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Phenomenological renormalization and scaling behaviour of SU(2) lattice gauge theory *

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Near the deconfinement transition of SU(2) gauge theory the finite-size scaling behaviour of the order parameter, the susceptibility and the normalized fourth cumulant g_r is studied on $N_\sigma^3 \times N_\tau$ lattices with $N_\tau = 4$ and 6 and $N_\sigma = 8$, 12, 18, 24 or 26. For that purpose we have calculated new high-statistics data for $N_\tau = 6$ and re-evaluated previous results obtained for $N_\tau = 4$. In both cases we used the density of states method. We determine the critical coupling and with a new way of phenomenological renormalization the critical exponents. For $N_\tau = 6$ we find that $4/g^2_{c,\infty} = 2.4265(30)$. Using the results for the critical temperature obtained for different N_τ we examine the approach to asymptotic scaling.

1. Introduction

During the last few years finite-size scaling (FSS) techniques have been successfully applied to study the critical properties of lattice gauge theories [1–3] at finite temperature. The analysis of the second-order deconfinement transition in SU(2) lattice gauge theory in 3 + 1 dimensions showed a remarkable agreement of the critical exponents with those of the three-dimensional Ising model.

The improvement of the original density of states method [4–6] for the evaluation of data [7–9] allows now the application of FSS techniques requiring continuous input functions and not only single data points. It seems therefore worthwhile to re-evaluate existing data and to extend the analysis to new data.

We consider SU(2) gauge theory on $N_\sigma^3 \times N_\tau$ lattices using the standard Wilson action

$$S(U) = \frac{4}{g^2} \sum_p \left(1 - \tfrac{1}{2}\mathrm{Tr}\, U_p\right), \qquad (1)$$

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where U_p is the product of link operators around a plaquette. The number of lattice points in the space (time) direction $N_{\sigma(\tau)}$ and the lattice spacing a fix the volume and temperature as

$$V = (N_\sigma a)^3, \qquad T = 1/(N_\tau a). \tag{2}$$

On an infinite-volume lattice the order parameter for the deconfinement transition is the expectation value of the Polyakov loop,

$$L(x) = \tfrac{1}{2}\mathrm{Tr} \prod_{\tau=1}^{N_\tau} U_{\tau,x;0}, \tag{3}$$

or otherwise, that of its lattice average,

$$L = \frac{1}{N_\sigma^3} \sum_x L(x), \tag{4}$$

where $U_{x;0}$ are the SU(2) link matrices at four-position x in time direction.

Since, due to system flips between the two ordered states on finite lattices the expectation value $\langle L \rangle$ is always zero, we thus consider as the "order parameter" the expectation value of the modulus of the lattice average:

$$\langle |L| \rangle = \left\langle \left| \frac{1}{N_\sigma^3} \sum_x L(x) \right| \right\rangle. \tag{5}$$

Similarly we replace the true susceptibility by

$$\chi = N_\sigma^3 (\langle L^2 \rangle - \langle |L| \rangle^2). \tag{6}$$

A direct scaling function is obtained from the normalized fourth cumulant,

$$g_r = \frac{\langle L^4 \rangle}{\langle L^2 \rangle^2} - 3, \tag{7}$$

which we shall exploit to determine the infinite-volume critical coupling with high precision.

In sect. 2 we shall describe how one may obtain information on the infinite-volume limit of the thermodynamical quantities from a finite-size scaling analysis. We develop a new technique – the method of phenomenological renormalization – which we apply later to our data. This method has the additional advantage that no explicit functional form of the scaling functions has to be assumed. The improved density of states method (DSM) enables us to interpolate our data in the critical region. The relevant details for carrying out this program are contained in sect. 3. Sect. 4 presents the Monte Carlo data and their evaluation with the DSM. The results are then used to determine the ratios of the critical exponents, β/ν and γ/ν. The N_τ-dependence of the critical temperature and its connection to the β-function are then investigated with our new and previous results. Finally we summarize our findings and conclusions.

2. Finite-size scaling theory and phenomenological renormalization

For a second-order phase transition the behaviour of the thermodynamical quantities in the infinite-volume limit is determined by the critical exponents. In the neighbourhood of the critical temperature T_c one expects in the limit of large N_σ that

$$\langle L \rangle \sim (T - T_c)^{\beta} \quad \text{for } T \to T_c^{+}. \tag{8}$$

The behaviour near to T_c of the susceptibility χ and the correlation length ξ in the large-N_σ limit is expected to be

$$\chi \sim |T - T_c|^{-\gamma}, \qquad \xi \sim |T - T_c|^{-\nu}. \qquad (9), (10)$$

However, on finite lattices this limiting behaviour is modified. A quantitative analysis becomes possible by using the renormalization group theory. In this framework it has been shown [10] that the singular part of the free energy density has the following form:

$$f_s(x, h, N_\sigma) = N_\sigma^{-d} Q_{fs}\left(g_T N_\sigma^{1/\nu}, g_h N_\sigma^{(\beta+\gamma)/\nu}, g_i N_\sigma^{y_i}\right). \tag{11}$$

The scaling function Q_{f_s} depends on the temperature T and the external field strength h in the form of a thermal and a magnetic scaling field,

$$g_T = c_T x + \mathrm{O}(xh, x^2), \tag{12}$$

$$g_h = c_h h + \mathrm{O}(xh, h^2), \tag{13}$$

which are independent of N_σ and where x is the reduced temperature, which in the neighbourhood of the transition for a fixed value of N_τ can be approximated by

$$x = \frac{4/g^2 - 4/g_{c,\infty}^2}{4/g_{c,\infty}^2}. \tag{14}$$

Here the action contains a further symmetry breaking term $h \cdot \sum_x L(x)$. Also additional irrelevant scaling fields g_i with negative exponents y_i may be present.

