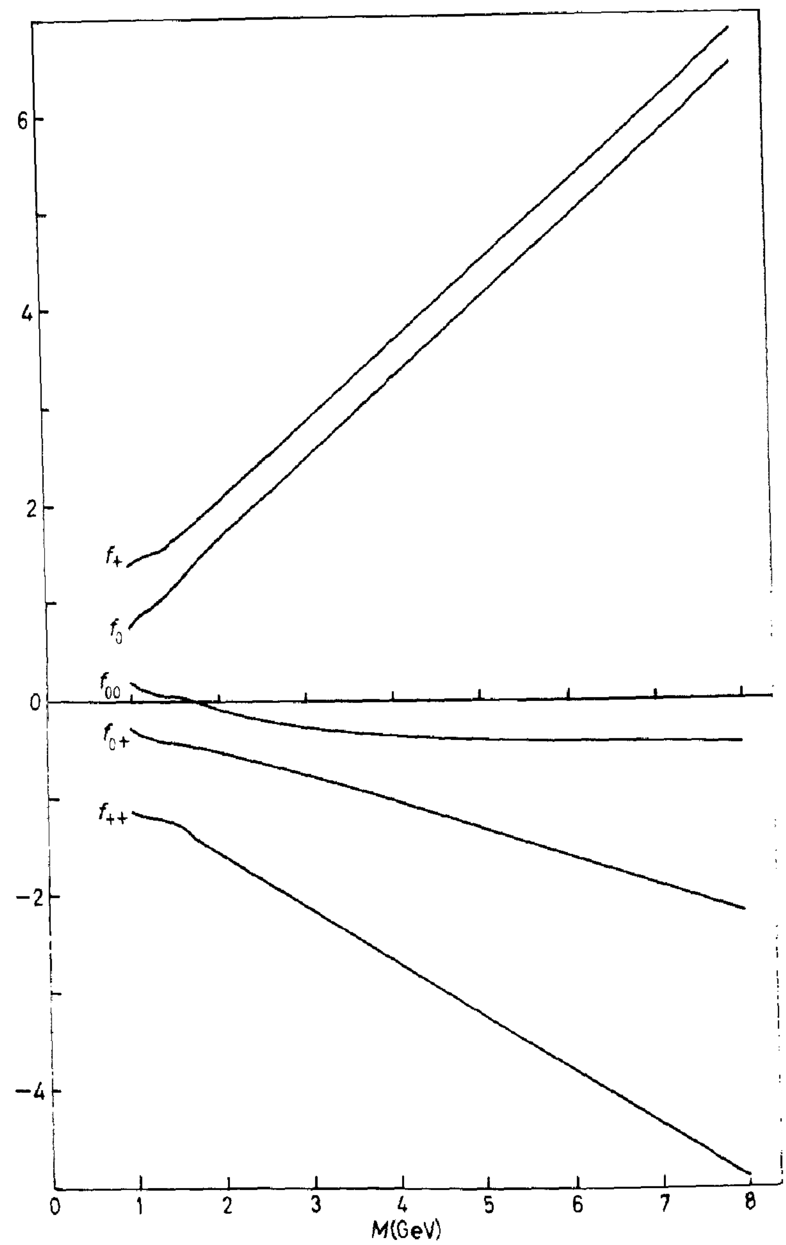

Fig. 5. – The charge correlation functions f_q and $f_{q_1q_2}$ predicted by the linear-bootstrap model for isovector fireballs only, eq. (2.4). The quantum numbers of the decaying fireball are $G = +$, $I = 1$, $I_3 = 0$.

which the fireballs are allowed to carry in the bootstrap. This result is corroborated by an analytical evaluation of the leading terms given in Appendix D (*).

It is remarkable that the predictions of the linear bootstrap (eq. (2.18)) coincide within one percent for $E \geqslant 5$ GeV with the results of a calculation

(*) Unfortunately, we cannot recover the results of KRIPFGANZ and ILGENFRITZ ([8]). Their asymptotic expressions for f_{--}, f_{-0}, f_{00} are different from ours, eq. (D.10), and they find positive values for these quantities.

