A REPRESENTATION RESULT FOR PREFERENCES

Walter TROCKEL, Jakob WEINBERG, Bonn

1. INTRODUCTION

It is known that, given two preferences, for every commodity bundle x one can find continuous utility functions u_1 and u_2, representing those preferences such that $u_1(x) < u_2(x)$. Moreover, it is known that for any preference and any real numbers $r_1 < r_2$ one can find a representing utility function such that for all commodity bundles x one has $r_1 < u(x) < r_2$.

We want to present a drastic sharpening of these two facts, which we became aware of in discussing results of Weinberg (1981) on the characterization of welfare measures. We show that even for utility representations u_1, u_2 which are comparable in the sense that minimal and maximal utilities are the same for both representations, one can arrange it to get $u_1(x)$ smaller than $u_2(x)$ for <u>all</u> commodity bundles $x \neq 0$ or vice versa.

Though the argument is straight forward, the result is new to our knowledge.

We first proved it only for continuous utility functions (cf. Trockel - Weinberg (1981a)). Following a suggestion of Gerard Debreu we extended our Proposition to the case of r times continuously differentiable utility function, $1 \leq r \leq \infty$ (cf. Trockel - Weinberg (1981b)).

In the proof of the continuous case we used the function which associates with every level set of one utility function the minimal value of the second utility function on this level set. One way to establish the continuity of this minimal-value function is to apply a version of the so-called Theorem of the Maximum (cf. Varian (1978)). There are also versions of this theorem for continuously differentiable functions which would build a possible tool for the extension to the differentiable case of our original result.

However, proofs of those versions rely on the Implicit Function Theorem and are based, therefore, on same rank conditions for the derivatives of the Lagrange function associated with the parametrized optimization problem.

Such a rank condition parallels the condition of non-vanishing Gaussian curvature of indifference curves in points of demand, which is used in Debreu (1972) to get continuous differentiability of the demand function.

In our case this kind of rank condition amounts to requiring that the Gaussian curvatures of the respective indifference hypersurfaces are different at the point of interest.

In addition to this restriction coming from the rank condition, a general differentiability result for the maximum-value function can be expected only in the case of unique maximizers, i.e., when preferences are strongly convex.

There is, however, a way to get the desired result even in the case of non-convex preferences and without a restricting rank condition. For this one has to replace the minimum-value function by a smooth (C^∞) function doing the same job. A standard mathematical technique of smoothing is the convolution with some smooth function (cf. Lang (1969)).

The essence of the proof in the present paper consists of the choice of a suitable function to convolve with: i.e., such that the resulting smooth function inherits all the properties of the minimum-value function which are necessary for the proof in the original continuous utility case.

At the end we discuss the consequence of our result for Rawlsian social choice.

2. RESULT

Consider $\ell \geq 1$ commodities. Let $\mathbb{R}_+^\ell$ be the consumption set for all agents. Assume the agents' preferences, $\precsim$, to be reflexive, transitive, complete, and continuous binary relations on $\mathbb{R}_+^\ell$.

Continuity means that for all $x \in \mathbb{R}_+^\ell$ the sets $\{y \in \mathbb{R}_+^\ell \mid x \precsim y\}$ and $\{y \in \mathbb{R}_+^\ell \mid y \precsim x\}$ are closed. Moreover, we assume that except for the origin, which is an absolute minimum for all preferences, there are no local extrema at all. In case of smooth preferences this is fulfilled whenever preferences are locally non-satiated.

Finally, we assume for each preference $\precsim$ the following perfect substitution assumption*):

For all $x \in \mathbb{R}_+^\ell$ and for all $h \in \{1,\dots,\ell\}$ there exists a positive number N such that

$x \prec Ne_h$, where e_h is the unit vector having all components zero except the h-th one.

If we speak of a preference in the following it is always understood that it has all the properties listed above.

We are going to prove the following

<u>Proposition</u>: Consider any pair u_1, u_2 of continuous utility functions representing the preferences $\precsim_1$, $\precsim_2$, respectively.

If $u_1 \in C^r(\mathbb{R}_+^\ell)$,**) $0 \le r \le \infty$, then there is a utility function $\bar{u}_1$, equivalent to u_1 , such that

i) $\bar{u}_1 \in C^r(\mathbb{R}_+^\ell)$

ii) $\sup\limits_{x\in\mathbb{R}_+^\ell} \bar{u}_1(x) = \sup\limits_{x\in\mathbb{R}_+^\ell} u_2(x)$

iii) $\inf\limits_{x\in\mathbb{R}_+^\ell} \bar{u}_1(x) = \inf\limits_{x\in\mathbb{R}_+^\ell} u_2(x)$

iv) $\bar{u}_1(x) < u_2(x)$ for every $x \in \mathbb{R}_+^\ell$, $x \neq 0$.

