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Problem-Orientated Geometry Teaching With Consideration Of Computers

Summary:

The task of geometry teaching in regular schools cannot be the supplying of scientifical knowledge or the preparation for the study of a special subject like mathematics. That is why I will begin with the explanation of general objectives of geometry teaching at school. After that I will talk about methods of teaching geometry which fit together with these objectives. With this it will come out that no axiomatic way will be suitable, the main point of the method should be rather working on problems.
With such a conception a computer presents a good help. On one hand it can spare drawing work during experimentations with figures and on the other hand it animates to look out for new questions.

In the second part then I will present and discuss several problems as illustration of the before presented conception. Three examples have to do with polygons (shapes of triangles, angles in regular polygons, combinatoric with polygons) and concern mostly the junior secondary school while two examples about curves concern the senior secondary school.

1. General objectives of geometry teaching at regular schools

The aim of mathematics teaching at school cannot be the learning of mathematics for itself. Just so I don't see the task of geometry teaching only in the mediation of geometric knowledge. Also a geometric course which tries to keep on a strong scientifical systematic with a far extending completness of proves is not suitable in my view. On one hand the experiences showed that for pupils an adequate formal course is not practicable and on the other hand with such a course a lot of important general objectives can't be attained.

Therefore I want to talk about general objectives before I begin to talk about ways of geometry teaching at school which promise to be possible and fruitful. The following categorisation of general objectives naturally is only one characterisation of the task of geometry teaching at school and also the categories are not exactly distinct.

(1) Pragmatic orientated objectives

Some *geometrical knowledge and skills* – e. g. as for symmetry, measures of areas and scale-drawings – *are necessary or at least helpful in the everyday world* of each person in our society. And if you see not only the direct personal tasks and problems but also the social ones and the imaginable future tasks and problems of single persons as well as the whole society you will find a lot of geometrical knowledge and skills which should be taught at school. In this view it is important of course to teach these attainments on basis of solving problems of real life. (More to this topic see G. Graumann 1987 and 1988a.)

(2) On cultural foundation orientated objectives

Besides knowledge and skills which can be used directly each person should have *basic knowledge of the important achievements of the human culture* especially those of our civilization. The comprehension of our world and the ability of forming the future world together with others can't be done well without knowledge about our "roots". (Cf. also Heymann, 1989, p. 5/7 and Kirsch, 1980, p. 231/232).

In respect to geometry teaching this concerns first the role of geometry for the kind of thinking of the old greek, for the change of thinking during the beginning of modern times (especially with Decartes) and for the two thousand years lasting discussion about the axiom of parallels which finally led to the modern way of thinking in mathematics. Secondly with this topic it is concerned the experience that out of pragmatic and vivid reflections the human being can come to pure theoretical questions, as for example reflections about parquetry patterns led to the platon solids or to questions of group theory.

(3) Language and thinking promoting objectives

a) A lot of geometry concepts – like triangle, angle, plane, intersection, symmetry, rotation, cylinder and cube – are not only used in mathematics but also in everyday life. If it is our intension that all people can catch things and relations of the world and can communicate as best as possible then the school has the task to teach the children *founded and differentiated concepts*. Different applications of geometric concepts, different visual characterizations of such concepts and the put out of the practical, technical and theoretical function of geometric forms (cf. Bender/Schreiber, 1985, p. 34 ff.) then are important for geometry teaching.

b) For the development of a versatile set of geometric concepts and of a good spatial faculty of thinking it is necessary to *promote spatial imagination, the ability to discover patterns in complex structures and the ability to mathematizise situations of everday life* with geometric concepts and knowledge. Moreover the ability to *find logical relations and keep on argumentations* also can be trained very good in geometry.

c) In our present and also future world with its complex problems a thinking in narrow, from single sciences preformed ways won't do it any longer; rather *it is important to discuss any theme by different aspects and with many relations to other themes*? In geometry you have often to do with figures which have a lot of aspects and relations to other figures and geometry also has many connexions to other subjects like arts or physics; *therefore geometry teaching is proper to develop a thinking in connexions*. (See also Graumann 1984).

d) Last but not least a good faculty of thinking requires a *general ability of problem solving*. Also for training this ability geometry offers a lot of possibilities.

There exist a lot of problems out of real life as well as out of pure geometry which are understandable and interesting for children and also not to simple.

