AGGREGATION OF DEMAND IN CASE OF
NONCONVEX PREFERENCES

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1. Introduction

Economists commonly believe that the consistency of individual decisions in a purely competitive economy is achieved by the price mechanism. However, the consistency of individual decisions cannot be expected if the aggregate decision is not uniquely determined by the price system. If consumers' preferences exhibit nonconvexities, individual demand need not be uniquely determined by the price system. A first question then is: When will aggregate demand of the consumption sector be uniquely determined? Because of the upper hemi-continuity of the mean demand correspondence this amounts to the question of when mean demand will be a continuous function.

But even a continuous aggregate demand function is not suitable to explain the consistency of individual demand decisions if it has extremely steep slopes. If aggregate demand is too sensitive with respect to prices, very small price variations lead to a considerable deviation from equilibrium. Furthermore, one would like to be able to show that equilibria are generically regular. So a next question is: When will aggregate demand be a differentiable function? Our approach in this paper has been motivated by this question.

The problem of smoothing demand by aggregation was posed by Debreu (1972) and W. Hildenbrand (1974). The continuity and differentiability of aggregate demand has been studied by Sondermann (1975, 1976, 1978) and by Araujo and Mas-Colell (1978). A major difference between their work and ours is that they stipulate a finite-dimensional manifold structure on the space of preferences considered, an assumption which we want to avoid. The manifold structure is used to formulate the notion of dispersed preferences. The conditions in the papers just mentioned imply that mean demand is a continuous function, but they do not yield differentiability everywhere. For the study of continuity of mean demand without the use of derivatives, see Mas-Colell and Neuefeind (1977), Neuefeind (1978), Yamazaki (1978), and W. Hildenbrand (1978).

In our approach to the problem of differentiability of aggregate demand we concentrate on smooth preferences. Since the space of consumers' characteristics consists of two components, namely the space of preferences, which has little structure, and the wealth space, which has the structure of the real line, we decided to carry out the aggregation in two steps, first with respect to wealth, then with respect to preferences. In this paper we deal with the first step only and show that aggregation with respect to wealth already brings about a considerable smoothing effect.

Section 2 introduces the model. In section 3 we show that for 1 large subset of utility functions of a certain class and for all price systems the mean demand of all consumers whose tastes are represented by a given utility function in that

subset is a uniquely determined bundle of commodities. This together with the well-known upper hemi-continuity of the mean demand correspondence implies that the mean demand of the consumption sector is a continuous function. This result is essentially a consequence of the multijet transversality theorem (Golubitsky and Guillemin (1975)).

In section 4 we point out that for a fixed preference relation, aggregation with respect to a continuous income distribution leads to a continuously differentiable demand function except for prices in a closed null set. The prices in this null set correspond to three types of difficulties: vanishing Gaussian curvature of indifference surfaces, critical jumps, and multiple jumps.

Points with vanishing Gaussian curvature can be considered as points where a catastrophe occurs. In case of a cusp catastrophe we show that vanishing Gaussian curvature does not destroy differentiability of mean demand of a fixed preference as long as no other disturbance occurs simultaneously.

2. The Model

Let us consider the consumption sector of an economy with $\ell \geq 2$ commodities. The commodity space is $\mathbf{R}^{\ell}$. We consider prices in

$$S = \{p \in \mathbf{R}^{\ell} |\ p \gg o,\ ||p|| = 1\}.$$

$||\cdot||$ denotes the Euclidean norm, $p \gg o$ means $p_h > o$ for all $h \in \{1,\ldots,\ell\}$. Every consumer has the consumption set

$$X = \{\ x \in \mathbf{R}^{\ell}\ |\ x \gg o\}.$$

Let $\mathcal{U}$ denote the set of C^{∞} utility functions $u : X \to \mathbf{R}$ satisfying assumptions U1), U2), U3) below.

U1) $Du\ (x) \gg o$ for all $x \in X$ (monotonicity).

The following boundary assumption keeps demand away from the boundary of X.

U2) cl $(u^{-1}(u(x))) \subset X$ for all $x \in X$.

Let $g(x) = Du(x)\ ||Du(x)||^{-1}$. The third assumption makes the sets $g^{-1}(p), p \in S$, smooth one-dimensional manifolds.

U3) $g : X \to S$ is a submersion.

This assumption rules out the simultaneous vanishing of two or more principal curvatures of the indifference hypersurfaces, it does not preclude, however, the Gaussian curvature of these surfaces to become zero.

Let $\mathcal{P}$ denote the set of preference relations $\leq$ representable by utility functions in $\mathcal{U}$, i.e. there is $u \in \mathcal{U}$ such that $x \leq y \iff u(x) \leq u(y)$.