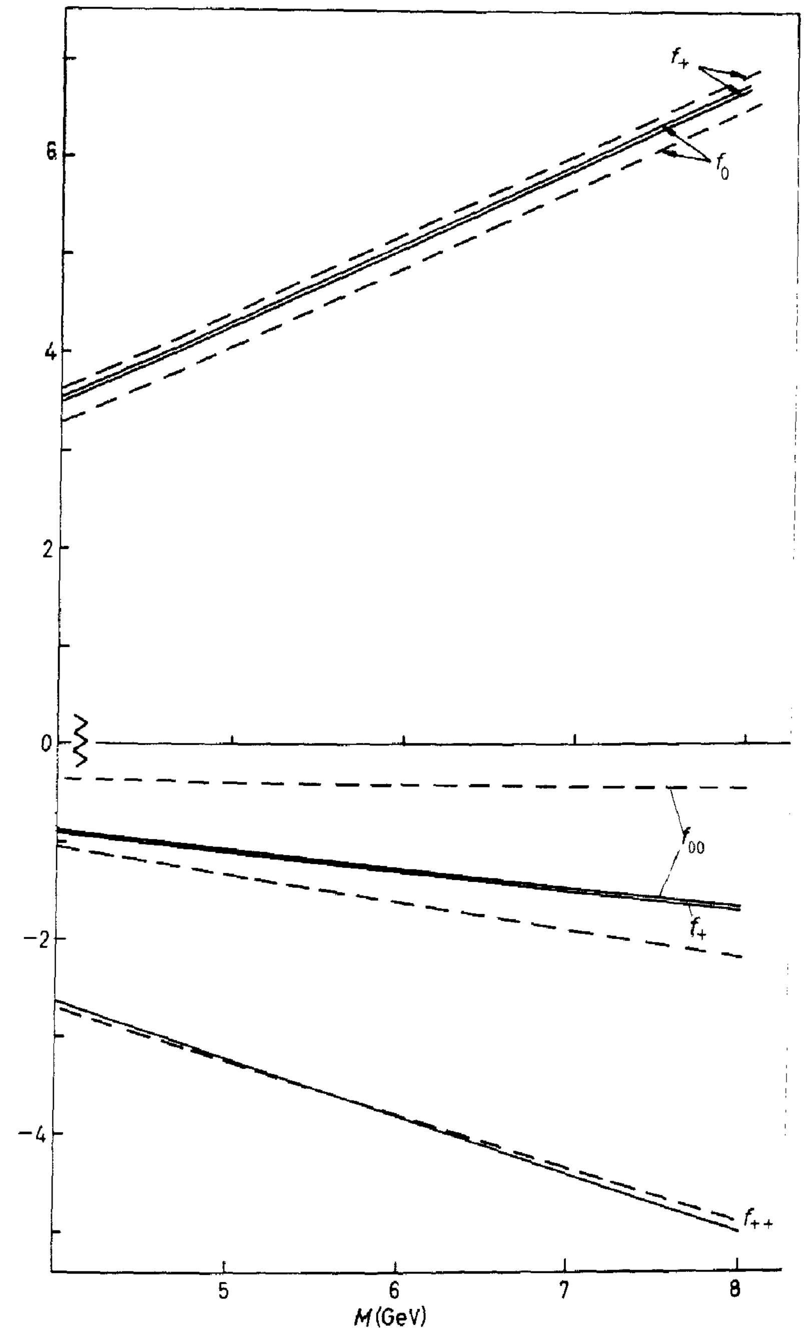

Fig. 6. – Comparison of the asymptotic charge correlations from the linear-bootstrap model with (eq. (2.20), full lines) and without (eq. (2.4), broken lines) isoscalar fireballs. The quantum numbers of the decaying fireball are $G = +$, $I = 1$, $I_3 = 0$.

which neglects isospin in the bootstrap and uses Cerulus coefficients to obtain the partial widths $\Gamma^{(I)}_{N_+N_0N_-}$ from $\Gamma_{N_{tot}}$

$$\Gamma^{(I)}_{N_+N_0N_-} = \frac{{}^*P^{(I)}_{N_+N_0N_-}}{\varrho^{(I)}_{N_{tot}}}\,\Gamma_{N_{tot}}\,, \qquad N_{tot} = N_+ + N_0 + N_-\,. \tag{4.4}$$

This does not hold, however, for the simplified linear bootstrap.

