A Measure Theoretical Problem in Mean Demand Analysis

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1. Introduction

In this paper a new alternative approach to the problem of uniqueness of mean demand of an economy's consumption sector is suggested.

Decentralized decision making of individual agents can clear the markets and thus result in equilibrium only if aggregate demand of an economy is single valued rather than multivalued. The individuals' tastes, however, are described by non-convex preference relations which result in multivalued individual demand sets, given prices and wealths.

Thus, we can state the economic problem as follows:

Is multivalued individual demand compatible with single valued mean demand?

It is generally agreed in the literature that only sufficient diversification of tastes can yield a unique mean demand. This sufficient diversification has to be made precise in terms of measure theory.

In the so called parametric framework due to Sondermann (1975) the space of preferences is a subset of the set of all preferences which is parametrized by a subset of a finite dimensional Euclidean space. For the parametric approach see also Araujo - Mas-Colell (1978), Yamazaki (1978) and Hildenbrand (1978).

The dispersedness assumptions in the quoted literature essentially imply that the "bad behavior" occurs in a set of positive codimension only. The uniqueness of mean demand, therefore, depends on the special family of preferences and of its parametrization.

In this paper we are interested in the dispersedness properties of the set of all preferences, considered as its own parameter space.

We shall reduce in the following the economic problem to a problem in the theory of Hausdorff measure and of Hausdorff - Besicovitch dimension.

2. The Model

We consider the consumption sector of an economy with ℓ commodities which is defined by its constituting consumers.

The consumption set for each consumer is $\mathbb{R}^{\ell}_{+}$, the nonnegative orthant of the commodity space, $\mathbb{R}^{\ell}$.

A consumer is described by the pair $(\succsim, w)$, of his consumption characteristics where $\succsim$ is a binary relation on the consumption set, $\mathbb{R}^{\ell}_{+}$, and w is a positive real number. The relation $\succsim$ represents the consumer's taste and is called preference relation, the number w describes the consumer's wealth.

The set P of preference relations can be topologized in different ways. We will not become specific about properties of $\succsim$ and about the topology on P.

The space of <u>price systems</u> is the set

$$S := \{p \in \mathbb{R}^{\ell} \mid p \gg 0 , \; \| p \| = 1\}$$

Here any norm in $\mathbb{R}^{\ell}$ is good.

Each pair $(p, w) \in S \times \mathring{\mathbb{R}}_{+}$ determines a <u>budget set</u>

$$B_{p,w} := \{x \in \mathbb{R}^{\ell}_{+} \mid px \leq w\} .$$

The set of $\succsim$ - maximal elements in $B_{p,w}$ is the <u>demand set</u> of consumer $(\succsim, w)$ at the price system p. Since consumers' preferences are not assumed to be convex the demand sets cannot be expected to be singletons. Therefore we are led to a <u>demand correspondence</u>

$$\varphi : P \times S \times \mathring{\mathbb{R}}_{+} \longrightarrow \mathbb{R}^{\ell}_{+} .$$

Our economic problem can now be stated as follows: <u>Which probability μ on the space $P \times \mathring{\mathbb{R}}_{+}$ of consumption characteristics has the property that for any $p \in S$</u>

$$\# \varphi(\succsim, p, w) = 1 \quad \underline{\mu\text{ - almost everywhere on } P \times \mathring{\mathbb{R}}_{+}} \;?$$

For shortness and simplicity we fix $p \in S$ and $w \in \mathring{\mathbb{R}}_+$ for the following and concentrate on measures on $\mathcal{P}$.

3. Hausdorff Measure and Hausdorff - Besicovitch Dimension

We mentioned above that the "bad" set of preferences in the articles working with the parametric framework correspond to manifolds of positive codimension in a finite dimensional linear space or manifold. But even "bad" sets filling out much more of a space of preferences than any manifold would do, could still have measure zero.

Looking for a more general concept of dimension one is first led to the topological dimension. For a definition see Hurewicz and Wallmann (1941). Since homeomorphisms, which preserve the topological dimension of a set, do not necessarily preserve the measure zero, however, this notion turns out to be useless for our purpose.

There is, however, a different notion of dimension being only a metric invariant rather than a topological one. This notion is intimately related to the concept of Hausdorff outer measure.

These two concepts are suitable for the characterization of the degree of dispersedness or scatteredness of sets in separable metric spaces.

Denote by A and $\bar{\mathbb{R}}$ a separable metric space and the extended real numbers, respectively. Recall that an <u>outer measure</u> on A is a nonnegative, extended real-valued, monotone, σ - subadditive mapping μ^* defined on the power set of A . It is a <u>metric outer measure</u> if for all $T, U \subset A$:

$$d(T, U) > o \Rightarrow \mu^*(T \cup U) = \mu^*(T) + \mu^*(U) \ .$$

Here d denotes the metric on A and
$d(T, U) := \inf \{d(t, u) | t \in T , u \in U\}$.

If μ^* is a metric outer measure on A a set $T \subset A$ is called <u>μ^* - measurable</u> if and only if for every $U \subset A$

$$\mu^*(U) = \mu^*(U \cap T) + \mu^*(U \backslash T) \ .$$

The μ^* - measurable subsets of A build a σ - algebra on A including the Borelian subsets of A . The restrictions of μ^* to this σ - algebra or to the Borel σ - algebra are measures in its usual meaning. For details see Caratheory (1918).