The *wealth* of a consumer is a number $w \in]o, \infty[$. For an agent, described by his wealth $w \in]o, \infty[$ and his preference $\leq \in \mathcal{P}$, the demand at price system $p \in S$ is

$$\varphi(\leq, p, w) = \{x \in X | px \leq w, x \prec y \Rightarrow py > w\}.$$

The *mean demand* of all consumers is the integral of the demand correspondence φ with respect to a measure on the space $\mathcal{P} \times]o, \infty[$ of consumers' characteristics. The *integral of a correspondence* is defined as follows: Let $(\Omega, \mathcal{A}, \nu)$ be a mea-

sure space, $\psi : \Omega \to \mathbf{R}^m$ be a correspondence. The integral of ψ with respect to ν is the set

$\int \psi \, d\nu := \{\int s \, d\nu \in \mathbf{R}^m | s \in L^1 (\Omega, \mathcal{A}, \nu), s (\omega) \in \psi (\omega) \nu\text{-a.e. in } \Omega\}$.

For details see Hildenbrand (1974).

3. Continuous mean demand

Let $\delta_{\leq}$ be the wealth distribution for the fixed preference $\leq \in \mathcal{P}$. Let $\Phi_{\leq}(p) = \int_0^\infty \varphi_{\leq} (p,w) \delta_{\leq}(dw)$ denote the mean demand at price system p with respect to the measure $\delta_{\leq}$ on $(]o, \infty[, \mathcal{B} (]o, \infty[))$ given the preference $\leq \in \mathcal{P}$ where $\mathcal{B}$

denotes the Borel σ-algebra. Clearly, $\Phi_{\leq}(p)$ is a singleton if and only if $\varphi_{\leq}(p,w)$ is a singleton for $\delta_{\leq}$ - almost every wealth w. To prove this last property for any measure $\delta_{\leq}$ which is absolutely continuous with respect to Lebesgue measure λ, it suffices to prove it for λ on $]o, \infty[$. Note that the single - valuedness of $\Phi_{\leq}$ at every $p \in S$ implies that $\Phi_{\leq}: S \to \mathbf{R}^\ell$ is a continuous function.

Theorem 1: *In the space $\mathcal{U}$ of utility functions endowed with the C^∞ Whitney topology there is a residual subset $\mathcal{U}_{res}$ such that each element in $\mathcal{U}_{res}$ represents a preference relation $\leq$ for which $\varphi_{\leq}(p,w)$ is a singleton λ- almost everywhere on $]o, \infty[$.*

We sketch the proof of the theorem [1]. First we show that

$\lambda(\{w = g(x)x | g(x) = p, \mathcal{K}(x) = o\}) = o$,

where $\mathcal{K}(x)$ denotes the Gaussian curvature of $u^{-1}(u(x))$ at x. The rank condition U3) makes $g^{-1}(p)$ a one - dimensional differentiable manifold. The manifold $g^{-1}(p)$ is tangent to the budget hyperplane through $x \in g^{-1}(p)$ with normal vector p, if the Gaussian curvature of the indifference surface $u^{-1}(u(x))$ vanishes at x. Hence x is a critical point of the mapping $x \mapsto g(x)x$ defined on $g^{-1}(p)$. The set of critical values has Lebesgue measure zero.

It suffices to show that for any u in a residual subset $\mathcal{U}_{res}$ of $\mathcal{U}$ and for any $p \in S$ there is, outside that null set, only a set of isolated points $w \in]o,\infty[$ for which demand fails to be single - valued. Take any $u \in \mathcal{U}$ and suppose that the demand set at (p,w) does not contain any point with vanishing Gaussian curvature. The demand set at (p,w) is contained in the intersection of the manifold $g^{-1}(p)$ with the budget hypersurface $B_{p,w} = \{x \in X | px = w\}$ corresponding to (p,w). This intersection is transversal because of the nonvanishing Gaussian curvature. Therefore, and because of boundary assumption U2), $B_{p,w} \cap g^{-1}(p)$ consists of finitely many points. For fixed p any point in $B_{p,w} \cap g^{-1}(p)$ can be traced locally if w

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1) For a detailed proof see Dierker, Dierker, Trockel (1978a).

varies. This means that there are $\epsilon > 0$ and r smooth functions
$h_i :]w - \epsilon, w + \epsilon[\rightarrow X$, $i=1, \ldots, r$, $r = \# (B_{p,w} \cap g^{-1}(p))$, such that for any w'
$\epsilon]w - \epsilon, w + \epsilon[$ the set $B_{p,w'} \cap g^{-1}(p)$ equals $\{h_1(w'), \ldots, h_r(w')\}$.