5. – Summary and conclusions.

Our results can be summarized in short:

1) We formulated bootstrap equations for isoscalar and isovector mesonic fireballs with strangeness zero and solved them explicitly. It was shown that conservation of G-parity and isospin does not alter the asymptotic behaviour of the fireball densities.

2) The leading terms of the correlation functions of fireball decay f_N are predicted to have the same value within all bootstrap models considered.

3) Charge correlations of secondary pions depend sensitively on the isospin of the fireballs admitted in the bootstrap.

4) In our case of pion secondaries only, the onset of asymptotics, characterized through straight-line behaviour in M, occurs at $M = 2$ GeV for the average multiplicities, at $M = 3$ GeV for two-particle charge correlations and at 4 GeV for three-particle charge correlations. The quantity f_{+-} is the only positive two-particle correlation.

5) For not too small energies we have found numerically that imposing charge weight *à la* CERULUS to the final states obtained from the isospin-free linear bootstrap leads to the same results as given by the linear bootstrap that includes isospin. This might be useful for practical calculations. We expect the same to hold for the full-bootstrap case.

6) The most relevant test for the validity of the statistical bootstrap model applied to fireball decay is the linear dependence of all correlation functions on the fireball mass M for large M. This linear dependence does not hold for other statistical models ([18]).

7) We remind the reader that the numerical results depend on the value of T_0, which is not too well determined. Moreover, inclusion of kaons and baryons leads to modifications.

APPENDIX A

G-parity in the bootstrap.

In this Appendix we want to prove eqs. (2.27) and (2.34). Denote the even (odd) G-parity triplet contributions by $Z^{(e)}$ ($Z^{(o)}$) and the singlet terms by $S^{(e)}$ ($S^{(o)}$). If we bear in mind that pions carry odd G-parity, eq. (2.20) is

([18]) A. JABS: University of Kaiserslautern preprint, Sept. 1972; S. J. ORFANIDIS and V. RITTENBERG: *Nucl. Phys.*, **59** B, 570 (1973).

modified to

$$\text{(A.1)}\qquad \begin{cases} Z^{(o)} = t + (Z^{(e)} + S^{(e)})\,t\,, \\ Z^{(e)} = (Z^{(o)} + S^{(o)})\,t\,, \\ S^{(o)} = Z^{(e)}t\,, \\ S^{(e)} = Z^{(o)}t\,. \end{cases}$$

These equations can be written more symmetrically

$$\text{(A.2)}\qquad \begin{cases} \mathfrak{Z}^{\pm} = \pm t \pm t(\mathfrak{Z}^{\pm} + \mathfrak{S}^{\pm})\,, \\ \mathfrak{S}^{\pm} = \mathfrak{Z}^{\pm}(\pm t)\,, \end{cases}$$

with the abbreviations

$$\text{(A.3)}\qquad \begin{cases} \mathfrak{Z}^{\pm} = Z^{(e)} \pm Z^{(o)}\,, \\ \mathfrak{S}^{\pm} = S^{(e)} \pm S^{(o)}\,. \end{cases}$$

We observe that $\mathfrak{Z}^{\pm}$, $\mathfrak{S}^{\pm}$ obey the previous bootstrap equations, eqs. (2.20), if the sign of the pion inhomogeneity t is properly chosen to be $\pm t$, respectively. Therefore, we finally arrive at

$$\text{(A.4)}\qquad \begin{cases} \tau_{+}^{1(e)} \pm \tau_{+}^{1(o)} = \sum\limits_{\substack{N\geqslant 0\\ \nu\leqslant N/2}} \binom{N-\nu}{\nu} B^{N}(-1)^{N+1}\Omega_{N+1}(Q^2)\,, \\[2ex] \tau^{0(e)} \pm \tau^{0(o)} = \sum\limits_{\substack{N\geqslant 0\\ \nu\leqslant N/2}} \binom{N-\nu}{\nu} B^{N+1}(-1)^{N}\Omega_{N+2}(Q^2)\,, \end{cases}$$

from which eq. (2.27) follows immediately.

