Uniqueness of Mean Maximizers via an Ergodic Theorem

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Summary: In the present paper known and well established facts and methods of ergodic theory are used to prove a theorem on uniqueness of mean maximizers. The problem arises in economic theory. As a corollary of our present result one gets single-valuedness, hence continuity, of market demand as a relation of prices. The statement and proof of the theorem is anteceded by an extensive economic motivation.

1. Introduction to the Economic Problem

The idea of decentralization of consumption and production decisions by means of a system of equilibrium prices is basic for economic equilibrium theory. The explanatory value of such an equilibrium price system depends crucially on the conception that it uniquely determines an economy's total demand and total supply. We are concerned with the consumption side of an economy. The importance of the problem does not depend on whether we look at the consumption sector of a competitive market economy or a centrally planned economy.

Consider, for example a blue print of a socialist economy as described by Oskar Lange (1938). Assume the central planning board gives a price system of consumption goods to a household. Now, unless we do not make the heroic and unjustified assumption of strictly convex preference relations for every household, we cannot expect that the households' optimization problems, given their wages and the price system, will have unique solutions. Accordingly, the total demand, i.e. the aggregate demand of all households in the economy, will in general fail to be unique. Hence the mere information on what the prices for consumption commodities are, does not suffice to clear the market, i.e. total demand may fail to meet total supply. Consequently Lange's competitive solution could not work. Now, the non-uniqueness of a household's demand for consumption goods is a phenomenon, which occurs only at certain exceptional combinations of wage and price system. If the number of households is very large one may expect that these exceptional budget situations are different for the various households. If there is enough *diversification* of households' characteristics, i.e. of preferences and wages, one may hope that for any given budget situation, i.e. pair of wage and price

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system, the result of *most* households' optimization problems are unique bundles of consumption commodities. As a consequence the aggregate demand set at that price system will be small, i.e. will contain only very similar commodity bundles. In the idealized case of a continuum of households, which expresses in a convenient way the large number of households, aggregate demand, which is an integral rather than a sum, can be hoped for to be unique. This idea was already expressed by AUGUSTIN COURNOT (1838) and referred to later on by WALRAS (1874), who contributed this regularizing effect of aggregation to the "Law of large numbers".

In (1972) GERARD DEBREU conjectured that a "suitable diffusion" on a measure space of consumers' characteristics might imply that aggregate demand is a C^0 or even C^1 function of prices. This conjecture initiated a series of articles concerned with the problem of "smoothing demand by aggregation".

A major difficulty, conceptually as well as formally, was it how to formalize "suitable diffusion". Equal distribution as formalized via LEBESGUE measure seemed to be an ideal candidate to express suitable diffusion. Unfortunately the space of preferences is not an EUCLIDean space. In his parametric approach SONDERMANN (1975) coped with this difficulty by considering only spaces of preferences parametrized by open sets of an EUCLIDean space. This made the LEBESGUE measure (locally) available. However this did not suffice to yield sufficient diversification. Preferences may change with a finite set of parameters without changing demand behavior at all. Therefore SONDERMANN had to make an additional assumption to guarantee richness of demand behavior for his finite dimensional families of preferences. This was performed in terms of a differential-topological transversality assumption. A more general version, independent of the differential context, was given by HILDENBRAND (1980). This parametric approach has two drawbacks, however. The first is that the concentration on finite dimensional subsets of the huge space of preferences is very restrictive. The second disadvantage is that suitable diversification is not expressed in measure theoretical or, more specifically, in probalistic terms.

On one hand it is clear that not all properties of preferences which might possibly be considered are of any relevance for the analysis of the demand. On the other hand one had to find a natural substitute or generalization of what LEBESGUE-continuous probabilities are on subsets of an Euclidean space. This led to the idea that the space of preferences or, more generally of preferences and wages, could be modelled as a G-space, the acting group being the group of price systems or, more generally of budgets. The HAAR measure transported in this way to the orbits under the action generates just the diffusion or diversification which corresponds to the economic intuition. Moreover, it turned out to yield uniqueness of aggregate demand (cf. TROCKEL (1980)) and, if combined with methods of differential topology, also continuous differentiability of the aggregate demand as a function of prices (cf. DIERKER; DIERKER; TROCKEL (1980) and (1981)).