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*) In technical terms: the indifference surfaces have to be bounded.

**) $f \in C^r(X)$ means that f is continuous and r times continuously differentiable on X .

The proof of the Proposition relies on the following two lemmas.

<u>Lemma 1</u>: Let $\precsim_1$, $\precsim_2$ be two preferences represented by continuous utility functions u_1, u_2, respectively. Let $a = u_1(0)$ and $b = \sup_{x \in \mathbb{R}^\ell_+} u_1(x)$. Then the function $V: [a,b) \to \mathbb{R}$ defined by $V(t) = \inf\{u_2(x) \mid x \in u_1^{-1}(t)\}$ has the following properties:

i) V is continuous

ii) V is increasing, i.e. $(t > t') \Rightarrow (V(t) > V(t'))$

iii) $V(a) = u_2(0)$

iv) $\sup_{t \in [a,b)} V(t) = \sup_{x \in \mathbb{R}^\ell_+} u_2(x)$

v) $V(u_1(x)) \leq u_2(x)$ for all $x \in \mathbb{R}^\ell_+$.

For a proof see Trockel-Weinberg (1982).

<u>Lemma 2</u>: Let $V: [a,b) \to \mathbb{R}$ be continuous and increasing, $-\infty < a < b \leq \infty$. Then there is a function $\tilde{V}: [a,b) \to \mathbb{R}$ such that

i) $\tilde{V} \in C^\infty([a,b))$

ii) $\tilde{V}$ is increasing

iii) $\tilde{V}(a) = V(a)$

iv) $\sup_{t \in [a,b)} \tilde{V}(t) = \sup_{t \in [a,b)} V(t)$

v) $\tilde{V}(t) < V(t)$ for all $t \in (a,b)$.

<u>Proof</u>: Let $\lambda, \mu: \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$\lambda(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

$$\mu(t) = \frac{\lambda(t)\lambda(1-t)}{\int_0^1 \lambda(s)\lambda(1-s)ds}$$

and let $\bar{V}: (-\infty,b) \to \mathbb{R}$ be defined by

$$\bar{V}(t) = \begin{cases} V(t) & t \in (a,b) \\ V(a) & t \le a \end{cases}$$

Define the function $\tilde{V}: (-\infty,b) \to \mathbb{R}$ by

$$\tilde{V}(t) = \int_0^1 \mu(s) \cdot \bar{V}(t-s)ds \ .$$

The function $\tilde{V}$ restricted to $[a,b)$ has the properties i), ii), iii) and v), independently of the specification of b .
It is C^∞ since it is defined by convolution with a C^∞ function (cf. Lang (1969)).
It is increasing on $[a,b)$ by definition since $\bar{V}$ is so .

$$\tilde{V}(a) = \int_0^1 \mu(s)\bar{V}(a-s)ds = \int_0^1 \mu(s)V(a)ds = V(a) \ .$$

$\tilde{V}(t) < V(t)$ on (a,b) because V is increasing.
Now assume that $b = \infty$. Then we have for all t

$$\bar{V}(t-1) = \int_0^1 \mu(s)\bar{V}(t-1)ds \le \int_0^1 \mu(s)\bar{V}(t-s)ds = \tilde{V}(t) \le \int_0^1 \mu(s)\bar{V}(t)ds =$$

$$= \bar{V}(t) \ .$$

Since $\sup_{t\in[a,\infty)} \bar{V}(t-1) = \sup_{t\in[a,\infty)} \bar{V}(t)$, this chain of inequalities implies $\sup_{t\in[a,\infty)} \tilde{V}(t) = \sup_{t\in[a,\infty)} \bar{V}(t) = \sup_{t\in[a,\infty)} V(t)$.
Hence also property iv) holds in this case.

Consider now the case $b < \infty$.
We want to define a second function $\tilde{\tilde{V}}$ which is defined on some interval $(b-\varepsilon,b)$ and has properties i), ii), iv) and v).
Once we have this function $\tilde{\tilde{V}}$ we will glue together $\tilde{\tilde{V}}$ and $\tilde{V}$ in a smooth way and by this get a function having all of the five properties of Lemma 1.