(4) On critical use of reason and responsibility orientated objectives

A well educated person in my view should keep company with his-/herself and his/her world around so that nobody will be burden immoderate. This means e. g. that you have to estimate the effects of your actions and that you don't overrate your abilities. Just in our world of today - where nearly everything is determined by science and where we already can see the bad effects of extensive utilization of isolated knowledge - it is very important *to show the limitations of mathematizations and the limitation of mathematical reflections* already at school. Hans Werner Heymann in this connexion pointed out that a general education also has to include that you should use your reason critically, i. e. that you don't take statements and judgements as to value without analysis in respect to discrepancies and your own experiences. (See Heymann, 1989, p. 5)

With mathematics education in general and geometry teaching in special you can attain this aim for example if you discuss the mistakes which come out by mathematical modelling. But also the discovery of mistakes in seeming clear proofs can strengthen the critical reasoning if you discuss these mistakes not only in a technical manner. Moreover the discussion about the power as well as the limitation of visual geometric models (as for example the model of atoms of Bohr) can promote the consciousness of the limitation of mathematical thinking.

(5) Creativity, pleasure and self-confidence promoting objectives

We know that children normally work on geometric problems with pleasure because of their visual presentation and possibilities to act concrete and because in geometry the flow of work is not so fixed as in arithmetics. Therefore *geometry courses offer many possibilities to promote creativity. In geometry courses the pupils also can strengthen their self-confidence very good* because they can get a lot of positive reactions especially on visual level.

Finally I would like to mention that geometry teaching also offers good possibilities to develop the aesthetic perception. (See e. g. Graumann 1988b).

2. A methodical way of geometry teaching

As I already mentioned the orientation of geometry courses which try to follow the named objectives can't be found in the systematic of the scientifical geometry. Therefore *I propose to structure geometry teaching by working on problems resp. fields of problems* whereat from time to time surveying and systematizing discussions take place (cf. also Wagenschein, 1965, p. 18). The scientifical systematic then is no longer the guide for method but the background only for the teacher. Single theorems arise during problem solving as necessary tools. A clear and good founded development of concepts turn out to be necessary or at least helpful for the understanding and solving of problems. Different representations of geometric subjects and several different definitions or characterizations of geometric concepts have to be treated. Ways how we come to further reaching or sometimes also totally new problems have to be shown and discussed, as e. g. generalizations, analogies and other relations with earlier treated problems.

With such a method the pupils will be led to independent thinking and research-like work. By the retrospective view and the production of relations in a systematic way after each problem solving phase the pupils part by part win a digest of geometry. In some cases the problems can also be chosen by the pupils but mostly this is an important task of the teacher who have to keep the whole development in mind. During working on particular problems yet the pupil should do their own decisions as far as possible. Discussions with the teacher and all pupils as well as between single pupils or in small groups make up an important aspect of this method because aspectful concepts, different strategies for problem solving, correct regulars and the ability to find interesting theorems or problems can be built best in a communicative situation.

Further I would like to mention that with this scetched methodical way the tools should not be fixed like compasses and ruler in classical geometry. Chequered paper, pin boards, card paper with fixed shapes, coloured cubes, murmels or even packing boxes e. g. are helpful tools for geometry teaching. Also *a computer is a very good tool for the method of geometry teaching I propose.* First you can use much more geometric forms by which the geometry at school can be presented more interesting and aspectful. Secondly by "playing" with special programs questions about concepts or new experiences can be stimulated. Moreover the creative work in geometry can be stimulated with a computer much more than with the classical tools.

In this connection I would like to remark that the acquisition of knowledge about computer science and of skills about programming computers is not to be seen as task of geometry teaching. Therefore it is not forbidden to use special software without knowing something about the programming. As in everyday world we use the software like a machine from which we don't know the construction but only the function. (This does not exclude, that the pupils construct simple programs by their own.) However the discussion about the use of computers in general and the reflection about the limitation of the use of computers in special is to be seen as task of such a geometry teaching.

3. Examples for the practice of teaching

In the following I want to illustrate the above general discussion with some examples. First I will talk about three examples out of the domain "polygons" which mostly concerns the junior secondary school. And secondly I will describe two different aspects out of the domain "curves" which concerns the senior secondary school.

<u>1. Example:</u> *Shapes of triangles*

At the beginning of geometry teaching in the secondary school the children have to obtain a wide visual foundation about the possibilities of shapes of triangles. Thus they should draw a lot of different formed triangles (see e.g. Wittenberg 1963, p. 78 ff). Such exercises also can be combined with problems of art. Here I would like to discuss the following problem which is connected with the Pythagorean way of thinking (i. e. structuring the world by simple ratios).