The order parameter $\langle L \rangle$, the susceptibility χ and the renormalized coupling g_r are obtained from f_s by taking derivatives with respect to h at $h=0$. The general form of the scaling relations derived in this way is

$$O(x, N_\sigma) = N_\sigma^{\omega/\nu}\, Q_O\big(g_T N_\sigma^{1/\nu},\, g_i N_\sigma^{y_i}\big). \tag{15}$$

Here O is $\langle L \rangle$, χ and g_r with $\omega = -\beta$, γ and 0. Taking into account only the largest irrelevant exponent y_1 and expanding the scaling function Q_O to first order at $x=0$ results in the following equation:

$$O(x, N_\sigma) = \big\{c_0 + (c_1 + c_2 N_\sigma^{y_1})\, x N_\sigma^{1/\nu} + c_3 N_\sigma^{y_1}\big\} N_\sigma^{\omega/\nu}. \tag{16}$$

Standard finite-size scaling (FSS) methods are based on the evaluation of eq. (15) in the neighbourhood of the infinite-volume critical coupling $4/g^2_{c,\infty}$. Using the linear expansion in eq. (16) we get

$$\omega/\nu = \ln \frac{O(0, N_\sigma)}{c_0 + c_3 N_\sigma^{y_1}} \Big/ \ln N_\sigma. \tag{17}$$

Even in this linear approximation we have four unknown parameters $4/g^2_{c,\infty}$ (for the definition of x), c_0, c_3 and y_1 which have to be determined by measuring O for various lattice sizes N_σ and then fitting the parameters. These difficulties in the usual fits arise on one hand from the incomplete information on the β-dependence (where $\beta = 4/g^2$ is the inverse coupling, not to be confused with the critical exponent β) of the scaling fields $g_{T,i}(\beta)$ and on the other hand from the unknown functional form of the scaling functions.

A more elegant way avoiding the mentioned problems and including possible irrelevant scaling fields is the method of phenomenological renormalization. The existence of a scaling function Q allows us to develop a procedure to renormalize the coupling by the use of two different lattice sizes N_σ and bN_σ. Formally this phenomenological renormalization is defined by the equation

$$Q\big(g_T(\beta) N_\sigma^{1/\nu},\, g_i(\beta) N_\sigma^{y_i}\big) = Q\big(g_T(\tilde\beta) b^{1/\nu} N_\sigma^{1/\nu},\, g_i(\tilde\beta) b^{y_i} N_\sigma^{y_i}\big). \tag{18}$$

It expresses that the scaling function Q remains unchanged if the lattice size is rescaled by a factor b and the inverse coupling β is shifted to $\tilde\beta(\beta, N_\sigma, b)$ simultaneously. Of course the arguments of Q on the left- and right-hand side of eq. (18) are then equal separately. As a result eq. (18) is valid for Q_L, Q_χ and Q_{g_r} with a common coupling $\tilde\beta$.

The procedure for the calculation of the critical exponents is then the following: first the phenomenologically renormalized inverse coupling $\tilde\beta(\beta, N_\sigma, b)$ is determined by eq. (18) using the fact that g_r is a scaling function directly. We do this by comparing the two curves $g_r(\beta, N_\sigma)$ and $g_r(\tilde\beta, bN_\sigma)$ determined for two different lattice sizes N_σ and bN_σ. Inserting $\tilde{\beta}(\beta, N_\sigma, b)$ in eq. (15), taking the logarithm of the ratio then results in the following expression for the exponent ω:

$$\omega/\nu = \ln\frac{O(\tilde{\beta}, bN_\sigma)}{O(\beta, N_\sigma)}/\ln b. \tag{19}$$

In practice the phenomenological renormalization of the coupling β is most easily done by measuring $O(\beta, N_\sigma)$ and $g_r(\beta, N_\sigma)$ simultaneously and then plotting O as a function of g_r.

The infinite-volume critical coupling β_c^∞ can be extracted from the fixed points $\beta_c = 4/g_c^2(N_\sigma, b)$ of the renormalization transformations $\tilde{\beta}(\beta, N_\sigma, b)$ for finite lattices. The equation for a fixed point reads:

$$\tilde{\beta}(\beta, N_\sigma, b) = \beta \equiv \beta_c(N_\sigma, b). \tag{20}$$

The effective, critical couplings $\beta_c(N_\sigma, b)$ are determined by the intersection points of two curves $g_r(\beta, N_\sigma)$ and $g_r(\beta, bN_\sigma)$.

Using the expansion (16) at $\beta = \beta_c$ for $g_r(\beta, N_\sigma)$ gives for $N_\sigma \gg 1$:

$$\beta_c(N_\sigma, b) = \beta_c^\infty\left(1 + \frac{a_3}{a_1}\epsilon\right), \tag{21}$$

where

$$\epsilon = N_\sigma^{y_1 - 1/\nu}\frac{1 - b^{y_1}}{b^{1/\nu} - 1}. \tag{22}$$

By plotting $\beta_c(N_\sigma, b)$ as a function of ϵ it is possible to determine β_c^∞ as β_c at $\epsilon = 0$, if the values of y_1 and ν are already known. If $\beta_c(N_\sigma, b)$ is known for at least three different pairs of N_σ and b then for fixed ν one can estimate the exponent y_1 by a fit to the data such that $\beta_c(N_\sigma, b)$ becomes a linear function of ϵ.

3. The density of states method

The density of states method was introduced [4–6] for partition functions, which may be written in the form

$$Z(K) = \int W(S)\exp(-KS)\,\mathrm{d}S. \tag{23}$$

Here, $W(S)$ is the density of states, K the coupling – in our case $4/g^2$ – and KS corresponds to the total action in eq. (1). We recapitulate the essential formulae of the method in the way it was proposed by Ferrenberg and Swendsen [8,9] and as we have actually used it.