The introduction of G-parity into the full bootstrap scheme, eq. (2.30), proceeds very similarly. In terms of $\mathfrak{Z}^{\pm}$, $\mathfrak{S}^{\pm}$, the bootstrap equation reads

$$\text{(A.5)}\qquad \begin{cases} 2\mathfrak{Z}^{\pm} = \pm t + \exp[\mathfrak{S}^{\pm}]\sum\limits_{N=1}^{\infty} \dfrac{\varrho_N^{(1)}}{N!}\,\mathfrak{Z}^{\pm N}\,, \\[2ex] 2\mathfrak{S}^{\pm} = \exp[\mathfrak{S}^{\pm}] - 1 + \exp[\mathfrak{S}^{\pm}]\sum\limits_{N=2}^{\infty} \dfrac{\varrho_N^{(0)}}{N!}\,\mathfrak{Z}^{\pm N}\,. \end{cases}$$

Again the problem is diagonal and has the same form as previously, with proper sign of t. This proves eq. (2.34).

Appendix B

Recursion relations.

Near $t = 0$ the bootstrap equations (3.1)

$$\text{(B.1)}\qquad \begin{cases} 2Z = t + \exp[S]\,\lambda'(Z)\,, \\ 2S = \exp[S]\,\lambda(Z) - 1 \end{cases}$$

with

$$\lambda(Z) = 1 + \sum_{N=2}^{\infty} \varrho_N^{(0)} \frac{Z^N}{N!}$$

have solutions that allow a power series expansion

$$S = \sum_{n=0}^{\infty} b_n t^n , \qquad Z = \sum_{n=0}^{\infty} c_n t^n . \tag{B.2}$$

The physical solutions are distinguished by the conditions

$$c_0 = 0 \qquad \text{and} \qquad b_0 = b_1 = 0 ,$$

which are equivalent to

$$\tau^1(Q^2) = 0 \qquad \text{if} \qquad Q^2 < \mu^2 ,$$

$$\tau^0(Q^2) = 0 \qquad \text{if} \qquad Q^2 < 4\mu^2 .$$

By inserting the ansatz (B.2) into the equation (B.1) one derives the following recursion formulae for the coefficients b_n and c_n:

$$b_n = \Lambda_n + E_n + \sum_{k=2}^{n-2} (b_k + E_k) \Lambda_{n-k} , \tag{B.3}$$

$$c_n = L_n + E_{n-1} + b_{n-1} + \sum_{k=2}^{n-2} (b_k + E_k)(c_{n-k} + L_{n-k}) , \tag{B.4}$$

with

$$E_n = \sum_{m=2}^{n} \sum_{\substack{l_2, \dots, l_{n-1} \\ \sum_{j=2}^{n-1} l_j = m \\ \sum_{j=2}^{n-1} j l_j = n}} \frac{b_2^{l_2} \dots b_{n-1}^{l_{n-1}}}{l_2! \dots l_{n-1}!} , \tag{B.5}$$

$$\Lambda_n = \sum_{m=2}^{n} \varrho_m^{(0)} \sum_{\substack{l_1, \dots, l_{n-1} \\ \sum_{j=1}^{n-1} l_j = m \\ \sum_{j=1}^{n-1} j l_j = n}} \frac{c_1^{l_1} \dots c_{n-1}^{l_{n-1}}}{l_1! \dots l_{n-1}!} , \tag{B.6}$$

$$L_n = \sum_{m=2}^{n} \varrho_{m+1}^{(0)} \sum_{\substack{l_1, \dots, l_{n-1} \\ \sum_{j=1}^{n-1} l_j = m \\ \sum_{j=1}^{n-1} j l_j = n}} \frac{c_1^{l_1} \dots c_{n-1}^{l_{n-1}}}{l_1! \dots l_{n-1}!} . \tag{B.7}$$

Appendix C

Analytic properties.