Let us first assume that $d = \sup_{t\in[a,b)} V(t) < \infty$.

Denote $V(a) = c$. Define $V^*: [a,b) \to \mathbb{R}$ by

$$V^*(t) = d + c - V(t) .$$

Obviously, $V^*([a,b)) = (c,d]$. Since V^* is decreasing on $[a,b)$ the inverse function $V^{*-1} = U : (c,d] \to \mathbb{R}$ exists and is increasing. Now define $\bar{U} : (-\infty,d) \to \mathbb{R}$ by

$$\bar{U}(\tau) = \begin{cases} U(\tau) & \tau \in (c,d) \\ \inf\limits_{\sigma\in(c,d)} U(\sigma) = b & \tau \leq c \end{cases}$$

Define $\tilde{U} : (-\infty,d) \to \mathbb{R}$ by

$$\tilde{U}(\tau) = \int_0^1 \mu(\sigma)\bar{U}(\tau-\sigma)d\sigma .$$

Obviously, $\tilde{U}$ is increasing and C^∞ on (c,d) since defined by convolution with the C^∞ function μ. We also have $\tilde{U}(\tau) < U(\tau)$ for all $\tau \in (c,d)$ and $\tilde{U}(c) = \bar{U}(c) = b = \inf\limits_{\sigma\in(c,d]} U(\sigma)$.

Since $\tilde{U}$ is increasing on (c,d) the inverse function $\tilde{U}^{-1}$ is defined and decreasing on $(a,b) = \tilde{U}((c,d))$.

Next we will show that also $\tilde{U}^{-1}$ is C^∞. For this it suffices to show that the derivative $\tilde{U}'$ of $\tilde{U}$ does not vanish on (a,b). We use the following two facts. First, for the derivative of $\tilde{U}$, which is defined by convolution, we have at every $\tau \in (c,d)$

$$\tilde{U}'(\tau) = (\mu * \bar{U})'(\tau) = (\mu' * \bar{U})(\tau)$$

i.e.

$$U'(\tau) = \int_0^1 \mu'(\sigma)\bar{U}(\tau-\sigma)d\sigma .$$

Secondly, by the symmetry of the function μ we have $0 < -\mu'(\sigma) = \mu'(1-\sigma)$ for all $\sigma \in (1/2,1)$.

Hence we get by substituting $1-\sigma$ for $\sigma \in [1/2,1]$

$$\int_0^1 \mu'(\sigma)\bar{U}(\tau-\sigma)d\sigma = \int_0^{1/2} \mu'(\sigma)\bar{U}(\tau-\sigma)d\sigma - \int_0^{1/2} \mu'(\sigma)\bar{U}(\tau-(1-\sigma))d\sigma$$

$$= \int_0^{1/2} \mu'(\sigma)[\bar{U}(\tau-\sigma) - \bar{U}(\tau-(1-\sigma))]d\sigma \, .$$

Since $\bar{U}$ is increasing on (c,d) for every $\tau \in (c,d)$ both factors of the integrand, hence the integral, is positive. Now, define the C^∞ function $\tilde{\tilde{V}} : (a,b) \to (c,d)$ by

$$\tilde{\tilde{V}}(t) := d + c - \tilde{U}^{-1}(t) \, .$$

$\tilde{\tilde{V}}$ is increasing since $\tilde{U}^{-1}$ is decreasing on (a,b).

$$\tilde{\tilde{V}}(t) = d + c - \tilde{U}^{-1}(t) < d + c - V^*(t) = V(t)$$

$$\sup_{t\in(a,b)} \tilde{\tilde{V}}(t) = d = \sup_{t\in(a,b)} V(t) \, .$$

Hence, in the case $b < \infty$ the function $\tilde{\tilde{V}}$ has the desired properties.

Consider now the case $[a,b] = [a,\infty)$.

First (a,∞) is mapped onto $(0,\infty)$ by the increasing C^∞ diffeomorphism $t \mapsto t-a$. Then $(0,\infty)$ is mapped by the increasing C^∞ diffeomorphism $t \mapsto \arctan t$ onto $(0,\pi/2)$. Composition of $\tilde{V}$ with these two maps reduces our problem to the case of finite b which we treated above. In fact, now we have the situation $a = 0$, $b = \pi/2$.

Now, we will glue together the functions $\tilde{V}$ and $\tilde{\tilde{V}}$.