Now suppose $x_1 = h_1(w)$ and $x_2 = h_2(w)$ are demanded at (p,w). Then, in par-
ticular, $u(x_1) = u(x_2)$, and $Du(x_1)$ is proportional to $Du(x_2)$. If $Du(x_1)$ exceeds
$Du(x_2)$, then a slight increase of wealth from w to w' prevents $h_2(w')$ from belon-
ging to the demand set at (p,w'), because $u(h_1(w')) > u(h_2(w'))$. Similarly, a
slight decrease of wealth from w to w'' prevents $h_1(w'')$ from belonging to the de-
mand set at (p,w'').

However, the case $Du(x_1) = Du(x_2)$ cannot be excluded, not even in the case
of only two commodities. Therefore, assume now $Du(x_1) = Du(x_2)$. Then one is led
to consider the second order variation of u at x_i along h_i. If the second order
increase of u at x_1 along h_1 exceeds that of u at x_2 along h_2, then a similar
reasoning as in the first order case shows that a slight variation of wealth pre-
vents one of the commodity bundles from belonging to the demand set. If the first
and the second order increase of utility at x_1 and x_2 along h_1 and h_2, respecti-
vely, happen to coincide, apply a similar argument to the third order increase of
utility, and so on.

The condition that all utility increases up to the order k coincide becomes
more and more restrictive for growing k. It turns out that there is a residual
set of utility functions for which it is impossible that all utility increases
up to the order ℓ at x_1 and x_2 coincide. However, taking derivatives for any order
into account, one can let the exceptional set of utility functions shrink much
more.

4. Towards Differentiability

In this section we want to show that, for a fixed preference ordering $\leq$,
aggregation with respect to wealth leads to a mean demand $\phi_{\leq} : S \rightarrow \mathbb{R}^\ell$ which is
almost everywhere C^1. The preference ordering $\leq$ is assumed to be represented by
the utility function $u : X \rightarrow \mathbb{R}$ which satisfies U1) to U3). We need two additional
assumptions on u.

The first of these assumptions concerns the behavior of u in a neighborhood
of a point $\bar{x}$ whose associated indifference surface has vanishing Gaussian curva-
ture $\mathcal{K}$ at $\bar{x}$. Let $\mathcal{K}(\bar{x}) = 0$, $\bar{p} = g(\bar{x})$, $\bar{w} = \bar{p}\bar{x}$. Consider the family of budget hyper-
surfaces $B_{p,w} = \{x \in X | px = w\}$. The function $u|B_{\bar{p},\bar{w}}$ has a degenerate critical
point at $\bar{x}$. The family $u|B_{p,w}$, $(p,w) \in S \times]0, \infty[$, can be regarded as an unfolding

of the germ of $u|B_{\bar{p},\bar{w}}$ at $\bar{x}$. We require this unfolding to be stable or, equivalently, to be versal (cf. Bröcker (1975)). To be more specific, define $U: \mathbb{R}_+^{\ell-1} \times S \times \mathbb{R}_+ \to \mathbb{R}$ by

$$U(x_1, \ldots, x_{\ell-1}, p, w) = u(x_1, \ldots, x_{\ell-1}, x_\ell(p,w)),$$

where $x_\ell(p,w) = (w - \sum_{h=1}^{\ell-1} p_h x_h)(1 - \sum_{h=1}^{\ell-1} p_h^2)^{-\frac{1}{2}}$.

We assume:

   U4) The unfolding U of $U(\cdot, \bar{p}, \bar{w})$ is stable.

Furthermore we postulate:

   U5) Let $(x,y,z) \in X \times X \times X$, $x \neq y, x \neq z, y \neq z$. Suppose $u(x) = u(y) = u(z)$,
      $\alpha Du(x) = \beta Du(y) = \gamma Du(z) = p$ for $\alpha, \beta, \gamma \in \mathbb{R}$, $px = py = pz$, and $x-y = \lambda(y-z)$. Then $\lambda \neq \frac{\alpha-\beta}{\beta-\gamma}$.

This assumption implies that triples (x, y, z) of pairwise distinct, collinear commodity bundles which are demanded simultaneously are isolated. Counting equations and unknowns makes it plausible that U5) is fulfilled in "most" cases.

Furthermore, we need the following assumption on the wealth-distribution of agents with preference ordering $\leq$:

   M) The probability measure $\delta_\leq$ on $(]0, \infty[, \mathcal{B}(]0, \infty[))$ has a continuous density, $h_\leq$, with respect to Lebesgue measure λ, and the support of $\delta_\leq$ is contained in a compact interval $[\underline{w}, \bar{w}] \subset]0, \infty[$.