Here we study the analytic properties of the physical solution of the bootstrap equations

(C.1a) $$2Z = t + \exp[S]\,\lambda'(Z)\,,$$

(C.1b) $$2S = \exp[S]\,\lambda(Z) - 1$$

with

$$\lambda(Z) = 1 + \sum_{n=2}^{\infty} \frac{\varrho_n^{(0)}}{n!} Z^n\,.$$

Statement C. The physical solutions (*) $Z(t)$ and $S(t)$ of the system (C.1) are analytic functions regular in a circle $|t| \leqslant t_0$: $Z(t)$ and $S(t)$ are both singular at $t = t_0$ on the positive real axis. The singularity is of square-root type.

To prove this statement we shall use repeatedly a theorem on the inversion of analytic functions [19]: A power series

$$W(Z) = \sum_{n=0}^{\infty} a_n (Z - Z_1)^n$$

has an inversion

$$P(W) = \sum_{n=0}^{\infty} b_n (W - W_1)^n$$

with $W_1 = W(Z_1)$ and $P(W(Z)) \equiv Z$, if and only if $a_1 \neq 0$.

Proof of statement C. We divide the proof into a number of steps:

1) Equation (C.1*b*)

$$\lambda(Z) = (2S + 1)\exp[-S]$$

may be inverted for $S < \frac{1}{2}$, because $\lambda[S]$ is regular and $\mathrm{d}\lambda/\mathrm{d}S \neq 0$ for $S < \frac{1}{2}$. So $S[\lambda]$ defined by eq. (C.1*b*) is regular near $\lambda = 1$.

2) As $\varrho_n^{(0)} < 3^n$, $\lambda(Z)$ is an entire analytic function of Z. Therefore

$$S[\lambda(Z)] = S(Z)$$

is a regular function of Z in the circle $|Z| < Z_1$ (Z_1 is defined by $S[\lambda(Z_1)] = \frac{1}{2}$).

(*) See Appendix B.

[19] H. Behnke and F. Sommer: *Theorie der Funktionen einer komplexen Veränderlichen*, IV. Kapitel (Berlin, 1955).

3) The result of step 2) shows that

$$t = 2Z - \exp[S]\lambda'(Z) \qquad \text{(eq. (C.1}a\text{))}$$

is regular in the circle $|Z| < Z_1$.

Moreover

$$\left.\frac{dt}{dZ}\right|_{Z=0} = 1 .$$

So $Z(t)$ is regular near $t = 0$.

4) The bootstrap equations, eqs. (C.1a) and (C.1b), show that the coefficients of the power series

$$Z(t) = \sum_{n=1}^{\infty} c_n t^n$$

are all positive (as they have to be in a physical model). This implies that the singularity of $Z(t)$ neighbouring $t = 0$ lies on the positive real axis.

5) Now we show that there is a singularity of $Z(t)$ on the positive real axis.

First we note that $S(Z)$ increases monotonically from 0 to $\frac{1}{2}$ and $S'(Z)$ increases monotonically from 0 to ∞, if Z goes from 0 to Z_1.

This is an obvious consequence of eq. (C.1b). Therefore

$$\frac{dt}{dZ} = 2 - \exp[S](S'\lambda' + \lambda'')$$

decreases monotonically from 1 to $-\infty$ in the interval $0 < Z < Z_1$.

So there is a zero of dt/dZ on the positive real axis at $Z = Z_0$ with $0 < Z_0 < Z_1$.

From

$$\frac{d^2t}{dZ^2} = -\exp[S](S'^2\lambda' + 2S'\lambda' + S''\lambda' + \lambda''') < 0 \qquad \text{if } 0 < Z < Z_1 ,$$

we find that

$$\left.\frac{d^2t}{dZ^2}\right|_{Z=Z_0} \neq 0 .$$

So the function $Z(t)$ is regular in the circle $|t| < t_0$ and has a square-root branch point at $t = t_0$ (t_0 = real positive).

6) $S(Z)$ is regular for $Z < Z_1$.

As $Z(t_0) = Z_0 < Z_1$ (see step 5)) the singularity of $Z(t)$ at $t = t_0$ generates the lowest singularity of $S(Z(t))$.

This completes the proof of Statement C.

APPENDIX D

Asymptotic expressions for higher moments and correlations of the multiplicity distribution.