The aim of the method is to determine the (unnormalized) density $W(S)$ by Monte Carlo measurements at one or more coupling values and then to interpolate between the input values or even to extrapolate from those values. For that purpose the S-range is subdivided into N_s bins. The partition function is then approximated by

$$Z(K) = \sum_{i=1}^{N_s} W(S_i) \exp(-KS_i). \tag{24}$$

Assumed we have measured S and any observable O at r couplings K_m, $m = 1, \ldots, r$ with n_m measurements each, then [9]

$$W(S_i) = \frac{\sum_{j=1}^{r} g_j^{-1} N_j(S_i)}{\sum_{m=1}^{r} g_m^{-1} n_m \exp(-K_m S_i + f_m)}, \tag{25}$$

where $N_j(S_i)$ gives the frequency distribution of S for K_j in the N_s bins of the S-range. The contributions to $W(S_i)$ of the different couplings are weighted by factors g_m, where g_m is two times the integrated autocorrelation time. The quantities f_m are the free energies

$$f_m = -\ln Z(K_m), \tag{26}$$

and have to be determined iteratively from eqs. (24) and (26). On the other hand the expectation value of S is [11,12]

$$\langle S \rangle = -\frac{\partial \ln Z}{\partial K}, \tag{27}$$

so that integrating $\langle S \rangle$ over K leads to $f(K)$ up to an integration constant. We have used this fact to find excellent start values for the self-consistent iteration of the f_m-values. To do that we first order the couplings K_m in ascending size and set then

$$\begin{aligned} f_1 &= 0, \\ f_m &= f_{m-1} + \tfrac{1}{2}(K_m - K_{m-1})(\langle S \rangle_m + \langle S \rangle_{m-1}); \end{aligned} \tag{28}$$

for $m = 2, \ldots, r$, i.e. we use the trapezoidal integration rule. The following iteration is considerably accelerated with these start values and yields as final result for the f_m only slightly different values.

To calculate the expectation value of an observable O as a function of K we use the following procedure. First we determine the weighted average of the observable in bin i:

$$O(S_i) = \frac{\sum_{j=1}^{r} g_j^{-1} \sum_{k=1}^{N_j(S_i)} O_{jk}}{\sum_{m=1}^{r} g_m^{-1} N_m(S_i)}, \tag{29}$$

where O_{jk} is the value of the observable measured at coupling K_j for the kth S-value falling into bin i. The expectation value of O is then obtained in the usual way:

$$\langle O \rangle = \frac{\sum_{i=1}^{N_s} W(S_i) O(S_i) \exp(-KS_i)}{\sum_{j=1}^{N_s} W(S_j) \exp(-KS_j)}. \tag{30}$$

By applying this method we avoid the construction of two-dimensional histograms and besides that we can compute expectation values of different observables at the same time.

4. Data and error analysis

We re-evaluated our existing data for $N_\tau = 4$ [1] and performed additional simulations for $N_\tau = 6$ in order to get results located closer to the asymptotic scaling regime. For the FSS analysis we used lattices of size $N_\sigma^3 \times N_\tau$ with $N_\sigma = 26, 18, 12, 8$ for $N_\tau = 4$ and $N_\sigma = 24, 18, 12, 8$ for $N_\tau = 6$. In the case of $N_\tau = 4$ we ran 100 000 to 450 000 iterations for each coupling, while for $N_\tau = 6$ we used similarly 300 000 to 500 000 iterations. The first 1000 ($N_\tau = 4$) and 2000 ($N_\tau = 6$) iterations were discarded for thermalization. The integrated autocorrelation time for the expectation value of the modulus of the lattice averaged Polyakov loop is listed in table 1.

The use of the DSM allows us to compute $\langle | L | \rangle$, χ and g_r as continuous functions of the inverse coupling. We have convinced ourselves that the histograms of the action calculated at neighbouring values of the inverse coupling were overlapping. In table 1 we give the number R of data points with overlapping histograms and the β-range of their couplings for each lattice.

The calculation of the errors was carried out according to the Jackknife method dividing the entire sample into 8 blocks. In figs. 1–3 we show the measured values

TABLE 1
Number of overlapping histograms with the range of β and $\tau_{\rm int}$

N_σ	N_τ	R	β-range	$\tau_{\rm int}$-range
26	4	4	2.290–2.310	19– 88
18	4	20	2.270–2.350	9– 56
12	4	24	2.260–2.350	9– 26
8	4	30	2.240–2.360	8– 14
24	6	8	2.410–2.445	280–505
18	6	6	2.400–2.450	120–295
12	6	10	2.415–2.460	32– 60
8	6	10	2.400–3.000	20– 35

of the lattice order parameter $\langle |L| \rangle$, the susceptibility χ and the renormalized coupling g_r. For clarity only the continuous curves are shown for $N_\tau = 4$. The curves are fully consistent with the individual points shown in ref. [1]. The measured order parameter and the susceptibility show the approach to the asymptotic behaviour described by eqs. (8) (9) with increasing spatial lattice size N_σ. For $N_\tau = 4$ the renormalized coupling g_r shows the expected fixed point with remarkable precision. However for $N_\tau = 6$ we observe especially for $N_\sigma = 8$ differences in the intersection points.

4.1. THE CRITICAL COUPLING

The infinite-volume limit of the critical coupling is determined from the intersection points of $g_r(4/g^2, N_\sigma)$ for different spatial lattice size N_σ. The DSM has the advantage that direct and accurate measurement of these crossing points are possible.

The position of the intersection points summarized in table 2 is shown in fig. 4. For $N_\tau = 4$ we observe a clear fixed point. Within error bars all curves intersect in a single point. Neither is there a significant influence from irrelevant scaling fields nor is there a noticeable correction from the regular part of the free energy density. As a consequence of the relatively small number of iterations and the number of only four β-values which determine the curve for $N_\sigma = 26$ we get the best estimate for the critical coupling extrapolated to the infinite-volume limit from the intersection point of the two curves for $N_\sigma = 12$ and $N_\sigma = 18$. This corresponds to taking the highest value for β_c^∞ and appears to be reasonable since usually the intersection points approach the infinite-volume limit from below [13]. In this way we get a value of $\beta_c^\infty = 2.2986(6)$ for $N_\tau = 4$.