Find all shapes of triangles (that means all triangles apart from similar ones) which can be built with 1, 2, 3, 4, 5 *as measures of the sides.*

Such a problem offers besides the experiences about shapes of triangles additional facilities. After some experimental work on this problem the children find a theorem about any triangle – namely the theorem that the sum of two side-measures always must be bigger than the measure of the remaining side. The given problem then can be transformed into the question to find a triple (a/b/c) of numbers out of $\{1,2,3,4,5\}$ which hold the inequalities $a \leq b \leq c < a + b$. For the case $a = 1$ it can be concluded that $b = c$. For the other cases it comes out very difficult to get a formula, especially because the triples of similar triangles appear only once. In any way the children get experiences with systematical proceedings. With variations of the given number-set these experiences can be deepend. In upper classes you also can ask for types of triangles which angle-measures have simple ratios (e. g. belongs to the angle-ratio $1:1:1$ the equilateral triangle, $1:1:2$ the right-angled isoceles triangle, $1:2:2$ and $1:1:3$ a triangle out of the regular pentagon, $1:2:3$ the right-angled triangle which make one half of the equilateral triangle). Secondly we can combine the given problem with the systematization of shapes of triangles by setting the task to classify the triangles respectively isosceles and equilateral triangles on one hand and acute, right, obtuse-angled triangles on the other hand. In upper classes you also can ask for a formular (the generalization of the theorem of Pythagoras by using inequalities).

<u>2. Example:</u> *Constant angles in polygons*

The regular pentagon already offers a lot of interesting questions, from the construction to questions about partial figures and ratios. Another very promising theme, which you find in several text books has to do with the angle-measures of all regular polygons and puzzles with regular polygons. Here I will discuss the following problem:

What happens if you pull down repeated a constant angle-measure and a constant length ?