The definition of the higher moments

$$\langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0} \rangle \qquad (\alpha_{\pm,0} = \text{integers})$$

leads directly to the relation

$$\text{(D.1)} \qquad \mathscr{L}\big(\langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0}\rangle B_i \tau_i(Q^2)\big) = \left(t_+ \frac{\partial}{\partial t_+}\right)^{\alpha_+} \left(t_- \frac{\partial}{\partial t_-}\right)^{\alpha_-} \left(t_0 \frac{\partial}{\partial t_0}\right)^{\alpha_0} Z_i \Big|_{t_+=t_-=t_0},$$

where $\mathscr{L}$ denotes the Laplace transform.

In order to calculate the leading term for $Q^2 \to \infty$ of the moments we have to separate the strongest singularity of the r.h.s. of eq. (D.1). Using $Z_i = \mathscr{N}_i/\Delta$, we find

$$\text{(D.2)} \qquad \mathscr{L}\big(\langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0}\rangle B_i \tau_i(Q^2)\big) = = (-1)^n \frac{n!\, t^n \mathscr{N}_i}{\Delta^{n+1}} \left(\frac{\partial \Delta}{\partial t_+}\right)^{\alpha_+} \left(\frac{\partial \Delta}{\partial t_-}\right)^{\alpha_-} \left(\frac{\partial \Delta}{\partial t_0}\right)^{\alpha_0} \Big|_{t=t_+=t_-=t_0=\tilde{t}} + \text{less singular terms}$$

with $n = \alpha_+ + \alpha_- + \alpha_0$.

It is important to note that

$$\frac{\partial \Delta}{\partial t_+} = \frac{\partial \Delta}{\partial t_-} = \frac{\partial \Delta}{\partial t_0} \qquad \text{if } t_+ = t_- = t_0 = \tilde{t}.$$

This is true for the model defined in Subsect. 3·3.1 and also for the linear bootstrap model for isovector fireballs only.

The consequence is that the moments $\langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0}\rangle$ are charge independent.

Inverse Laplace transformation of eq. (D.2) leads to the result

$$\text{(D.3)} \qquad \langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0}\rangle \approx \left(\frac{1}{3} \frac{M}{\mu} \frac{K_1(\mu\beta_0)}{K_2(\mu\beta_0)}\right)^n$$

with

$$n = \alpha_+ + \alpha_- + \alpha_0,$$

or

$$\text{(D.4)} \qquad \langle N_+ \rangle = \langle N_- \rangle = \langle N_0 \rangle,$$

and

$$\langle N_+^{\alpha_+} N_-^{\alpha_-} N_0^{\alpha_0}\rangle = \langle N_+ \rangle^n$$

valid for the leading terms.

A more detailed description of the multiplicity distribution is given by the correlations f_N, which follow from a generating function

$$\ln \Psi(Q^2, z) = \sum_{N=1}^{\infty} \frac{(z-1)^N}{N!} f^N(Q^2) . \tag{D.5}$$

For bootstrap models the generating function is

$$\Psi(Q^2, z) = c \sum_{N=1}^{\infty} d_N (zB)^N \Omega_N(Q^2) . \tag{D.6}$$

So we find

$$f_N = \frac{d^N}{dz^N} \ln \Psi(Q^2, z) \Big|_{z=1} = B^N \frac{d^N}{dB^N} \ln (B\tau(Q^2)) . \tag{D.7}$$

If the observed particles carry charge 0, ± 1, a slight generalization leads to

$$f^{(I,m)}_{N_+N_-N_0} = B^N \frac{d^N}{dB_+^{N_+}\, dB_-^{N_-}\, dB_0^{N_0}} \ln (B_m \tau^I_m(Q^2)) \Big|_{\substack{B_+=B_-=B_0=B \\ N=N_++N_-+N_0,\, m=N_+-N_-}} . \tag{D.8}$$

If we introduce the asymptotic form of

$$\tau^I_m(Q^2) \approx cM^{-3} \exp[\beta_0(B_+, B_-, B_0)M]$$

in eq. (D.8), the leading terms of the correlations follow:

$$f^{(I,m)}_{N_+N_-N_0} \approx MB^N \frac{d^N}{dB_+^{N_+}\, dB_-^{N_-}\, dB_0^{N_0}} \beta_0(B_+, B_-, B_0) \Big|_{\substack{B_+=B_-=B_0=B \\ m=N_+-N_-}} . \tag{D.9}$$

In the following we apply the general relation (D.9) to the linear-bootstrap models.