For $N_\tau = 6$ there exist obvious deviations from a fixed point. Therefore it is necessary to examine the influence of an irrelevant scaling field. Inserting the value $\nu = 0.63$ in eq. (22) we use eq. (21) to estimate both the critical coupling in the infinite-volume limit β_c^∞ and the largest irrelevant exponent y_1. The two

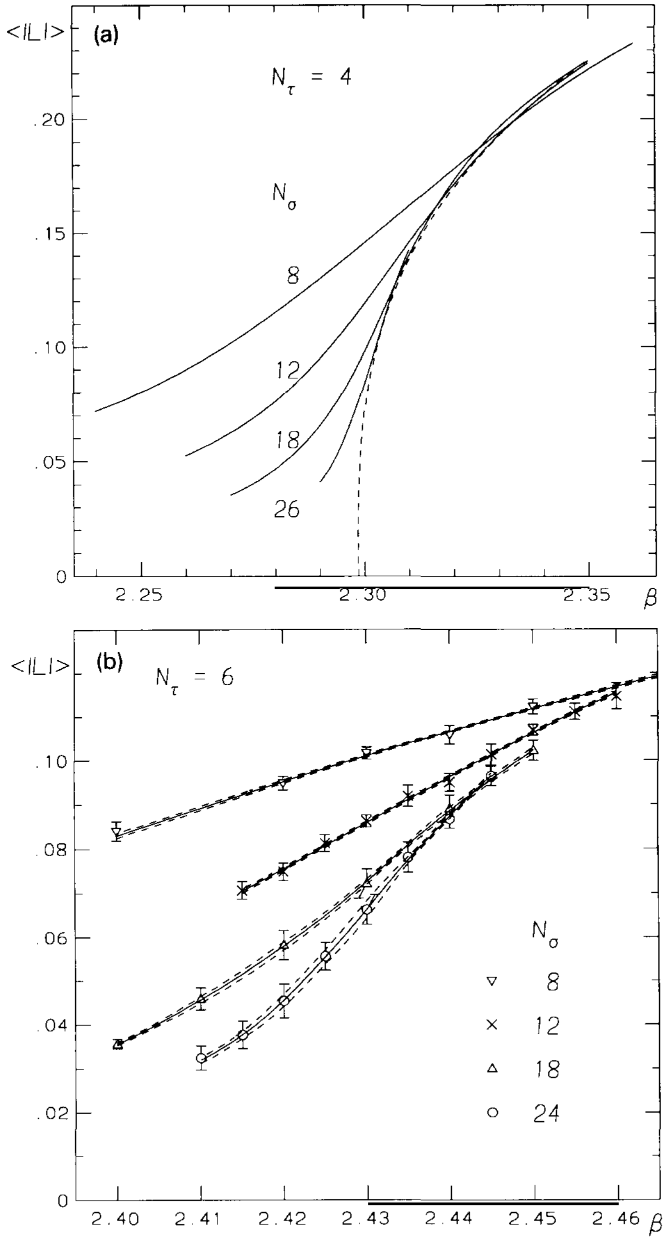

Fig. 1. The expectation value of the modulus of the lattice averaged Polyakov loop as a function of $\beta = 4/g^2$ for $N_\tau = 4$ and 6. The dashed line marks the infinite-volume behaviour according to eq. (8) with the critical exponent $\beta = 0.325$.

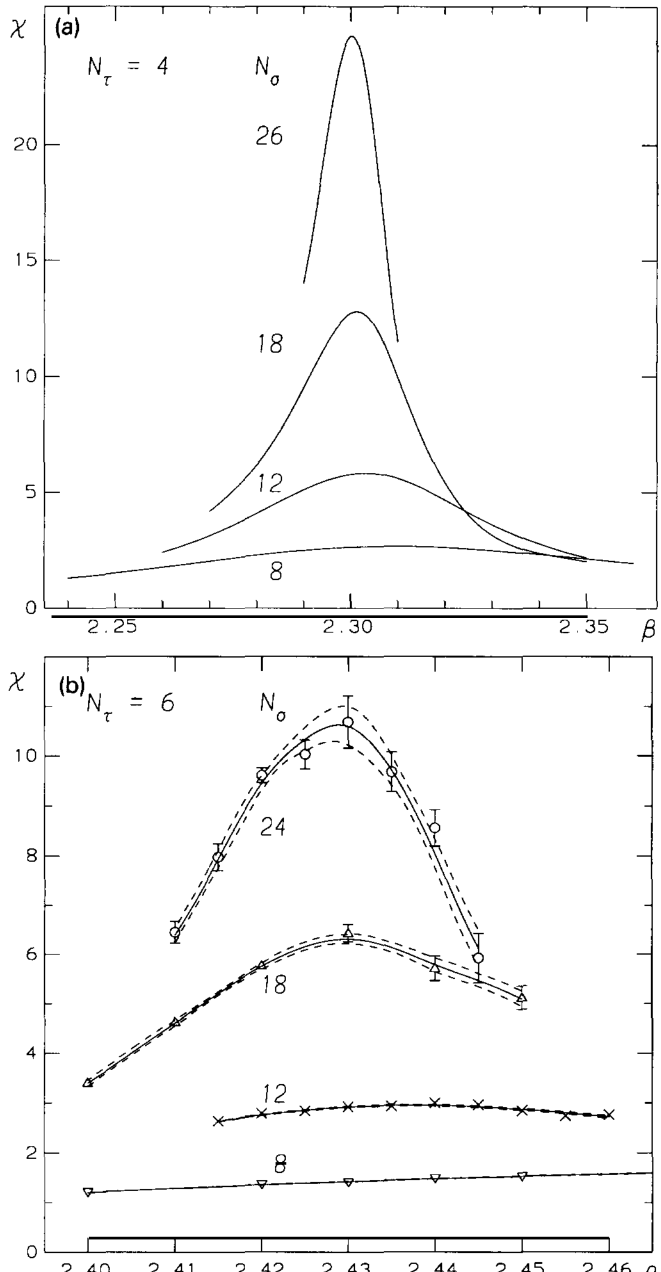

Fig. 2. The susceptibility as a function of β for $N_\tau = 4$ and 6.