The mean demand $\Phi_\leq : S \to \mathbb{R}^\ell$ of all agents with preference ordering $\leq$ is defined by

$$\Phi_\leq(p) = \int_{\underline{w}}^{\bar{w}} \varphi_\leq(p,w) \, h_\leq(w) \, \lambda(dw).$$

Theorem 2: *Let $\leq$ be represented by the utility function u satisfying U1) to U5) and let the wealth-distribution $\delta_\leq$ satisfy M). Then there is a closed null set $N_\leq$ in the price space S such that the restriction of the mean demand $\Phi_\leq$ to $S\backslash N_\leq$ is a C^1 function.*

We sketch the idea behind the theorem.[2]

There are three phenomena which may prevent $\Phi_\leq$ from being C^1 : vanishing Gaussian curvature, critical jumps, and multiple jumps.

To study the first of these phenomena, let X_0 be the set of points $x \in X$ such that $\mathcal{K}(x) = 0$, $\underline{w} \leq g(x) x \leq \bar{w}$, and x is a maximum of $u|B_{g(x),g(x)x}$.
      The latter requirement is fulfilled whenever x is in the demand set $\varphi_\leq(g(x), g(x)x)$. Points in X_0 correspond to catastrophes of corank 1, codimension ≥ 2. Hence the set $g(X_0)$ is null.

Two different commodity bundles x and y are demanded simultaneously at price

2) For a detailed proof see Dierker, Dierker, Trockel (1978b).

system p only if $g(x) = g(y) = p$, $u(x) = u(y)$, and $px = py$. This system of equations is used to define an $(\ell - 1)$- dimensional differentiable manifold of triples (p, x, y), which are called *jumps*. A jump is *critical* if it is a critical point of $(p, x, y) \mapsto p$. According to Sard's theorem critical jumps give rise to a null set of prices.

Mean demand $\phi_\leq$ need not be differentiable at a price system p associated with a noncritical jump if more than two commodity bundles are demanded at p. Thus one is led to consider the following system of equations:

$$g(x) = g(y) = g(z) = p, \quad u(x) = u(y) = u(z), \quad px = py = pz.$$

Its solutions are points (p, x, y, z) which are called *multiple jumps*. Multiple jumps form an $(\ell - 2)$- dimensional manifold and thus give rise to a null set of prices.

Let $N_\leq$ be the union of the three null sets corresponding to vanishing Gaussian curvature, critical jumps, and multiple jumps, respectively. Due to the compactness of $[\underline{w}, \bar{w}]$ and the boundary assumption U2) the set $N_\leq$ is closed. Let $\bar{p} \in S \backslash N_\leq$. For p near $\bar{p}$, individual demand $\varphi_\leq$ (p,w) is C^1 for all but a finite number of w's that correspond to jumps which are neither critical nor multiple. It follows that mean demand $\phi_\leq$ is C^1 at $\bar{p}$.

One would like to find reasonable assumptions yielding a mean demand which is C^1 everywhere on S. First observe that the vanishing of Gaussian curvature does not necessarily destroy differentiability. For the case of the cusp catastrophe we have:

Theorem 3: *Let $\leq$ be represented by the utility function u satisfying* U1), U2), U3) *and let* $\delta_\leq$ *satisfy* M). *Suppose* $\varphi_\leq$ $(\bar{p},\bar{w}) = \{\bar{x}\}$ *and* $\mathcal{K}(\bar{x}) = 0$. *Let* $g(\bar{x}) = \bar{p}$, $\bar{w} = \bar{p}\bar{x}$. *Furthermore, assume that the unfolding* U *associated with* $u|_{B_{\bar{p},\bar{w}}}$ *is equivalent to* f *defined by* $f(y_1, \ldots, y_{\ell-1}, v_1, v_2, \ldots,$

$$v_\ell) = -(y_1^4 + v_2 y_1^2 + v_1 y_1 + y_2^2 + \ldots + y_{\ell-1}^2).$$

Then there is $\epsilon > 0$ *such that*

$$\int_{\bar{w}-\epsilon}^{\bar{w}+\epsilon} \varphi_\leq (\cdot, w) \, h_\leq (w) \lambda(dw) \text{ is } C^1 \text{ at } \bar{p}.$$

For a proof see Dierker, Dierker, Trockel (1978b.)

5. Concluding Remarks

To deal with the lack of smoothness caused by critical and by multiple jumps one has also to aggregate with respect to preferences. At this point one needs assumptions expressing that preferences are sufficiently dispersed. As Lebesgue measure is not available on the space of preferences it is not clear what precise-

ly dispersion of preferences is supposed to mean. Multiple jumps involving no critical jumps lead to kinks of the mean demand function ϕ_s. To get differentiability by aggregation with respect to preferences one would require that, for every $p \in S$, the set of preferences exhibiting such a jump is null. Critical jumps, however, require a more sophisticated analysis. Moreover, situations where several disturbing phenomena, vanishing Gaussian curvature, critical, and multiple jumps, occur simultaneously have to be studied. It is not known, at present, which conditions must be satisfied so that the mean demand of an economy becomes everywhere C^1.

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