The present result extends to a considerably more general situation than needed

in the specific economic framework described above. However, it is likely that a general result of this kind will turn out useful for different parts of economic theory. For example, in the theory of rational expectations equilibria it is a fundamental problem how to describe diversification of information among the members of an economy. Again the spaces involved exclude the direct applicability of LEBESGUE or even HAAR measure. However, again a treatment via G-spaces seems possible. It is crucial to point out that the group structure cannot be chosen in an ad hoc way but has to be present in an inherent way in the economic model under consideration.

In the next section I shall briefly describe how the group of prices acts on commodities and preferences in the context of aggregate demand and what kind of assumption yields uniqueness of aggregate demand. The precise statement and the proof can be found in TROCKEL (1980). The result can also be derived directly from a slightly modified version of our present theorem, to be proved below.

2. An Example: Mean Demand

We shall concentrate on the case of two commodities which is no restriction and simplifies notation. Moreover it allows for graphical illustration.

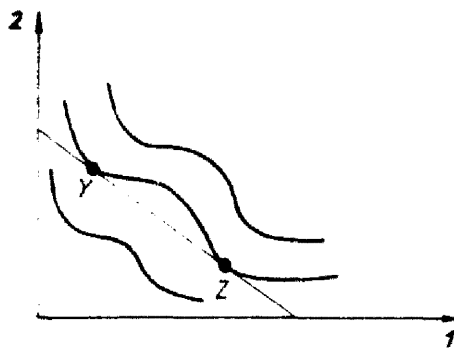Consider a situation as pictured in Figure 1.

Figure 1

Every consumer has the *consumption set* $\mathbb{R}^2_+$, which is a subset of the *commodity space* $\mathbb{R}^2$. Both commodities are assumed to be perfectly divisible. Any consumer (or household) is described by a positive real number, w, his *wealth* (or wage) and by a *preference relation*, $\geqslant$, on $\mathbb{R}^2_+$, i.e. a reflexive, transitive, complete binary relation. The preference is moreover assumed to be continuous and monotone. This can most easily be formalized by representability by a continuous function $u : \mathbb{R}^2_+ \to \mathbb{R}$ such that

i) $x \geqslant y \Leftrightarrow u(x) \geqq u(y)$ and

ii) $(x_i \geqq y_i,\ i = 1, 2;\ x \neq y) \Rightarrow (u(x) > u(y))$.

Such a function is a *utility function*. Clearly, for every increasing function $i : \mathbb{R} \to \mathbb{R}$, the utility function $i \circ u$ represents $\geqslant$ as well. A *price system*, p, is a positive linear functional on $\mathbb{R}^2$ restricted to $\mathbb{R}^2_+$. The price system p and the wealth w determine the budget constraint $px \leqq w$, under which an optimal commodity bundle with

respect to $\geqslant$ has to be chosen. Monotony of $\geqslant$ resp. u reduces the inequality $px \leqq w$ to the equality $px = w$. The set $q(\geqslant, w, p)$ is the demand set at the price system p of a household described by $(\geqslant, w)$. In Figure 1 we have $q(\geqslant, w, p) = = \{y, z\}$. The line through y and z is the budget line $px = w$, where $w = py = pz$. The curved lines are *indifference curves* of $\geqslant$, i.e. iso-utility lines for u.

Since $\varphi(\geqslant, w, p) = q(\geqslant, tw, tp)$ for any $t > 0$, i.e. since demand is positively homogeneous of degree zero, one may choose $p = (p_1, 1)$, i.e. the space of price systems can be chosen $S = \dot{\mathbb{R}}_+ \times \{1\}$ rather than $\dot{\mathbb{R}}_-^2$. Clearly, S is a group under coordinatewise multiplication.

