speed, provide reasonable answers, and be done mentally. Children also need to develop a broad repertoire of strategies allowing them to use estimation flexibly depending on the context.

The decisions children need to make were considered and issues such as when to estimate and the need for 'close' answers were discussed. The importance of imbedding experiences in real world contexts for motivation and application was recognised. Work with estimation and mental computation needs to start early to be thoroughly integrated with all computational work.

One program on estimation was presented using estimation skills including: front-end calculation (for example to add 465 to 275: 400 + 200 = 600, the rest makes about 100 more, so 700), rounding, grouping to make easier numbers to work with, clustering about assumed means or averages, and using compatible numbers.

It was noted that estimation skills are not easily learned by children and that teachers require curriculum support materials in order to make a sustained instructional effort in this area.

2. Ratio, Decimals and Fractions

Convenors: Friedhelm Padberg (FRG) and Leen Streefland (Netherlands)

The papers and discussions focussed on the important components of concepts of ratio, decimals and fractions, the extent to which these concepts are being learned by students, and the way the concepts and models are used in moving to more formal means of expression with algorithms and/or computational rules. Data from research studies and comments from a wide variety of people from various countries were the basis of most of the conclusions reached in the sessions.

Ratio: Gerard Vergnaud (France) introduced ratio in a broad sense by tracing the development from the beginning idea of 'the more you buy, the more you pay' to the additive properties for age 7 or 8 and more advanced linear concepts at age 10 or 11. The importance of proportions was evident in his opening statement. 'It is not multiplication and division that make young students understand proportion. It is rather proportion that makes multiplication and division meaningful to them.' He noted the long time it takes for children and adolescents to extend the concept of ratio between magnitudes of different kinds, such as distance and time. He asked, 'What does it mean to divide a distance by a duration?'

Stephen Willoughby (USA) gave examples of ratio from his textbooks at the early and later grades using number-cube games. Streefland presented alternative examples for introducing ratio using a picture of a building and having children draw themselves to scale, arguing for less pre-structuring from the point of view of mathematics. Visual reality was modified by showing a similar picture of a scale model of the building. He suggested that visual conflicts help to make the intuitive notions of children more explicit. He called this a qualitative approach to problems with estimation as an intermediate tool on the way to numerical precision, an approach to be incorporated into an elementary school program on ratio. Other examples included density at an early stage, based on the idea of intertwining the separate learning sequences for counting large quantities and measuring area (a small and a large cake with different numbers of ginger pieces and also different ginger taste per bite), ratio tables, multiscale number lines, and stick shadows to illustrate ratio invariance. Ratio should be considered a key notion in elementary school mathematics, as it is a key to applications in physics, chemistry, biology and specific areas of mathematics including probability, similarity and trigonometry. An overall criticism was that the existing curriculum places too much emphasis on the mathematics rather that relating the reality of the child to the mathematical concepts.

Decimals: Diana Wearne (USA) presented a paper prepared with James Hiebert (USA) on the meanings of decimals held by students in grades 4,5,6,7 and 9. They gave written tests to 700 students and interviewed 150 of these. They found that less than half of the students at the end of grade 6 could correctly write a decimal for 3/10, with a fourth of the students making the error of 3.10. About the same number wrote 0/9 or 9/0 when asked to write a fraction for .09. Students tended to treat decimals as whole numbers rather than as quantities, ignoring the decimal point. Only 14% of grade 6 and 37% of grade 7 students thought that .5 was larger than .42. Where 4 of 100 equal parts of a region were shaded, only half of the students were able to write the decimal correctly at the end of grade 9, with 4.100 the common error.

Data show that many students have not connected decimal symbols with either place value or part-to-whole concepts of common fractions. If they are meaningless symbols, then decimal computation turns into a mechanical application of symbol manipulation rules, with an abundance of predictable procedural flaws. The most urgent instructional problem is to help students create meaning for decimal numerals.

Fractions: Friedhelm Padberg (FRG) reported on his study which investigated whether it is better to begin with addition or multiplication of fractions. His results with 28 classes of grade 7 students showed clear superiority for doing addition first. For students who did multiplication first, 19% gave 5/13 for 3/8 + 2/5 while only 9% of those who did addition first made the same error. He argued for addition first because 3 × 4/5 can be done with repeated addition, 4/5 + 4/5 + 4/5 and because addition situations occur more often in daily life than multiplication situations. The most difficult level of exercise on multiplication was 2/11 × 5 where 10/55 was given as the answer by 26% of the students, while 15% of the students gave 15/7 as the answer to 5/7 × 3/7.

Earlier Vergnaud had expressed the view that we need to conceive of fractions as both operators (action of sharing) and quantities (result of sharing). Padberg reported on the changes in German texts where the formal idea of operator is used in multiplication (2/3 means multiply by 2 and divide by 3). Until 1976, both fractions were viewed as operators in a problem such as 2/3 × 4/5. Texts now treat one fraction as a quantity and one as an operator.

Joseph Payne (USA) reported on curriculum problems and issues for rational numbers. He discussed his research results, showing that the set model and number line model are more difficult than models with real objects and plane regions, thus suggesting that sets and the number line be delayed until concepts of objects and regions are firmly learned. Equivalent fractions and equivalent decimals were noted as difficult topics. He emphasised the need for increased instructional time on the meaning of fractions and decimals.

The discussion clearly showed the dominant practice in all countries represented is for students to learn a set of computational rules. Often the rules are mislearned with application of part of a rule for one operation applied to a completely different operation.

The issue of grade placement of computation topics was explored at some length. Suggestions included the delay of multiplication until grade 7, keeping computation relatively simple (small, easy denominators), doing a better job of developing the computation, and delaying the more complex work until grades 7 and 8. The importance of mixed numbers was raised, with the conclusion that operations on mixed numbers are of declining importance. Payne and Milton Behr (USA) suggested that concrete models be related more carefully to the verbal and symbolic rules for computation, illustrating the point using muliplication of fractions.

Summary

Faulty additive reasoning in solving ratio and proportion problems, the persistent peculiarities in the ways students operate with fractions and decimals, and the poor understanding the students possess show that purely numerical treatment and formalisation of rules enter too early in mathematics education. This premature formalisation often destroys the insights and ideas which pupils already have acquired from real life experiences. We should not teach rules too early because of the danger of students performing meaningless operations with meaningless symbols. Rules need to be derived carefully, making sure that the rules and models are related. Algorithms which are often confused need to be contrasted and compared. Above all, we need to develop quantitative feelings for ratio, fractions and decimals, always taking into account the insight and reality that students have. There is a strong need for developmental research to see how these important objectives can be achieved.

3. *Verbal Problem Solving.*

Convenors: Frank Lester (USA) and Marilyn Zweng (USA)

The principal questions addressed by this action group were:

- How do children solve problems?
- What makes problems difficult to solve?
- Problem-solving instruction: what do we know and what do we need to know?

In addition, presenters responded to the broader question:

- What are the problems in teaching problem solving?

Research reported by Terezinha Carraher (Brazil) showed that many children who are proficient in solving problems outside school (market situations) fail to solve equivalent problems when they are presented in school settings (story problems). Not only did the children perform better in market situations, 98% correct against 73% correct on the same problems given verbally later, they also used different strategies. In market situations the children manipulated quantities, while in school situations they more frequently manipulated symbols. In market situations the children used methods to solve problems that they had not learned in school. These methods were effective and demonstrated that the children had dealt intelligently with the quantities involved in the problem. Carraher contrasted these methods with the 'numeral pushing' taught in school.