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Strong Coupling $1/N_c$ Expansion in the Gluonic Sector of Lattice Quantum Chromodynamics

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Abstract. The vacuum state of gluonic quantum chromodynamics on the lattice is determined up to fifth order in a $1/N_c$ expansion (N_c = number of colours). The vacuum expectation value of the gluon field squared $F_{a\mu\nu}F_a^{\mu\nu}$ is deduced. The quark–antiquark and gluon–gluon potential is calculated in the same limit up to the $1/N_c^3$ order.

1. Introduction

The confidence in quantum chromodynamics (QCD) as a theory of strong interactions is based at present mainly on the good agreement of its perturbative ("asymptotic freedom") predictions with experiments. Much remains to be understood, however, concerning genuine strong interaction aspects like the hadronic spectrum ("confinement"). Among the methods proposed to investigate the strong forces in QCD the strong coupling expansion [1, 2], the semi-classical methods via instantons [3–5] and the $1/N_c$ expansion [6–8] are receiving the most attention.

A widespread opinion among theorists is that the confinement phenomenon prohibiting the appearance of free quarks and gluons is due to the complicated non-perturbative vacuum structure of QCD. In addition, it is also mostly believed that confinement follows from the properties of the gluonic (pure Yang–Mills) sector. (Although the breaking of chiral symmetry requires a non-trivial quark–antiquark content of the vacuum, too.) The simplest manifestation of the non-trivial gluonic vacuum structure is the non-zero vacuum expectation value of the gluon field (squared):

$$\langle v|F_a(o)_{\mu\nu}F_a(o)^{\mu\nu}|v\rangle = -4\langle v|L_{YM}(o)|v\rangle \cong 0.48\,g^{-2}\,\mathrm{GeV}^4. \quad (1.1)$$

Here $|v\rangle$ is the vacuum, $F_a(x)_{\mu\nu}$ the gluon field strength tensor, $L_{YM}(x)$ denotes the Lagrangian density of the pure Yang–Mills (gluon) field and g is the coupling constant. The numerical value given in (1.1) has been estimated in [9] comparing the sum rules for charm production with experimental data in e^+e^- annihilation.

Many qualitative features of the hadronic spectrum are reasonably well described by the dual string model (for references see [10]). Therefore, there has to be some relation between QCD and strings. Indeed, within the strong coupling approach to QCD on the lattice, Wilson [1] established the linear rise of the quark–antiquark potential (the "area law" for Wilson loop). This is certainly an important effect coming from the "elastic" string between quarks, but otherwise the expansion in $1/g^4$ does not seem to be a string expansion. However, as it was shown by 't Hooft [4], the large N_c limit of QCD exhibits also a diagrammatic structure with topologies reminescent of the dual string model. Moreover, as it was shown recently by Bars and Green [8], if both $N_c \to \infty$ and $g^2N_c \to \infty$ then the meson propagator is the same as in a string model with quarks at the ends. This demonstrates the dynamical equivalence of strings and QCD for large N_c and g^2N_c. These conclusions are strengthened by recent studies of the physical ($N_c = 3$) theory with the help of certain string variables showing the close relationship between classical chromodynamics and the quantum mechanics of relativistic strings [11, 12]. (For further references see [13]).

All that shows the interest in the study of the large N_c limit. The drawback of the $1/N_c$ expansion is, however, as observed by de Witt and 't Hooft [14], that certain quantities show an anomalous non-analytic or even singular behaviour as a function of N_c. These "anomalies" (together with the problem of baryons) may prevent the $1/N_c$ expansion from being actually a method of solution for the physical ($N_c = 3$) theory. In such a case the large N_c QCD can be considered to be a model only. Nevertheless, fundamental problems such as the quark confinement mechanism are likely to be independent of N_c.

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Besides, 1/3 may be small enough so that the spectrum obtained by the $1/N_c$ expansion will presumably resemble the physical spectrum.

In the present paper the vacuum state of the pure gluonic QCD on the lattice is investigated in the $1/N_c$ expansion keeping the bare coupling constant g fixed (Sect. IV). The static potential between external charges is determined in the quark–antiquark and gluon–gluon case (Section V). First, however, the basic ingredients of Hamiltonian QCD on the lattice are summarized in order to introduce our notations and conventions (Sect. II). In Sect. III the properties of the Hilbert space of physical states necessary for our calculations are derived. Section VI contains the summary and concluding remarks.

