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On Uniform Local Dispersion on a Family of G-Orbits*

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Consider a topological space T which is the union of a family of G-orbits, where G is a locally euclidean group G acting on T. On every G-orbit consider a probability which is absolutely continuous with respect to the image measure of the normalized restriction of the Haar measure on some compact neighborhood of the identity in G. Assume that the densities of the probabilities on the orbits have a common upper bound. Let μ be a probability on T which is the integral over the measures on the orbits with respect to some probability μ' on T. It is shown that this specific kind of integral representation of μ does not depend on the size of the compact neighborhood of the identity in G. © 1986 Academic Press, Inc.

1. INTRODUCTION

The concept of Haar measure on a locally compact group expresses perfectly the idea that mass is dispersed over this underlying group. Also images of Haar measures under actions of such a group on certain topological spaces are suitable to formalize the idea of dispersion. The basic spaces in this case are G-spaces or, more generally, G-orbits (cf. Mackey [4] and Furstenberg [3]).

Sometimes one is interested only in local dispersion around a certain point. This can be done by restricting the action to some compact neighborhood of the identiy. Moreover, by extending the class of considered measures to those having L_∞-densities with respect to Haar measure, one considers families of G-orbits and measures on these orbits having a uniform upper bound for their densities with respect to the image of the Haar measure restricted to a neighborhood of the identity.

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Since on a group itself every compact neighborhood is equally apt to describe local dispersion around the identity, the whole class of compact neighborhoods can be considered as describing local dispersion. This yields an immediate analogy with the concept of a germ of sets or functions.

The question arises whether this irrelevance of the size of neighborhood for the formalization of local dispersion is preserved when going from groups to G-orbits. Moreover, if this is the case, is it still true for a family of G-orbits if a uniform upper bound for the densities is postulated, as described before? This problem for locally Euclidean groups originated from a problem in economic theory (cf. Dierker, Dierker and Trockel [2]). See also Trockel [6, 7, 8].

2. Result

We assume that the group G is the additive group $\mathbb{R}^n$, $n<\infty$. T is a Polish space. $\mathscr{B}(X)$ is the Borel σ-field of a topological space X. Denote B, B' neighborhoods of O. The normalized restrictions of Lebesgue (i.e., Haar) measure to B and B' are λ and λ' respectively, i.e., $\lambda(B)=\lambda'(B)=1$. Denote by $\mathring{B}$ and $\mathring{B}'$ the interiors of B and B', respectively. We consider the following surjective measurable mappings: Let $\tilde{a}$ be a right action of G on T, i.e., a measurable map $\tilde{a}\colon T\times G\to T$: $(t,g)\mapsto t_g$. Denote by $[t]=a(\{t\}\times G)$ the G-orbit of t.

We shall need the following surjective measurable maps:

$$a=\tilde{a}|_{T\times B} \qquad a'=\tilde{a}|_{T\times B'} \qquad e=\tilde{a}|_{T\times(B\times\mathbb{Z}^n)};$$
$$b\colon T\times B\to B\cap\mathbb{Z}^n\colon (t,g)\mapsto g'$$

where $g'\in B\cap\mathbb{Z}^n$ if $g\in\mathring{B}'+\mathbb{Z}^n$, $g'=0$ otherwise;

$$c\colon T\times(B\cap\mathbb{Z}^n)\to T\colon (t,g)\mapsto(t,g'),$$

where $g'=g(\text{mod }\mathbb{Z}^n)$ if $g\in\mathring{B}'+\mathbb{Z}^n$, $g'=0$ otherwise.

The definition of b and c outside the sets $\mathring{B}'+g$, $g\in\mathbb{Z}^n$ concerns only pairs (t,g'), where g' is in a λ-null set and will, therefore, be unimportant for our purpose.

Proposition. *Let B be a neighborhood of $O\equiv id\in G$, let γ be a probability on $(T\times B,\ \mathscr{B}(T\times B))$ with disintegration*

$$\gamma=\int_T \xi'_t\gamma\circ a^{-1}(dt):$$

(i) *the probabilities* ξ'_t, $t \in \operatorname{supp} \gamma \circ a^{-1}$, *on* $(B, \mathscr{B}(B))$ *are equivalent*,

(ii) $\xi'_t \ll \lambda$ *for all* $t \in \operatorname{supp} \gamma \circ a^{-1}$,

(iii) $\{d\xi'_t/d\xi'_s, d\xi'_t/d\lambda\}_{t,s \in \operatorname{supp} \gamma \circ a^{-1}}$ *is a relatively weak* compact subset of* $L_\infty(B, \mathscr{B}(B), \lambda)$.