Consider the following action of S.

$$S \times \dot{\mathbb{R}}_+^2 \to \mathbb{R}_+^2 : ((q, 1), ((x_1, x_2)) \to (q \cdot x_1, x_2) \equiv : x^q .$$

Now let $\mathscr{S}$ be a Polish space of preferences. This can be shown to be the case for all usual specifications of $\mathscr{S}$ (cf. HILDENBRAND (1974)). We consider now the action of S on $\mathscr{S}$ defined as follows.

$$S \times \mathscr{S} \to \mathscr{S} : ((q, 1), \geqslant) \to \geqslant_q$$

where $\geqslant_q$ is defined by

$$x^q \geqslant_q y^q : \Leftrightarrow x \geqslant y .$$

The combined effects of the actions of S on itself, on $\mathbb{R}_+^2$ and on $\mathscr{S}$ are illustrated in Figure 2 for the acting price system $q = (2, 1)$.
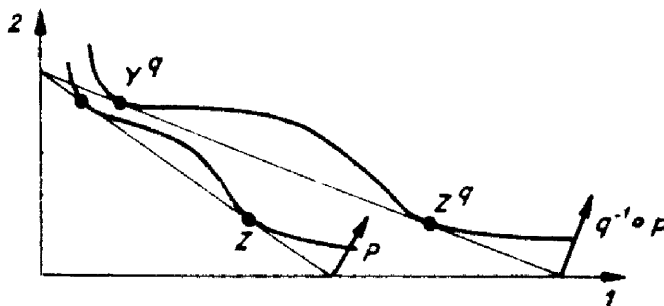

Figure 2

The indifference curves through y and z and through y^q and z^q correspond to $\geqslant$ and $\geqslant_q$, respectively. It can directly be concluded from Figure 2 that $q \circ \varphi(\geqslant, w, q \circ p) = \varphi(\geqslant_q, w, q^{-1} \circ p)$ or, equivalently, $q \circ \varphi(\geqslant, w, q \circ p) = \varphi(\geqslant_q, w, p)$. If one considers probabilities on $\mathscr{S}$ which are integrals of ergodic quasi-invariant probabilities on the orbits $S \circ \geqslant$ with respect to some probability on $\mathscr{S}$, one gets for LEBESGUE almost every $w \in \mathbb{R}_+$ that demand, integrated over preferences, is a singleton. Integration over wealth by LEBESGUE-continuous probabilities yields demand sets which are singletons for every price system.

The diversification of tastes, accomplished by the action of S, is obvious from Figure 2. Both households, $(\geqslant, w)$ and $(\geqslant_q, w)$, have demand sets which fail to be singletons, however at different price systems. At the price system p almost all $(\geqslant_q, w)$, for q near identity $\in S$, have single-valued demand. The key property is $q \circ \varphi(\geqslant, w, q \circ p) = \varphi(\geqslant_q, w, p)$, which asserts that one may analyze demand for varying preferences in an S-orbit at a fixed price system p by analyzing demand

for a fixed preference if prices vary in S. This "duality" property corresponds to the probability of replacing space average by time average in BIRKHOFF's Individual Ergodic Theorem. However here we are not concerned with equality of integrals of functions rather than with equality of integrals of cardinalities of set-valued mappings.

Let me conclude this section with the most simple example for a mean demand problem where a structure, as described above, yields uniqueness. We consider a continuum of consumers, say $(0, 1]$, which have to decide, whether to buy one unit of a certain commodity or not. Assume consumer t decides not to buy unless the price p of the commodity is smaller or equal that t. Accordingly, his (partial) demand function for this commodity is

$$f_t : (0, 1] \to \{0, 1\} : p \mapsto \begin{cases} 1 & \text{if } \quad p \le t \\ 0 & p > t. \end{cases}$$

Putting $f_t(t) = \{0, 1\}$ instead. makes f_t a correspondence, i.e. a set-valued mapping.