If the given angle-measure is equal to one of a regular polygon you get the regular polygon of course. But what happens with other measures? *In many cases you get a regular star-polygon.* After getting several examples a lot of questions are coming out: How are their angle-measures related to those of the normal regular polygons with the same or twice number of edges. How much iterations are necessary that the polygon closes? How many regular star-polygons with a fixed number of edges can we find? Does the figure close in all cases?
A computer program to test such questions is very simple in LOGO or COMAL, but also in BASIC or PASCAL it is not very difficult to find out such a program (see fig. 1).

```
 1 REM  * Angle-Iteration *
10 LPRINT CHR$(28);CHR$(37);REM
   * Graphik-Modus is on *
20 INPUT "ANGLE = ";A
30 INPUT "NUMBER OF ITERATIONS = ";B
35 IF B=0 THEN 80
40 INPUT "RADIUS = ";R
50 INPUT "NUMBER OF COLOUR = ";C
55 LPRINT "J";C;REM
   * Choosen Colour is on *
60 W=180-A
62 X=40+R*COS(90)
64 Y=-40+R*SIN(90)
66 LPRINT "M";X;",";Y;REM
* Move to starting-point *
70 FOR I=1 TO B
72 X=40+R*COS(90+I*W)
74 Y=-40+R*SIN(90+I*W)
76 LPRINT "D";",";X;",";Y;REM
   * Draw *
78 NEXT I
80 END
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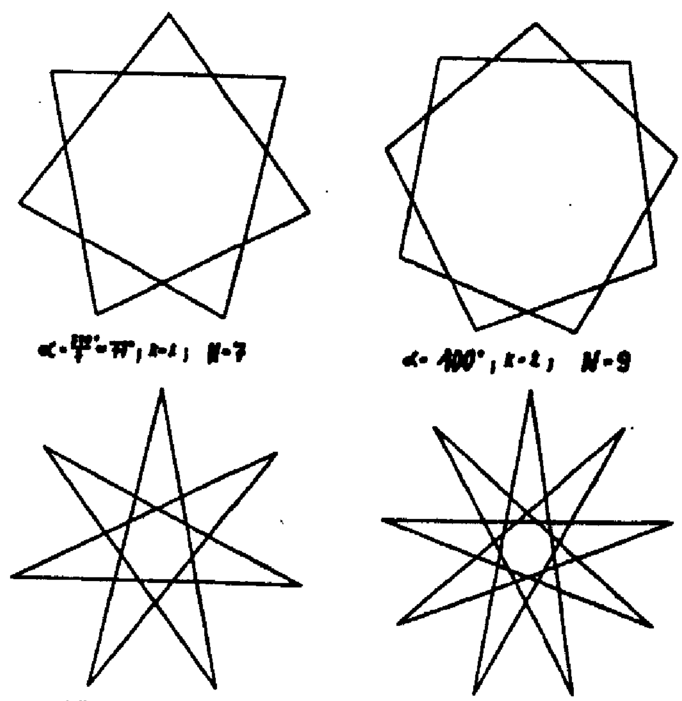


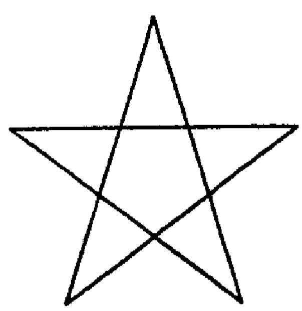


Fig. 1

By looking out for a formula you remark that besides the given angle-measure α and the number of edges n also the number k of 360°- turnings is important. (You can count them while the computer is plotting.) Further it is not very difficult to see that the angle-measure $(180° - \alpha)$ is the measure of turning at one point. With these reflections you come to the formula $n \cdot (180° - \alpha) = k \cdot 360°$ or $(180° - \alpha)/360° = k/n$ where k can be any integer smaller than $\frac{n}{2}$. Indeed it comes out that for $k = 1$ you get the normal regular polygon and only for k, n without common divisor you get different figures. Thus for $n = 3$ or 4 you get no star-polygon, for $n = 5$ you get one star-polygon, for $n = 7$ you get two different star-polygons and in general the number of star-polygons which can be drawn in the given way is equal to the number of integers k which hold $1 < k < \frac{n}{2}$ and have no common divisor with n.

By thinking about the star-polygon with 6 edges which is built by two equilateral triangles we get further types of star-polygons; they arise by combination of two or more polygons of the first type and could be denoted with the integers k, n where the two numbers have a common divisor.
By varying the given length after each step (e. g. halving or doubling) or by varying the given angle-measure after each step (e. g. alternating two measures) you can get a lot of more questions which lead again to new types of star-polygons or entire new themes.