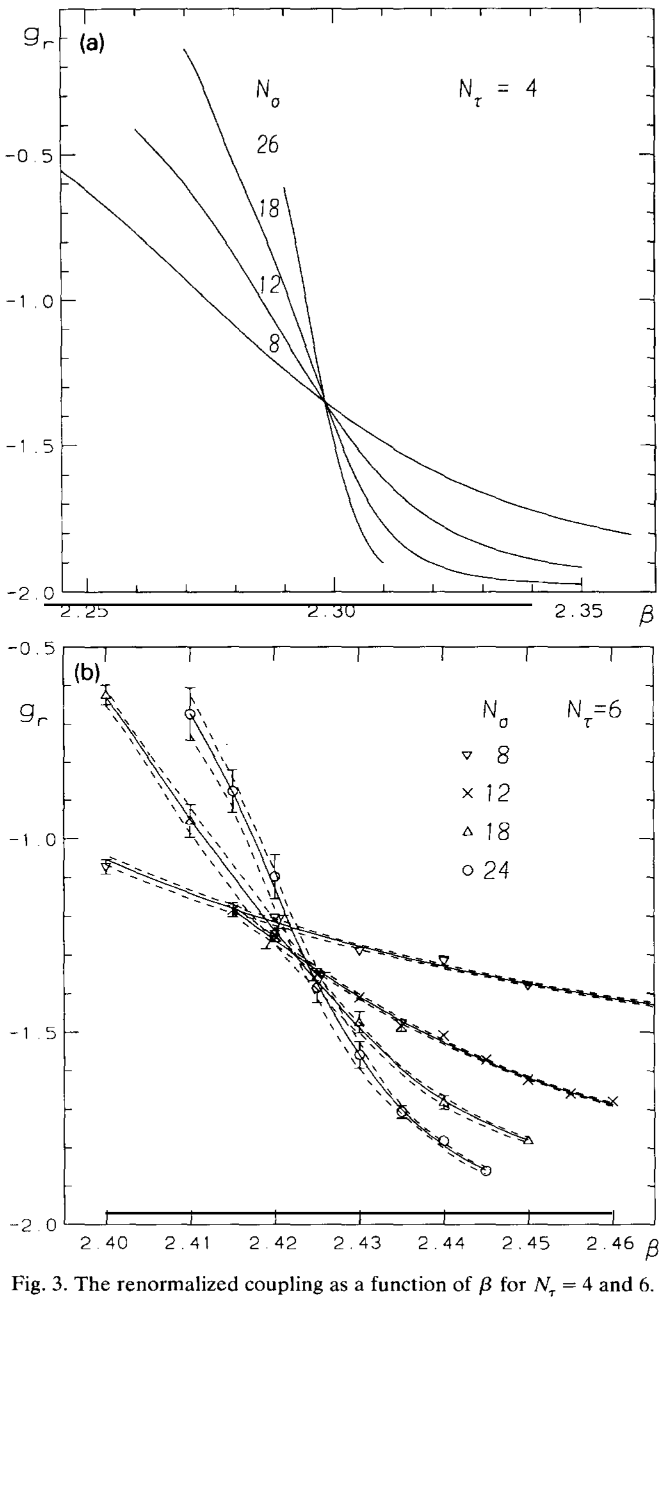

Fig. 3. The renormalized coupling as a function of β for $N_\tau = 4$ and 6.

TABLE 2
Intersection points of g_r

N_τ	N_σ^1	N_σ^2	$4/g_c^2$
4	8	12	2.2980 (7)
4	8	18	2.2984 (4)
4	8	26	2.2982 (6)
4	12	18	2.2986 (6)
4	12	26	2.2983 (7)
4	18	26	2.2981(13)
6	8	12	2.4149(40)
6	8	18	2.4189(30)
6	8	24	2.4219(10)
6	12	18	2.4222(40)
6	12	24	2.4238(15)
6	18	24	2.4247(30)

parameters β_c^∞ and y_1 are fitted to the data points such that a linear behaviour results when the intersection points are plotted as a function of the variable ϵ defined in eq. (22). This is shown in fig. 5. By extrapolating to $\epsilon = 0$ we determine the value of the infinite-volume critical coupling to be $\beta_c^\infty = 2.4265(30)$. The largest irrelevant exponent y_1 turnes out to be consistent with a value of $y_1 = -0.9$. This is in agreement with a value of $y_1 = -1$ found for the three-dimensional Ising model [13]. It shows that y_1 is in fact large and negative. Thus irrelevant contributions will disappear rather fast with increasing spatial lattice size.

4.2. PHENOMENOLOGICAL RENORMALIZATION

On condition that irrelevant scaling fields are negligible it is possible to apply the method of direct scaling fits to determine the ratios β/ν and γ/ν. This seems to be valid for $N_\tau = 4$ and we refer to ref. [1], where we assumed a linear form of the scaling function for the order parameter and the renormalized coupling and a quadratic form for the susceptibility.

In the present analysis we want to include higher-order terms of the reduced temperature by using a phenomenological renormalization. As already explained the method has the advantage that no knowledge about the critical coupling or the explicit form of the scaling function is required. To accomplish this we regard the order parameter and the susceptibility as a function of g_r for the values of N_σ under consideration. According to eq. (15) the resulting curves would be proportional to $N_\sigma^{-\beta/\nu}$ and $N_\sigma^{\gamma/\nu}$. The exponents are determined by comparing two curves for different N_σ at a fixed value of g_r.