In this example there is enough diversification of demand behavior. Hence the mean demand relation $F : (0, 1] \to \mathbb{R}$ defined by $F(p) = \int_0^1 f_t(p) \, dt = 1 - p$ is a continuous function of the commodity price.

3. Some Basic Facts and Notation

Consider a binary relation $B \subset S \times X$, where S is some locally compact commutative group and X is an EUCLIDean space, say $X = \mathbb{R}^n$.

Assume that if for every $p \in S$ the section $B_p = \{x \in X \mid (p, x) \in B\}$ is compact. Moreover, *assume* that B is *upper hemi-continuous*, i.e., for every open set $G \subset X$ the *strong inverse* $\{p \in S \mid B_p \subset G\}$ is open in S.

Let T be some Polish space. Consider a real-valued function f on $T \times X$, whose sections $f_t = f(t, \cdot)$ are continuous functions on X.

Let χ be a specification of the HAAR measure on the BOREL σ-field $\mathcal{B}(S)$ of S.

Assume that for χ-almost every $p \in S$ the function $f(t, p, \cdot) := f(t, \cdot) \mid_{B_p}$ has at most one maximizer on B_p.

We are interested in conditions on a probability on the BOREL σ-field $\mathcal{B}(T)$ on T which imply that the mean of the individual maximizers is a singleton for *every* $p \in S$.

Denote by $\varphi(t, p)$ the set of maximizers for the function $f(t, p, \cdot)$, $t \in T$, $p \in S$. For every $t \in T$ the set (Graph $\varphi)_t = \{(p, x) \in S \times X \mid x \in \varphi(t, p)\}$ defines again an upper hemi-continuous relation (cf. HILDENBRAND (1974)).

Denote by $\# M$ the cardinality of a set M. We define now three actions of the group S on S, T and X.

$$o : S \times S \to S : (q, p) \mapsto q \circ p = p \circ q$$
$$a : S \times T \to T : (q, t) \mapsto t_q$$
$$b : S \times X \to X : (q, x) \mapsto x^q$$

The actions $o \times b$ of S on $S \times X$ and $o \times a \times b$ of S on $S \times T \times X$ are defined by

$$o \times b: S \times (S \times X) \to S \times X : (q, p, x) \to (q \circ p, x^q)$$

and

$$o \times a \times b: S \times (S \times T \times X) \to S \times T \times X : (q, p, t, x) \to (q \circ p, t_q, x^q).$$

Let us state now the following *duality condition* on the actions o, a, and b. Note that $(o \times a \times b)_q \equiv o \times a \times b(q, \cdot)$.

D: 1. $o \times b \ (S \times B) = B$

          2. $\forall q \in S: f \circ (o \times a \times b)_q = f$.

This means that 1) whenever $x \in B_p$ then $x^q \in B_{q \cdot p}$ and that 2) $f_t \mid_{B_p(x)} = f_{t_q} \mid_{B_{q \cdot p}(x^q)}$.

Next let us recall some facts on quasi-invariant measures.

Let μ be a measure on $(T, \mathcal{B}(T))$. The *measure class* $[\mu]$ of μ is the set of all measures on $(T, \mathcal{B}(T))$ having the same null sets as μ has.

A measure μ on $(T, \mathcal{B}(T))$ is *quasi-invariant* under the action of group S, if for every $q \in S$, one has $\mu_q = [\mu]$. Here μ_q is defined by $\mu_q(E) = \mu(E_q)$. E, $E_q \in \mathcal{B}(T)$, $E_q = \{t_q \mid t \in E\}$. μ is *invariant* under S if for all $q \in S$ one has $\mu_q = \mu$.

The action of S on T is *ergodic* with respect to μ resp. μ is ergodic with respect to S, if every set $M \in \mathcal{B}(T)$ which is invariant under S, i.e. $M = M_q \ \forall q \in S$, has either full measure or measure zero.

If S acts on T, there exists an (almost everywhere) unique partition of T into disjoint Borel subsets, the *orbits*. on which the action of S is *transitive*, i.e., the orbits are the smallest subsets of T which are invariant under the action of S.