Then for every $\varepsilon > 0$ *there exists a neighborhood* B' *of* O *with* $\operatorname{diam} B' < \varepsilon$ *and a probability* γ' *on* $(T \times B', \mathscr{B}(T \times B'))$ *such that*

(a) $\gamma \circ \operatorname{proj}_T^{-1} = \gamma' \circ \operatorname{proj}_T^{-1}$,

(b) $\gamma' = \int_T \xi''_t \gamma' \circ a'^{-1}(dt)$,

(c) *the probabilities* ξ''_t, $t \in \operatorname{supp}(\gamma' \circ a'^{-1})$ *fulfill* (i), (ii), (iii) *above, when* B, λ, *and* ξ'_t, $t \in \operatorname{supp} \gamma \circ a^{-1}$ *are replaced by* B', λ', *and* ξ''_t, $t \in \operatorname{supp} \gamma' \circ a'^{-1}$.

Proof. It suffices to treat the case where $B' = [-\frac{1}{2}, \frac{1}{2}]^n \subsetneq B \subset [-\frac{3}{2}, \frac{3}{2}]^n$. Repeated application of the proof provides us with a probability on a cube with diameter smaller than ε. If B were not a subset of $[-\frac{3}{2}, \frac{3}{2}]^n$ we would replace B' by the largest cube $[-(m+\frac{1}{2}), m+\frac{1}{2}]^n$ contained in but not equal to B, while not changing at all the arguments to follow.

Define γ' on $(B', \mathscr{B}(B'))$ by $\gamma \circ c^{-1}$. Obviously, γ and γ' have the same marginal distribution on $(T, \mathscr{B}(T))$, say μ. Hence (a) is proved.

In the following let A always denote an element of $\mathscr{B}(T \times B')$. Define $\mu' := \gamma \circ a^{-1}$. We have

$$\begin{aligned}
\gamma'(A) = \gamma(c^{-1}(A)) &= \int_T \xi'_t(c^{-1}(A))\, \gamma \circ a^{-1}(dt) \\
&= \int_T \int_{B \cap \mathbb{Z}^n} \xi'_t(c^{-1}(A) \mid b^{-1}(g))\, \xi'_t \circ b^{-1}(dg)\, \mu'(dt) \\
&= \int_{T \times (B \cap \mathbb{Z}^n)} \xi'_t(c^{-1}(A) \mid b^{-1}(g))\, \rho(d(t, g)) \\
&= \int_{e^{-1}(T)} \xi'_t(c^{-1}(A) \mid b^{-1}(g))\, \rho(d(t, g)).
\end{aligned}$$

The probability ρ on $(T \times (B \cap \mathbb{Z}^n), \mathscr{B}(T \times (B \cap \mathbb{Z}^n)))$ is defined by

$$\rho(M \times N) = \int_M \xi'_t \circ b^{-1}(N)\, \mu'(dt), \qquad M \in \mathscr{B}(T),\ N \in \mathscr{B}(B \cap \mathbb{Z}^n).$$

Denote the $\mathscr{B}(T \times (B \cap \mathbb{Z}^n))$-measurable map $(t, g) \mapsto \xi'_t(c^{-1}(A) \mid b^{-1}(g))$ by f^A. Then

$$\gamma'(A) = \int_{e^{-1}(T)} f^A(t, g)\, \rho(d(t, g)).$$

Now we make use of the disintegration of ρ provided by the surjective, measurable map e due to the disintegration theorem (cf. Parthasarathy [5, p. 145]). We get

$$\rho(A)=\int_T \beta_t(A)\,\rho\circ e^{-1}(dt)=\int_{e^{-1}(T)} \beta_{e(t,g)}(A)\,\rho(d(t,g)).$$