From figs. 6 and 7 one can see that we get results close to the expected three-dimensional Ising values $\beta/\nu = 0.5180(57)$ and $\gamma/\nu = 1.9828(70)$ [13]. Espe-

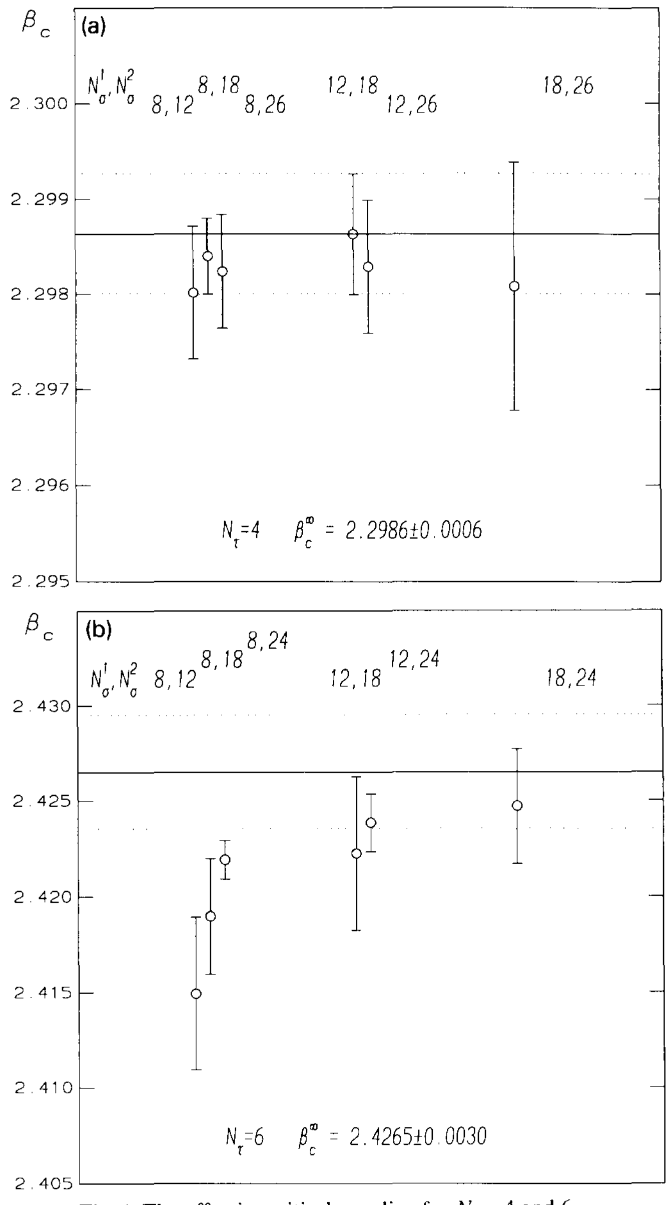

Fig. 4. The effective critical coupling for $N_\tau = 4$ and 6.

cially in the case of $N_\tau = 6$ the computed values are for β/ν systematically too large while for γ/ν they are correspondingly too small.

Although the values for $\langle |L| \rangle$, χ and g_r are evaluated with much higher precision the statistical errors on the quotients β/ν and γ/ν can reach twenty

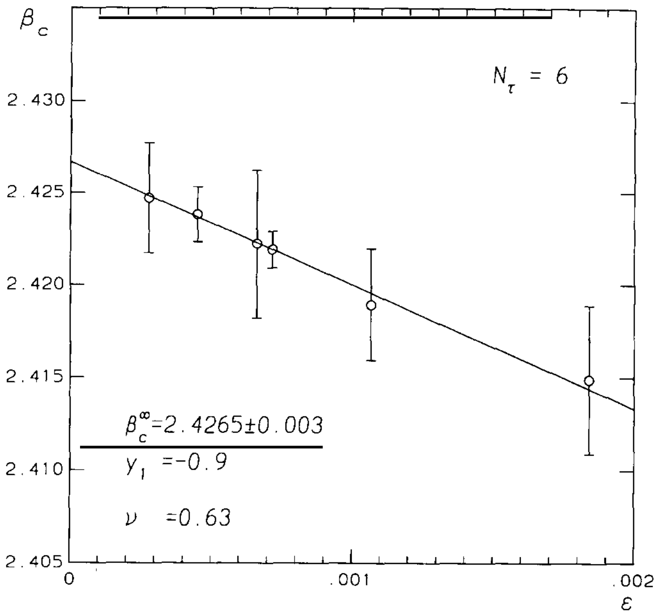

Fig. 5. Extrapolation of the critical coupling to the infinite-volume limit $\epsilon = 0$ for $N_\tau = 6$. The variable ϵ is defined in eq. (22).

percent. The largest deviations occur when the smallest lattice size $N_\sigma = 8$ is involved. Therefore the deviations are most significant for exponent ratios calculated involving the smallest lattice size $N_\sigma = 8$ and $N_\tau = 6$. The observed corrections can arise from irrelevant scaling fields and from the regular part of the free energy density. Both contributions may depend on the volume, so that it is here impossible to disentangle them. However we know that irrelevant contributions vanish for large volumes.