Define the C^∞ function $\gamma : \mathbb{R} \to [0,1]$ by

$$\gamma(t) = \int_t^b \lambda(s-b+\varepsilon)\lambda(b-s)ds \Big/ \int_{b-\varepsilon}^b \lambda(s-b+\varepsilon)\lambda(b-s)ds \, .$$

Define $\hat{V} : [a,b) \to \mathbb{R}$ by

$$\hat{V}(t) = \gamma(t)\tilde{V}(t) + (1-\gamma(t))\tilde{\tilde{V}}(t) \, .$$

i) $\hat{V}$ is C^∞ since $\tilde{V}$, $\tilde{\tilde{V}}$ and γ are so.

ii) $\tilde{V}$ and $\tilde{\tilde{V}}$ are increasing. For increasing t also the weight $(1 - \gamma(t))$ of $\tilde{\tilde{V}}(t)$ increases. Since $\tilde{\tilde{V}}(t) > \tilde{V}(t)$ for all $t \in (b-\varepsilon,b)$, the function $\hat{V}$ is increasing by definition.

iii) $\hat{V}(a) = \tilde{V}(a) = V(a)$

iv) $\sup\limits_{t\in[a,b)} \hat{V}(t) = \sup\limits_{t\in(a,b)} \tilde{\tilde{V}}(t) = \sup\limits_{t\in(a,b)} \bar{V}(t) = \sup\limits_{t\in(a,b)} V(t)$

v) $\hat{V}(t) < V(t)$ for all $t \in (a,b)$ since $\tilde{V}$ and $\tilde{\tilde{V}}$ have this property.

q.e.d.

<u>Proof of the Proposition</u>: According to i) and ii) of Lemma 1 the function $V : [a,b) \to \mathbb{R}$, $t \mapsto V(t) = \inf \{u_2(x) \mid x \in u_1^{-1}(t)\}$ satisfies the assumptions of Lemma 2.

Now define $\bar{u}_1$ as $\bar{u}_1 = \tilde{V} \circ u_1$, where $\tilde{V}$ is defined as in Lemma 2. Now the properties of $\bar{u}_1$ claimed in the Proposition are an immediate consequence of Lemma 1 and Lemma 2.

q.e.d.

<u>Remark</u>: Our result allows for an alternative symmetric formulation. For simplicity's sake we will state this only for the case of continuous utility functions. The extension to the smooth case is immediate.

<u>Corollary</u>: For any two preferences $\precsim_1$, $\precsim_2$ and for any $a \in (0,\infty)$ there exist continuous utility functions u_1 and u_2 representing $\precsim_1$ and $\precsim_2$, respectively such that

i) $u_1(0) = \inf_{x \in \mathbb{R}_+^\ell} u_1(x) = \inf_{x \in \mathbb{R}_+^\ell} u_2(x) = u_2(0) = 0$

ii) $\sup_{x \in \mathbb{R}_+^\ell} u_1(x) = \sup_{x \in \mathbb{R}_+^\ell} u_2(x) = a$

iii) $u_1(x) < u_2(x)$ for all $x \in \mathbb{R}_+^\ell \setminus \{0\}$.

The corollary is an immediate consequence of the proposition and of the following Lemma 3. The proof of Lemma 3 is straightforward and can be found in Trockel-Weinberg (1981a).

Lemma 3: For any preference and any $a \in (0,\infty]$ there is a continuous utility representation u such that $\inf_{x \in \mathbb{R}_+^\ell} u(x) = 0$ and $\sup_{x \in \mathbb{R}_+^\ell} u(x) = a$.

3. CONCLUDING REMARKS

Our result allows for an application in social choice theory. This is shown for a generalization of the continuous case in Trockel-Weinberg (1982). Consider the following game for individuals, having preferences on some choice set of alternatives. Each individual's strategy space is the set of normalized cardinal utility representations of his true preference. Given the Rawlsian social choice function, which associates with each tupel of strategies a feasible alternative such that the least individual utility level is maximal, the outcome is determined by the utility of that alternative.

The result shows that there does not exist a Nash-equilibrium (even in the case where the true (ordinal) preferences are revealed). Given the representation of the others, each individual may choose a representation guaranteing him dictatorship. This shows that the Rawlsian Rule, which was introduced for ethical reasons - making the least privileged member of the society best off - is not practicable due to an inherent conceptual weakness.

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Walter Trockel and Jakob Weinberg,
Abt. Wirtschaftstheorie II, Universität Bonn
Adenauerallee 24, 5300 Bonn 1, Fed. Rep. Germany