3. Example: *Combinatoric questions with polygones*
The question about the number of all union-lines of 3, 4, 5 or 6 given points you often find in text books for the junior secondary school. By experience I know that pupils like such problems. But to train combinatoric abilities and to find structure qualities you also should work on similar problems like the question about the numbers of possible polygons with 3, 4, 5, or 6 given points or the question about the number of intersections with 3, 4, 5, 6 ... given lines / circles / planes or the question about the union-lines and union-planes with spatial given points. With the first question in case of general position of the points the pupils should learn the different ways of counting at which you can get the formula about the sum of all integers from 1 to a fixed integer. By the looking out for formulas in other cases they can learn to work about more difficult combinatoric problems. Some other interesting and not too difficult problems at which the concept of a general polygon can be deepened is the following:

How many diagonals does a polygon with n edges have? Can you build new interesting figures with all or some diagonals of such a polygon? Or how many areas arise from the diagonals? How many intersection-points arise from the diagonals or the sides resp. how many diagonal points does a polygon have?

The computer again is a good help for these problems just as computing-help in case of bigger numbers or as tool for getting a lot of different drawn examples.

4. Example: *The family of sinus-curves*
A family of curves is characterized by curve-equations which have one or more parameters. For example all lines through one point or generally all lines built a family of curves. All parabolas with equation $y = a \cdot x^2$ built a family too.
The sense of working with families of curves on one hand is to see a curve in connexion with others and on the other hand to find attributes by systematic resp. dynamic varying. The above named families yet are not very proper for these aims. Therefore I will discuss the family of sinus-curves here; it has also relevance in function theory, electro dynamics, oscillation theory and music.

a) We know that we can get the cosinus-curve from the sinus-curve by transformation $x \rightarrow x + c$.

What do we get by transformating the sinus-curve by $x \rightarrow c \cdot x$ with e. g. $c = 1, 2, 3, 4$ or π? What relations can we find between these curves?