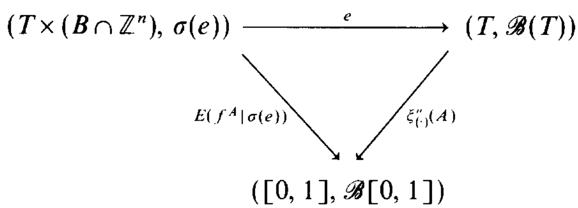
The probabilities β_t, $t\in T$, on $(T, \mathscr{B}(T))$ live on the fibres $e^{-1}(t)$ for ρ—a.e. $t\in T$. Moreover, for any $(t,g)\in T\times(B\cap\mathbb{Z}^n)$ we have

$$\beta_{e(t,g)}(A)=\rho(A\mid\sigma(e))\,(t,g) \qquad \rho\text{—a.e. on } T\times(B\cap\mathbb{Z}^n).$$

Consequently, we get

$$E(f^A\mid\sigma(e))(t,g)=\int_{e^{-1}(T)} f^A(t',g')\,\beta_{e(t,g)}(d(t',g')).$$

By the factorization of conditional expectations (cf. Bauer [1, p. 319]) there exists a $\mathscr{B}(T)$-measurable map $t\mapsto\xi''_t(A)$, which makes the following diagram commutative:



The transformation formula for integrals yields

$$\begin{aligned}\gamma'(A)&=\int_{e^{-1}(T)} f^A(t,g)\,\rho(d(t,g))=\int_{e^{-1}(T)} E(f^A\mid\sigma(e))(t,g)\,\rho(d(t,g))\\&=\int_T \xi''_t(A)\,\rho\circ e^{-1}(dt).\end{aligned}$$

Now $d((\rho\circ e^{-1})/d(\gamma'\circ a'^{-1})$ equals 1 $\gamma'\circ a'^{-1}$—a.e. (and therefore also $\rho\circ e^{-1}$—a.e.). To see this consider $S=\{t\in T\mid (d(\rho\circ e^{-1})/d(\gamma'\circ a'^{-1})(t))<1\}$ and assume $\gamma'\circ a'^{-1}(S)>0$. We have

$$\begin{aligned}\gamma'(S\times B)&=\int_T \xi''_t(S\times B)\,\gamma'\circ a'^{-1}(dt)\\&=\int_S \xi''_t(S\times B)\,\gamma'\circ a'^{-1}(dt).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\gamma'(S\times B) &= \int_T \xi''_t(S\times B)\,\rho\circ e^{-1}(dt)\\ &= \int_S \xi''_t(S\times B)\,\rho\circ e^{-1}(dt)\\ &= \int_S \xi''_t(S\times B)\,\frac{d(\rho\circ e^{-1})}{d(\gamma'\circ a'^{-1})}(t)\,\gamma'\circ a'^{-1}(dt)\\ &< \textstyle\int_S \xi''_t(S\times B)\,\gamma'\circ a'^{-1}(dt).\end{aligned}$$

This contradiction shows that the measures $\rho\circ e^{-1}$ and $\gamma\circ a'^{-1}$ coincide. Hence (b) is proved.

To finish the proof we have to give a uniform upper bound for the family $\{d\xi''_t/d\xi''_s,\ d\xi''_t/d\lambda'\}_{t,s\in\operatorname{supp}\gamma'\circ a'^{-1}}$. Before we carry this out, we prove the following

CLAIM. $[M\in\mathscr{B}(T\times B),\ \inf_{t\in\operatorname{supp}\gamma\circ a^{-1}}\xi'_t(M)=0]$ *implies*

$$[\sup_{t\in\operatorname{supp}\gamma\circ a^{-1}}\xi'_t(M)=0].$$

Proof of the Claim. By assumption (iii) there is some $k>0$ such that for all $t, s\in\operatorname{supp}\gamma\circ a^{-1}$ we have

$$\left\|\frac{d\xi'_s}{d\xi'_t}\right\|_\infty,\qquad \left\|\frac{d\xi'_t}{d\xi'_s}\right\|_\infty<k<\infty.$$

Hence, if $\xi'_t(M)=0$ for some $t\in\operatorname{supp}\mu'$ then also for all $t\in\operatorname{supp}\mu'$.