<u>Definition</u>: For any positive number K the <u>K - dimensional</u> (Hausdorff (1919) (1927)) <u>Hausdorff</u> (outer) <u>measure</u> is the metric outer measure μ_K^* on A defined by:

$\forall \ T \subset A$:

$$\mu_K^*(T) := \sup_{\varepsilon > o} \inf \{ \sum_{i=1}^{\infty} (\text{diam } T_i)^K |$$

$$\bigcup_{i=1}^{\infty} T_i = T \text{ , diam } T_i < \varepsilon \text{ , } i \in \mathbb{N} \} \text{ .}$$

The Hausdorff outer measure is regular in the following sense. For every $T \subset A$ there exists a decreasing sequence $(U_n)_{n \in N}$ of open sets containing T such that $\mu_K^*(T) = \mu_K^*(\bigcap_{n \in N} U_n)$.

We shall use this concept of Hausdorff measure to define the dimension of a set in a separable metric space. This is possible because any set can have finite non-zero K - dimensional Hausdorff measure for at most one $K \in \mathbb{R}$. We note the following important fact:

<u>If $T \subset U$, $\mu_K^*(T) < \infty$ and $K' > K > o$, then $\mu_{K'}^*(T) = o$.</u>

For a given set $T \subset A$ there may be no real number $K > o$ such that $o < \mu_K^*(T) < \infty$. In any case the following definition does make sense.

<u>Definition</u>: For any $T \subset A$ the <u>Hausdorff - Besicovitch dimension</u> of T is

$$\dim T := \sup \{K > o | \mu_K^*(T) = \infty\} \text{ .}$$

This definition of dimension implies for any $T \subset A$:

$$\mu_K^*(T) = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \quad \text{for } K \begin{Bmatrix} > \\ < \end{Bmatrix} \quad \dim T .$$

For finite-dimensional A the Hausdorff - Besicovitch codimension of any subset T of A is defined by

$$\operatorname{codim} T := \dim A - \dim T .$$

The Hausdorff - Besicovitch dimension of a set $T \subset A$ does not depend on A but only on T considered as a metric space, i.e., any embedding of T into a different metric space will not change $\dim T$. Thus $\dim T$ is an intrinsic property of T .

The Hausdorff - Besicovitch dimension of a set is always at least as large as its topological dimension. Running through all topologically equivalent metrizations of the space and taking the infimum of the corresponding Hausdorff - Besicovitch dimensions yields the topological dimension.

The following properties of Hausdorff measures and of the Hausdorff - Besicovitch dimension might be important in an application to mean demand analysis.

The n-dimensional Euclidean space has not only linear and topological but also Hausdorff - Besicovitch dimension n .

The n-dimensional Hausdorff (outer) measure coincides on Lebesgue measurable sets with a measure which is equivalent to the Lebesgue measure. The n-dimensional Lebesgue measure gives mass one to a unit cube, whereas the n-dimensional Hausdorff measure gives mass one to the unit ball.

Any subset of n-dimensional Euclidean space has n-dimensional Hausdorff measure zero if its Hausdorff - Besicovitch dimension is smaller than n . For example, the Hausdorff - Besicovitch dimension of the irrationals in the unit interval is one, since its one dimensional Lebesgue and Hausdorff measure are one. On the other hand the irrationals have topological dimension zero.

The use of this fractional notion of dimension allows for the analysis of situations which, by the irregularities of the shapes involved, are excluded from the linear or differentiable analysis. Transversality conditions in the differential framework yield locally positive integral linear codimension and thus positive integral Hausdorff - Besicovitch codimension. A smaller fractional codimension allows for a much stronger concentration of the "bad" phenomenon in a neighborhood in the space, still small enough from the measure theoretical point of view.

4. Application To The Economic Problem

The mathematical work to do is to compute the Hausdorff - Besicovitch dimensions of the space of preferences, $\mathcal{P}$, and of its subset E of all preferences yielding multivalued demand at p , w . This is supposed to be a non-trivial task. Once it is done one only has to compare $\dim P$ and $\dim E$. If $\dim E < \dim P = n \in \overline{\mathbb{R}}$ then $\mu_n^*(E) = o$.

Then individual demand is unique μ_n^* - almost everywhere on P .

But still $\dim P = n$ can be infinite. Since we are interested in the existence of a natural probability measure on P in terms of which we can state our result and which allows forming a mean demand via integration, we would need

$$\mu_n^*(P) < \infty .$$

By normalization we would get the probability. But $\mu_n^*(P) < \infty$ presumes that $\dim P = n < \infty$. In this context it might be of interest that Larmann (1967a) (1967b) gave a characterization of spaces having finite Hausdorff - Besicovitch dimension. He also gave examples of such spaces being not Euclidean.

Summarazing one can say:

Prove: 1) $\dim P =: n < \infty$

2) $\dim P > \dim E$ ($\Rightarrow \mu_n^*(E) = o$)

3) $\mu_n^*(P) < \infty$

Properties 1) 2) 3) imply that mean demand with respect

<u>μ_n^* is unique</u>, given price p and wealth w.
But even a weaker version of 3) would help. If μ_n^* would be σ-finite or would have at least the finite subset property, one could replace P by a subset having finite μ_n^* - measure.

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