b) The sound of an instrument depends on the composition of the upper tones. *How does an equation which describes a basic tone with some upper tones look like? Are there some characterizations about the family of all curves with an equation $y = a \cdot sin(b \cdot x) + c \cdot cos(d \cdot x)$?*
The use of a computer with such questions is nearly indispensable (see fig. 2).

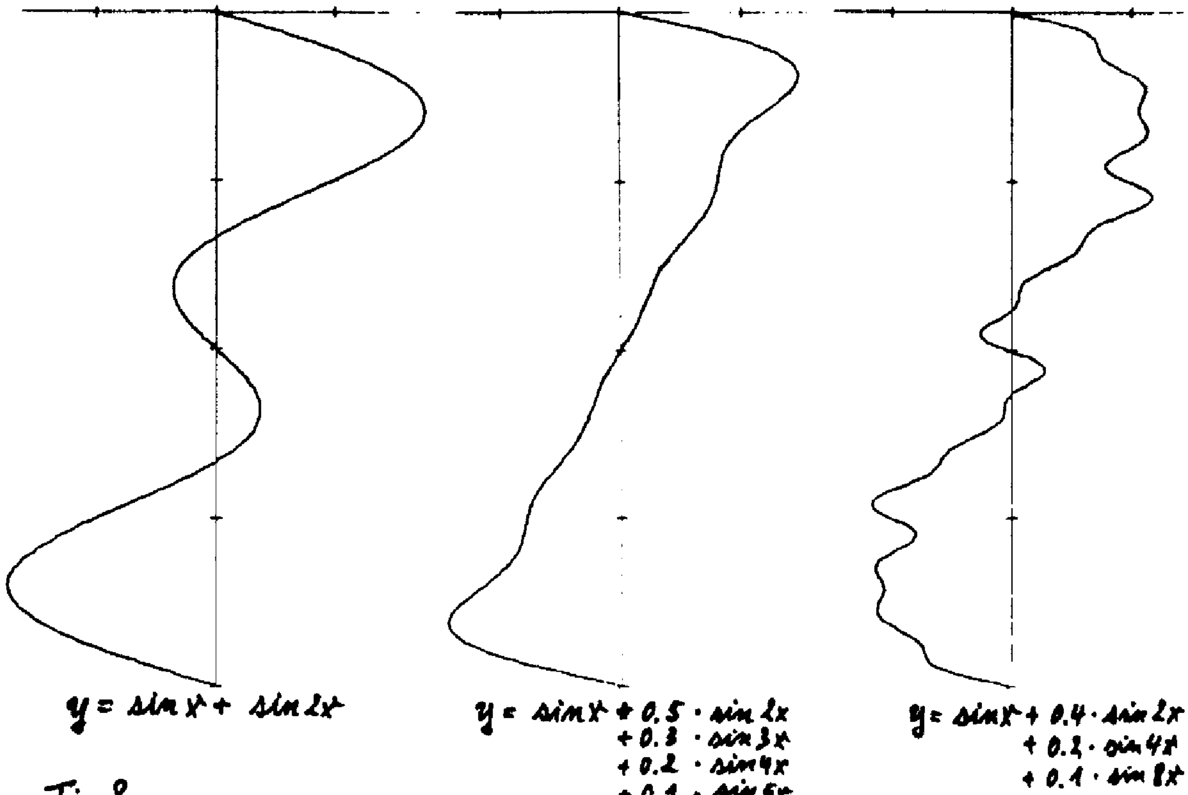
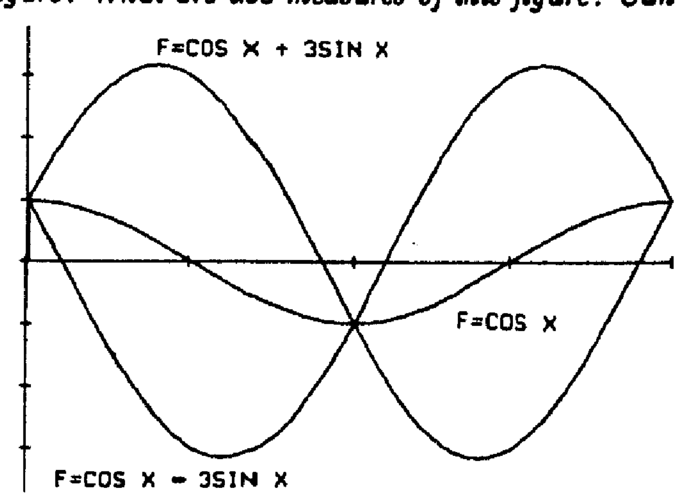


Fig. 2

c) The curves with equation $y = cos\, x, y = cos\, x + 3 \cdot sinx$ resp. $y = cos\, x - 3 \cdot sinx$ for $0 \leq x \leq \pi$ build a nice figure (see fig. 3).
What symmetries has this figure? What are the measures of this figure? Can you generalize these results?

Fig. 3



d) As generalization of sinus-curves you could use polar coordinates instead of cartesian coordinates. The experiences with curves can be enlarged by this way. Also new interesting questions and relations to art can be found.

How do the curves with equation $r = |a \cdot sin(b \cdot \varphi)|$ with $(r|\varphi)$ as polar coordinates look like? What properties have these curves? With which a, b are these curves not infinite?

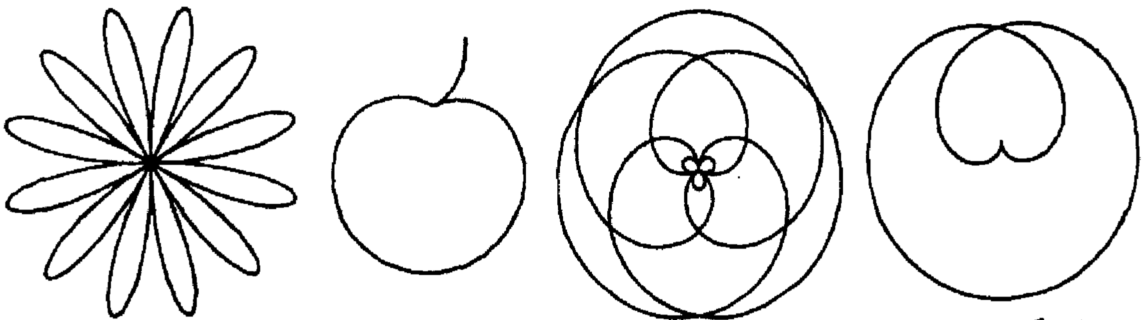
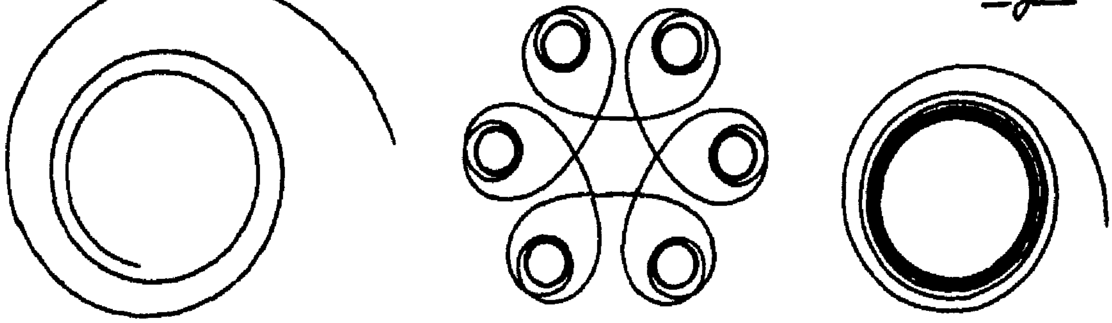


Fig. 4

That the use of a computer with plotter can be very helpful for these problems needs no discussion of course (e. g. see fig 4).

5. Example: *Curves with natural coordinates*
From differential geometry we know that we can describe curves only with inner coordinates (i. e. coordinates which someone who is walking on the curve can determine without outside attributes), namely the *curveture and the arc-length.* These coordinates are called natural coordinates. With given curveture and arc-length for each point of the curve you can generate the curve by integration. An approximate construction can be done in analogy to the approach of a curve with a polygon by given derivation in cartesian coordinates. That means with given curveture and arc-length for a row of curve-points you can approximate the curve with a curve build of parts of circles. This method however is efficient only if you have a lot of given curve-points whose distance is very small. Thus *a computer is necessary to generate curves in this way properly.*

Fig. 5



$K(B) = 10 \cdot B$ $\quad$ $K(B) = 25 \cdot |\sin B| + 1$ $\quad$ $K(B) = \log B$

The programming of a computer for such a task normally couldn't be done by pupils. The basic principle and the use of such a program however is a possible theme for pupils in senior secondary school and a chance to get experiences with entire different curves (see e. g. fig 5). They also can get an idea of a method mathematicians use for several problems, e. g. by working on a theory of dynamical systems (e. g. the fractales of Mandelbrot).

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