At fixed temperature the physical volume is given by $(N_\sigma/N_\tau)^3$. This is in accord with previous investigations of the heavy quark potential [14] where it has been

TABLE 3
Relative volume

N_τ	N_σ	V_r
6	8	1.0
6	12	3.4
6	18	11.4
6	24	27.0
4	8	3.4
4	12	11.4
4	18	38.4
4	26	115.9

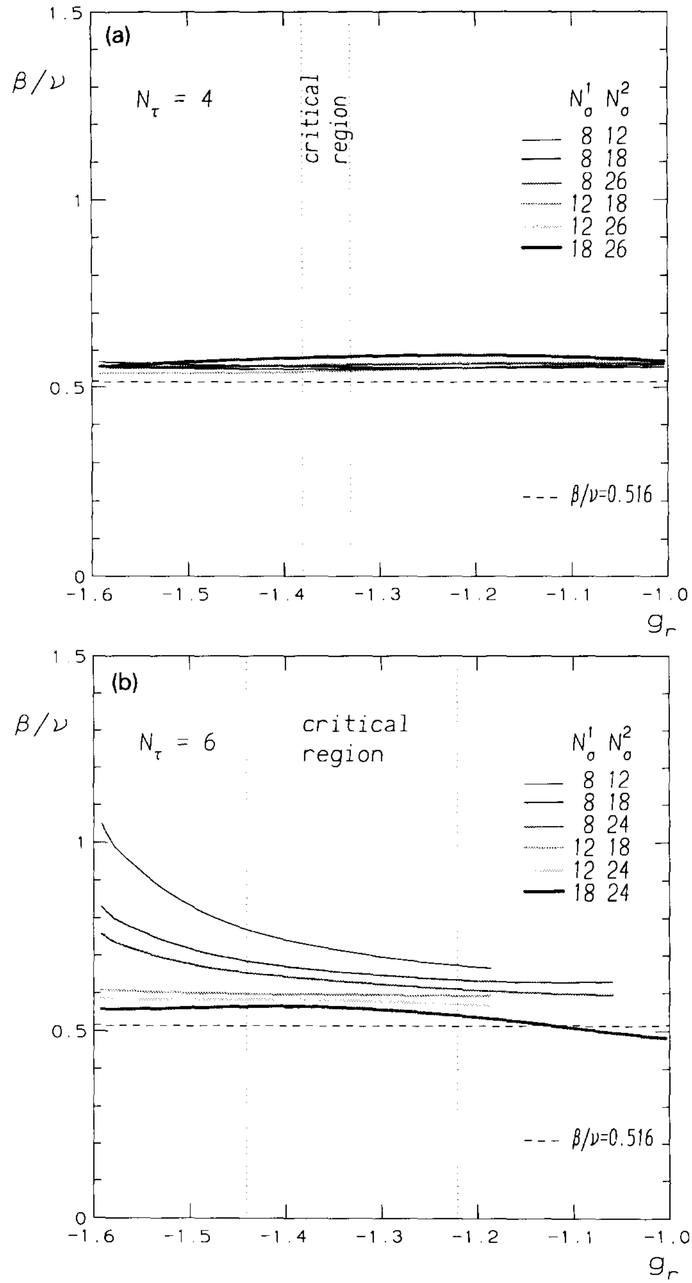

Fig. 6. The ratio of β/ν as a function of g_r for $N_\tau = 4$ and 6. The critical region is here defined as the interval containing the intersection points of g_r.

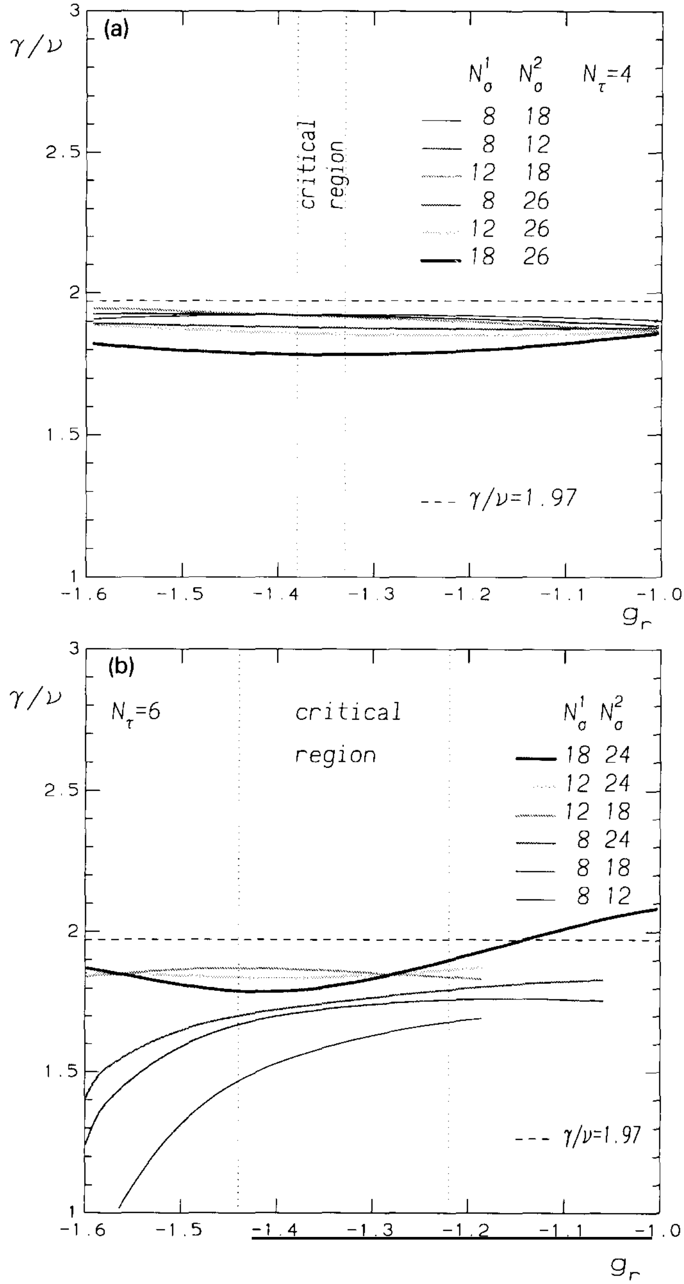

Fig. 7. The ratio of γ/ν as a function of g_r for $N_\tau = 4$ and 6.