Suppose $\xi'_s(M)\neq 0$ for some $s\in\operatorname{supp}\mu'$. We get

$$\begin{aligned}\inf_{t\in\operatorname{supp}\mu'}\xi'_t(M) &= \inf_{t\in\operatorname{supp}\mu'}\int_M \frac{d\xi'_t}{d\xi'_s}(t',g')\,\xi'_s(d(t',g'))\\ &\geqslant \inf_{t\in\operatorname{supp}\mu'}\ \inf_{(t',g')\in M}\frac{d\xi'_t}{d\xi'_s}(t',g')\,\xi'_s(M)\\ &\geqslant \xi'_s(M)\left[\sup_{t\in\operatorname{supp}\mu'}\ \sup_{(t',g')\in M}\frac{d\xi'_s}{d\xi'_t}(t',g')\right]^{-1}\\ &\geqslant\left(\sup_{t',t\in\operatorname{supp}\mu'}\left\|\frac{d\xi'_{t'}}{d\xi'_t}\right\|_\infty\right)^{-1}\xi'_s(M)>0\end{aligned}$$

This proves the claim.

Now we can estimate the densities $d\xi''_t/d\lambda'$, $d\xi''_t/d\xi''_{t'}$; $t, t'\in\operatorname{supp}\gamma'\circ a'^{-1}$. Define $\mu'':=\gamma'\circ a'^{-1}$.

For any $t \in \operatorname{supp} \mu''$ and any $g \in B \cap \mathbb{Z}^n$ we get

$$\begin{aligned}\frac{d\xi''_t}{d\lambda'} &= \frac{d\xi''_t}{d\lambda} \cdot \frac{d\lambda}{d\lambda'} \\ &= \frac{d\lambda}{d\lambda'} \cdot \frac{d}{d\lambda} \left[\int_{e^{-1}(t)} \xi'_{t'}(c^{-1}(\cdot) \mid b^{-1}(g'))\, \beta_t(d(t', g')) \right] \\ &= \frac{d\lambda}{d\lambda'} \sum_{-g \in B \cap \mathbb{Z}^n} \frac{d\xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))}{d\lambda}.\end{aligned}$$

In the case where for one, and hence for all, $t \in \operatorname{supp} \mu'$ we have $\xi'_t(b^{-1}(-g)) = 0$; we are free to define $\xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))$ at will. It does not affect the definition of ξ''_t anyway. Hence we can continue the estimation by assuming $\xi'_t(b^{-1}(-g)) > 0$. In this case we have

$$\begin{aligned}&\frac{d\xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))}{d\lambda} \\ &\quad = \frac{d(\xi'_{tg} \mid b^{-1}(-g))\, (c^{-1}(\cdot))}{d\lambda} \cdot \frac{1}{\xi'_{tg}(b^{-1}(-g))}.\end{aligned}$$

By the claim proved above we have

$$\alpha := \inf_{t \in \operatorname{supp} \mu'} \xi'_t(b^{-1}(-g)) > 0.$$

Hence we get

$$\begin{aligned}\frac{d\xi''_t}{d\lambda'} &= \frac{d\lambda}{d\lambda'} \sum_{-g \in B \cap \mathbb{Z}^n} \frac{d\xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))}{d\lambda} \\ &\leqslant \frac{1}{\alpha} \cdot 3^n \sup_{t' \in \operatorname{supp} \mu'} \left\| \frac{d\xi'_{t'}}{d\lambda} \right\|_\infty < \infty.\end{aligned}$$

As a last estimation we get for $t, s \in \operatorname{supp} \mu''$:

$$\begin{aligned}\frac{d\xi''_t}{d\xi''_s} &= \frac{d(\sum_{-g \in B \cap \mathbb{Z}^n} \xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))}{d(\sum_{-g \in B \cap \mathbb{Z}^n} \xi'_{tg}(c^{-1}(\cdot) \mid b^{-1}(-g))} \\ &= \frac{d(\sum_{-g \in B \cap \mathbb{Z}^n} \xi'_{tg} \mid b^{-1}(-g)\, (c^{-1}(\cdot))\, (\xi'_{tg}(b^{-1}(-g)))^{-1})}{d(\sum_{-g \in B \cap \mathbb{Z}^n} \xi'_{sg} \mid b-1(\,-g)\, (c^{-1}(\cdot))\, (\xi'_{sg}(b^{-1}(-g)))^{-1})} \\ &\leqslant \frac{1}{\alpha} \cdot 3^n \sup_{t', t \in \operatorname{supp} \mu''} \left\| \frac{d\xi'_t}{d\xi'_{t'}} \right\|_\infty.\end{aligned}$$

The last term is finite by assumption (iii). Both estimates together prove assertion (c) of the proposition. Q.E.D.

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