If S acts on T, and μ is a measure on T, quasi-invariant under the action of S, then μ has an integral representation $\mu = \int \xi_t \mu'(dt)$. such that the measure ξ_t, $t \in T$, on $(T, \mathcal{B}(T))$ are concentrated on the orbits $t_S = \{t_q \mid q \in S\}$ and μ' is a measure on $(T, \sigma(S))$. Here $\sigma(S)$ is the finest sub-σ-algebra of $\mathcal{B}(T)$ containing the orbits t_S, $t \in T$, as atoms. Moreover, the action of S on t_S is ergodic with respect to ξ_t.

I conclude this section with the remark that any nonempty measure class contains a probability measure. For more details I refer to Mackey [1968].

4. Result

Proposition: *Let B and q be defined as in section 3. Assume $\#q(t, p) = 1$ for χ-almost every $p \in S$. Let μ be some probability on $(T, \mathcal{B}(T))$ which is quasi-invariant under the action a of S. Then under the duality assumption D one has for every $p \in S$:*

$$\# \int_T q(t, p) \ \mu(dt) \leq 1 .$$

If f is continuous then $\# \int_T q(t, p) \ \mu(dt) = 1$.

The integral for the correspondence, i.e. set-valued mapping, q is defined as

follows.

$$\int \varphi(t, p)\, \mu(\mathrm{d}t) = \{\int g(t, p)\, \mu(\mathrm{d}t) \mid g_i(\cdot, p) \in \mathfrak{L}^1(T, \mathfrak{B}(T), \mu),$$
$$i = 1, \ldots, n,\ g(t, p) \in \varphi(t, p)\ \mu-\text{a.e. in } T\} .$$

Note that $\int \varphi \mathrm{d}\mu$ is well-defined without any measurability or integrability assumption on φ. However, $\int \varphi \mathrm{d}\mu$, may be empty. The μ-integral of φ is nonempty if φ has integrable selections. For example $\int \varphi \mathrm{d}\mu \neq \emptyset$ if φ is *integrably bounded* (i.e. $\exists g = (g_1, \ldots, g_n)$: $T \to X$ such that $g_i \in \mathfrak{L}^1(T, \mathfrak{B}(T), \mu)$, $i = 1, \ldots, n$ and $-g \leq \varphi(\cdot, p) \leq \leq g$) and has a *measurable graph*. If f is continuous it follows from Theorem 3, p. 29 and Lemma 1, p. 55 in HILDENBRAND (1974) that $\int \varphi \mathrm{d}\mu \neq \emptyset$. Hence $\# \int \varphi \mathrm{d}\mu \geq 1$ in this case.

Our concern, however, is whether $\int \varphi \mathrm{d}\mu$ has *not more* than one element. To show that $\# \int_T \varphi(t, p)\, \mu(\mathrm{d}t) \leq 1$ for every $p \in S$ we do not need any assumption as to how f depends on $t \in T$.

Proof: Since μ admits the disintegration $\mu = \int \xi_t \mu'(\mathrm{d}t)$ we get

$$\int_T \varphi(t, p)\, \mu(\mathrm{d}t) = \int_T \int_T \varphi(t', p)\, \xi_t(\mathrm{d}t')\, \mu'(\mathrm{d}t) .$$

Therefore, it suffices to show that μ'-almost everywhere on T one has

$$\# \int_T \varphi(t', p)\, \xi_t(\mathrm{d}t') \leq 1 .$$

Choose some $t \in T$ and keep it fixed for the following considerations. Since ξ_t lives on the orbit t_S, we have

$$\int_T \varphi(t', p)\, \xi_t(\mathrm{d}t') = \int_{t_S} \varphi(t', p)\, \xi_t(\mathrm{d}t') .$$