The derivatives of β_0 are calculated from

$$\frac{2\pi\mu^2 K_1(\mu\beta_0)}{\mu\beta_0} = \varphi(B_+, B_-, B_0)$$

and

$$\Delta(t_+, t_-, t_0) = \Delta(B_+\varphi, B_-\varphi, B_0\varphi) = 0 .$$

The results are

$$\left\{\begin{aligned} f_{++} &= 3\frac{M}{\mu}\left(\sigma^3 + \frac{\sigma^2}{\mu\beta_0} - c_i\sigma\right) . \\ f_{00} &= 3\frac{M}{\mu}\left(\sigma^3 + \frac{\sigma^2}{\mu\beta_0} + 4\left(\frac{1}{3} - c_i\right)\sigma\right), \\ f_{0+} &= 3\frac{M}{\mu}\left(\sigma^3 + \frac{\sigma^2}{\mu\beta_0} + (2c_i - 1)\sigma\right) \end{aligned}\right. \tag{D.10}$$

with

$$\sigma = \frac{1}{3}\frac{K_1(\mu\beta_0)}{K_2(\mu\beta_0)} = \frac{\langle N_+\rangle_{\text{asympt}}}{M/\mu}.$$

Equations (D.10) are valid for

a) the linear-bootstrap model for isovector fireballs only:

$$c_V = \frac{10}{27};$$

b) the linear-bootstrap model for isovector and isoscalar fireballs:

$$c_{VS} = \frac{19\sqrt{5}-25}{45}.$$

The numerical results for f_{++}, f_{00}, f_{+0} are different in the two models.
Whereas all bootstrap models lead to the same asymptotic form of f_N (no charge observed), this is not true for the correlations $f_{N_+N_0N_-}$ (see Fig. 6).

Appendix E

Expansion coefficients of $Z_m(t_+; t_0, t_-)$ in the linear-bootstrap model.

To obtain the expansion coefficients of the Laplace transforms $Z_{+,0,-}$ in eq. (3.12) we start by expanding Δ^{-1} (Δ is given in eq. (3.13))

$$\text{(E.1)} \qquad \Delta^{-1} = \sum_{N=0}^{\infty}\left(\frac{1}{2}t_0 + \frac{7}{6}x + \frac{1}{3}t_0^2 + \frac{2}{3}xt_0 - \frac{1}{6}t_0^3\right)^N$$

and $x = t_+t_-$. With

$$\text{(E.2)} \qquad \left(\frac{1}{2}t_0 + \frac{7}{6}x + \frac{1}{3}t_0^2 + \frac{2}{3}xt_0 - \frac{1}{6}t_0^3\right)^N = \sum_{\substack{n_1,\dots,n_5 \\ \sum n_i = N}} \frac{N!}{n_1!n_2!n_3!n_4!n_5!}\left(\frac{1}{2}t_0\right)^{n_1}\left(\frac{7}{6}x\right)^{n_2}\left(\frac{1}{3}t_0^2\right)^{n_3}\left(\frac{2}{3}xt_0\right)^{n_4}\left(-\frac{1}{6}t_0^3\right)^{n_5}$$

we get

$$\text{(E.3)} \qquad \Delta^{-1} = \sum_{N=0}^{\infty} N! \sum_{\substack{n_1,\dots,n_5 \\ \sum n_j = N}} \frac{2^{-n_1-n_2+n_4-n_5}\,7^{n_2}\,3^{-n_2-n_3-n_4-n_5}}{n_1!n_2!n_3!n_4!n_5!}(-1)^{n_5}x^{n_2+n_4}t^{n_1+2n_3+n_4+3n_5}.$$

It is convenient to rewrite this in the form

$$\Delta^{-1} = \sum_{n=0}^{\infty} \sum_{q=0}^{n/2} \bar{c}(q, n-2q) x^q t_0^{n-2q}. \tag{E.4}$$