TABLE 4
Critical couplings

N_τ	$4/g^2$	$T_c[\Lambda_L]$
2	1.8600(23)	28.30(16) [15]
3	2.1710(30)	41.11(30) [16]
4	2.2986 (6)	42.12(06)
5	2.3726(45)	40.58(53) [17]
6	2.4265(30)	38.73(29)

found that the finite-size dependence is a function of the ratio N_σ/N_τ and not of N_σ alone. From table 3 we see that the relative volume of the lattice with $N_\tau = 6$ and $N_\sigma = 8$ is more than three times smaller than the next larger volume. Even the volume $V_\tau = 27$ for $N_\tau = 6$ is four times smaller than the largest lattice for $N_\tau = 4$. This explains why the largest deviations occur for small N_σ and large N_τ where the physical volume is the smallest. Considering only volumes with $N_\sigma/N_\tau > 2$ in figs. 6 and 7 we see a convergence of the exponent ratios to the corresponding three-dimensional Ising values.

4.3. ASYMPTOTIC SCALING

An important point for the continuum limit of lattice gauge theories is the question where as a function of $4/g^2$ asymptotic scaling sets in. The determination

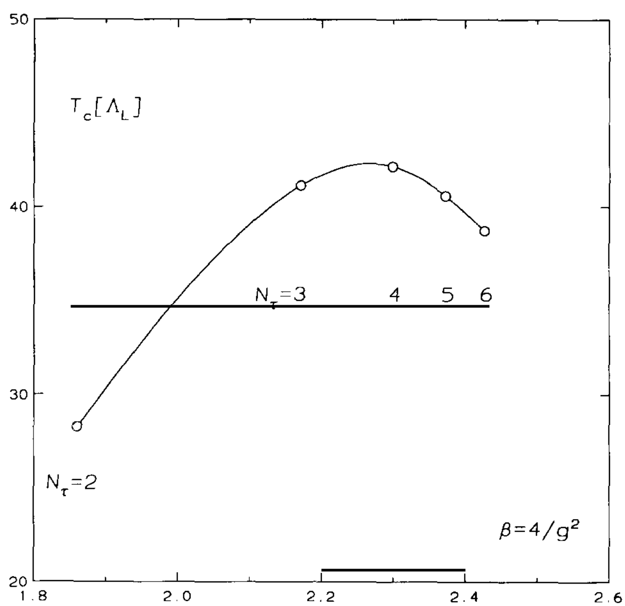

Fig. 8. The critical temperature as a function of β. The line is drawn just to guide the eye.

of the critical coupling for different values of N_τ allows us to look for violations of asymptotic scaling. Values for the critical coupling for $N_\tau = 2$, 3, 4, 5 and 6 are listed in table 4. The critical temperature is calculated using the two-loop approximation to the renormalization group equation.

By looking at the critical temperature as a function of N_τ it becomes obvious that the region of asymptotic scaling is not yet reached for $N_\tau = 6$ or equivalently at $4/g^2 = 2.4265$. From fig. 8 we see that T_c as a function of $4/g^2$ shows a maximum at $N_\tau = 4$ and goes to smaller values for larger values of N_τ. The change in the behaviour of $T_c(N_\tau)$ at $N_\tau = 4$ is probably due to the fact that the critical couplings for $N_\tau = 2$ and 3 still fall into the strong coupling region $4/g^2 < 2.2$. For $N_\tau = 6$ the critical temperature seems to be still falling, whereas one would expect a constant if asymptotic scaling is valid.

Although it is possible to construct a numerical beta function from the critical coupling as a function of N_τ [18], different operators may approach the continuum limit in a different way. Therefore it is important to know the region of asymptotic scaling where a universal beta function exists. The necessary calculations of the critical couplings for higher values of N_τ are already in progress [19].

5. Summary

In a comparative study we have investigated the finite-size scaling behaviour of SU(2) lattice gauge theory on various cubic spatial lattices with $N_\tau = 4$ and $N_\tau = 6$. Our high-statistics data were taken at selected values of the coupling $\beta = 4/g^2$ such that the critical region around the deconfinement transition was covered by histograms which were overlapping as a function of the action. Thus we were able to apply the density of states method, which we improved in some details, to obtain the relevant thermodynamic quantities as continuous functions of the coupling constant in the whole critical region. From the normalized fourth cumulant g_r we find the following precise values of the critical couplings:

$$4/g_c^2(N_\tau = 4) = 2.2986(6), \qquad 4/g_c^2(N_\tau = 6) = 2.4265(30). \tag{31}$$

In contrast to the case $N_\tau = 4$, where no influence of irrelevant scaling fields or other corrections could be observed at the critical point, we find that for $N_\tau = 6$ there are additional contributions which may be explained by an irrelevant exponent for which we estimate a value of $y_1 = -0.9$.

The value of g_r at the infinite-volume critical coupling is supposed to be a universal quantity. From the $N_\tau = 4$ data we find from the two largest lattices a value

$$g_r(4/g_{c,\infty}^2) = -1.38(5), \tag{32}$$

which is compatible with the $N_\tau = 6$ data and the value -1.41 [13] found for the three-dimensional Ising model.

With our method of phenomenological renormalization we examined the critical exponent ratios β/ν and γ/ν. As in ref. [1] we observe that for both N_τ values β/ν is somewhat larger and γ/ν is somewhat smaller but close to the expected three-dimensional Ising model value. The deviation is stronger for the $N_\tau = 6$ lattices. This is no big surprise, since one may argue that the true finite-size dependence is on N_σ/N_τ and not on N_σ alone. Then at fixed temperature the $N_\tau = 6$ lattices we used are by a factor of three smaller than our corresponding $N_\tau = 4$ lattices.

Finally we have investigated the N_τ or $4/g^2$ dependence of the critical temperature as obtained from the two-loop renormalization group equation. We conclude that for $N_\tau = 6$ or equivalently at $4/g^2 = 2.43$ the region of asymptotic scaling has not yet been reached.

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