Now the orbit t_S is homeomorphic to the factor space S/C_t via the map $h : t_S \to \to S/C_t : t_q \mapsto qC_t \equiv [q]_t$. Since t is fixed we drop the subscript t in $[q]_t$ for convenience and write $[q]$ instead of $[q]_t$. Denote $\xi_t \circ h^{-1}$ by $\bar{\xi}_t$. Then we have

$$\int_T \varphi(t', p)\, \xi_t(\mathrm{d}t') = \int_{S/C_t} \varphi(h^{-1}[q], p)\, \bar{\xi}_t(\mathrm{d}[q]) .$$

where $\bar{\xi}_t$ is ergodic and, in particular, quasi-invariant under the action of S on S/C_t. It remains to show that $\# \varphi(h^{-1}[q], p) = 1$, for every $p \in S$ and $\bar{\xi}_t$-almost everywhere on S/C_t.

Since S is locally compact and commutative it is unimodular. Hence, C_t is an invariant subgroup. Therefore, S/C_t is a topological group which is locally compact and commutative, hence unimodular, since S is so. As C_t is also closed it is unimodular, too (cf. NACHBIN (1965), p. 87). Accordingly, the Haar measure χ on S is the *homeomorphic product* of HAAR measures $\bar{\chi}_t$ and χ_t on S/C_t and C_t, respectively.

Due to assumption D we have $(x^q \in B_p) \Leftrightarrow (x \in B_{q^{-1} \circ p})$ and $f(t_q, p, x^q) = f(t, q^{-1} \circ \circ p, x)$, hence $\# \varphi(t_q, p) = \# \varphi(t, q^{-1} \circ p)$. As, by assumption, $\# \varphi(t, p) = 1$ for χ-almost every $p \in S$, we have $\# \varphi(t_q, p) = \# \varphi(t, q^{-1} \circ p) = 1$ for all $p \in S$ and χ-almost every $q \in S$. Since for $q \in C_t$, we have $\# \varphi(t, q^{-1} \circ p) = \# \varphi(t_q, p) = \# \varphi(t, p)$,

this implies that for $\bar{\chi}_t$-almost every $[q]_t \equiv [q] \in S/C_t'$ we have $\# q(h^{-1}[q], p) = 1$. By the uniqueness of HAAR measure the measures ξ_t and $\bar{\chi}_t$ must be equivalent. Therefore, $\# q(h^{-1}[q], p) = 1$ ξ_t-almost everywhere on S/C_t. ∎

5. Concluding Remarks

It can easily be seen from the proof that it suffices to assume, μ to be absolutely continuous with respect to a quasi-invariant measure rather than to be quasi-invariant itself. In particular, this allows a purely local reasoning, where some small neighborhood of the identity of the acting group plays the role of the group in determining the distribution on T. The class of so characterized probabilities μ reflects the idea, that observation of members of $t \in T$, in the model under consideration, can only be expected up to a certain degree of precision. In a certain neighborhood of the "true" t all t' are "almost equally likely" to be observed. This lack of precision in observation already suffices to yield uniqueness of mean maximizers.

Accordingly non-uniqueness of mean maximizers results only from an unduely sharp interpretation of the observed element t.

Clearly the space X which we assumed to be EUCLIDean may be any space on which a DANIEL integral exists. For example X may be a BANACH or only a locally convex topological vector space. The reasoning of the proof would not be affected at all.

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Zusammenfassung

Es ist ein Problem der ökonomischen allgemeinen Gleichgewichtstheorie nachzuweisen, daß die Marktnachfrage eine Funktion der Preise ist, obwohl individuelle Konsumenten bei manchen Preisen mengenwertige Nachfragen haben können. Die spezielle Struktur des ökonomischen Modells ermöglicht eine Lösung mittels eines Ergodensatzes. Diese Struktur kann in rein mathematischen Termen ohne Bezug zum ökonomischen Kontext formuliert werden.

Резюме

В настоящей работе применяются известные и хорошо обоснованные факты и методы эргодической теории для доказательства одной теоремы единственности максимизации в среднем. Это проблема экономической теории. Как слествие настоящего результата получается однозначность и непрерывность спроса в зависимости от цен.

Высказыванию и доказательству теоремы предшествует подробная экономическая мотивировка.