The $\bar{c}(q, n-2q)$ are obtained by eliminating n_3, n_4 and n_5 with

$$\sum_{i=1}^{5} n_i = N, \qquad q = n_2 + n_4, \qquad n-2q = n_1 + 2n_3 + n_4 + 3n_5. \tag{E.5}$$

As a result we find

$$\bar{c}(q, n-2q) = (-1)^{n+q} 2^{2q-n} \sum_{n/3 \leqslant N \leqslant n} N! \left(\frac{4}{3}\right)^N \sum_{2N-n \leqslant n_1 \leqslant (3N-n)/2} \frac{1}{n_1!} \left(-\frac{3}{4}\right)^{n_1} \cdot \sum_{l \leqslant n_2 \leqslant u} \frac{(-7/8)^{n_2}}{n_2!(3N-2n_1-n-n_2)!(q-n_2)!(n-q-2N+n_1+n_2)!} \tag{E.6}$$

and

$$\begin{cases} l = \max(0, 2N+q-n_1-n), \\ u = \min(q, 3N-2n_1-n). \end{cases} \tag{E.7}$$

For the $c_{N_+N_0N_-}$ of eq. (3.11) one finds then $(m = N_+ - N_-)$

$$c_{N_+N_0N_-} = \begin{cases} \bar{c}(N_-, N_0) + \frac{1}{2}\bar{c}(N_-, N_0-1) & \text{for } m = 1, \\ \bar{c}(N_+, N_0) + \frac{1}{2}\bar{c}(N_+, N_0-1) & \text{for } m = -1, \\ \bar{c}(N_+, N_0-1) - \frac{1}{2}\bar{c}(N_+, N_0-2) + \bar{c}(N_+-1, N_0) & \text{for } m = 0, \end{cases} \tag{E.8}$$

and all $\bar{c}(q, n-2q)$ with one or two negative arguments are understood to be zero. Finally we give the expansion coefficients of S

$$S = \sum_{\substack{N_+, N_-, N_0 \\ N_+ = N_-}} b_{N_+N_0N_-} t_+^{N_+} t_0^{N_0} t_-^{N_-}, \tag{E.9}$$

$$b_{N_+N_0N_-} = = \tfrac{1}{3}\{2\bar{c}(N_+-1, N_0) + 2\bar{c}(N_+-1, N_0-1) + \bar{c}(N_+, N_0-2) - \tfrac{1}{2}\bar{c}(N_+, N_0-3)\}. \tag{E.10}$$

Note added in proofs.

The linear-bootstrap problem for $I=1$, $G=-1$, $S=0$ mesons (ground state and excited pions) and $I=\frac{1}{2}$, $S=0$ baryons (ground state and excited nucleons) was solved recently by CSIKOR *et al.* [20].

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[20] F. CSIKOR, I. FARKAS, Z. KATONA and I. MONTVAY: *Nucl. Phys.*, **74** B, 343 (1974).

● RIASSUNTO (*)

Servendosi del modello statistico a bootstrap si studiano la densità dei livelli e il decadimento di fireball isoscalari e isovettoriali (con pioni di entrata). Si trova che per valori elevati della massa M dei fireball versioni diverse di questo modello danno la stessa distribuzione totale di molteplicità. Si deducono espressioni asintotiche per la correlazione di molteplicità integrata delle particelle cariche. Nell'intervallo 1 GeV $\leqslant M \leqslant$ 8 GeV si calcolano numericamente le correlazioni.

(*) *Traduzione a cura della Redazione.*

Статистические модели бутстрапа для образования и распада изоскалярных и изовекторных файрболов.

Резюме (*). — В рамках статистической модели бутстрапа исследуются плотность уровней и распад изоскалярных и изовекторных файрболов. Мы получаем, что различные варианты этой модели приводят к одинаковому полному распределению множественности для больших масс файрболов M. Выводятся асимптотические выражения для корреляции интегральной множественности заряженных частиц. В области 1 ГэВ $\leqslant M \leqslant$ 8 ГэВ численно вычисляются указанные корреляции.

(*) *Переведено редакцией.*