II. The Lattice Hamiltonian in QCD

In this section we shortly review the basic elements of lattice QCD. The Hamiltonian formulation of lattice QCD was introduced in [2]. It is defined on a cubic spatial lattice (with lattice constant a), leaving time continuous. In what follows, we shall denote lattice points by "middle" letters like $i, j, k, l, \ldots$. The directions of the links connecting two neighbouring lattice points will be denoted by "late" letters $r, s, t, \ldots$ (these can have values $\pm x, \pm y, \pm z$). A link is then given as $\ell \mathbf{r} k \equiv \ell \mathbf{r} \equiv \mathbf{r} k$ (see Fig. 1a). The short notation for a plaquette (in fact, plaquette boundary) starting from the point k first in the direction r and then s (that is $k - \mathbf{s} - \mathbf{r}\mathbf{s}\mathbf{r}k$) will be $k[rs]$ (Fig. 1b). In summations over plaquettes $[rs]$ takes on six different values with r, s positive. We shall assume periodic boundary conditions with respect to a cube of sides aN (N units on the lattice), therefore there are altogether $P \equiv 6N^3$ different plaquettes.

To every link on the lattice there is associated a "string operator" $U[\ell \mathbf{r} k]$ transforming like a $N_c \times N_c$ matrix under colour SU(N_c). That is, under a local gauge transformation U_k (in the point k) we have

$$U'[\ell \mathbf{r} k] = U_\ell^{-1} U[\ell \mathbf{r} k] U_k. \tag{2.1}$$

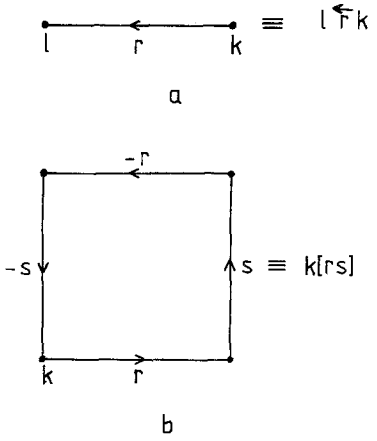

Fig. 1.a. A link on the lattice, **b** a plaquette

A curve on the lattice is a sequence of joined links. For instance, a closed curve starting and ending in the point i and consisting of $n+1$ links is

$$i \mathbf{r}_n i_n \mathbf{r}_{n-1} \ldots \mathbf{r}_1 i_1 \mathbf{r}_0 i \equiv i \leftarrow i. \tag{2.2}$$

Later on, we shall also use the notations $i \equiv i_0 \equiv i_{n+1}$ and $i_j \equiv i_{j+n+1}$, $r_j \equiv r_{j+n+1}$. The string operator belonging to this curve is the product of the string operators belonging to the links:

$$U[i \leftarrow i] = U[i \mathbf{r}_n i_n] U[i_n \mathbf{r}_{n-1} i_{n-1}] \ldots U[i_1 \mathbf{r}_0 i]. \tag{2.3}$$

The trace of it (over SU(N_c) indices) is the gauge invariant "Wilson-loop operator" (gluon loop operator):

$$A(i \leftarrow i) = \mathrm{Tr}\{U[i \leftarrow i]\}. \tag{2.4}$$

In particular, the string operator of the plaquette $k[rs]$ is $U[k[rs]]$ and the corresponding gluon loop operator is

$$A(k[rs]) = \mathrm{Tr}\{U[k[rs]]\}. \tag{2.5}$$

The Hamilton-operator H in the $A_0 = 0$ ("temporal") gauge is the following

$$H = H_E - H_A;$$
$$H_E = \frac{g^2}{4a} \sum_{k,r,b} E_{b,r}^{(k)} E_{b,r}^{(k)}; \tag{2.6}$$
$$H_A = \frac{1}{ag^2} \sum_{k[rs]} A(k[rs]).$$

Here $E_{b,r}^{(k)}$ is the colour electric flux operator in the point k and direction r. (The unusual factor 1/4 in front of the electric term H_E comes from our convention to sum r over both positive and negative directions.) b is the colour index taking on $N_c^2 - 1$ different values ("early" letters like $b, c, d, \ldots$ will be reserved for this). Note that following [15] we define the electric flux operator on lattice sites rather than on links. (The difference between the two choices vanishes for $a \to o$. For finite a we find our choice more natural.)

The commutation relation between the string operators and the electric flux operator is [15]:

$$[U[\ell \mathbf{r} k], E_{b,s}^{(i)}] = \frac{1}{2}(\delta_{rs} - \delta_{-rs}) \cdot \left\{ \delta_{ki} U[\ell \mathbf{r} k] \frac{\lambda_b}{2} + \delta_{\ell i} \frac{\lambda_b}{2} U[\ell \mathbf{r} k] \right\}. \tag{2.7}$$

λ_b ($b = 1, 2, \ldots, N_c^2$-1) are the generalized Pauli-matrices (or Gell-Mann matrices) for SU(N_c) normalized in such a way that

$$\mathrm{Tr}\{\lambda_b \lambda_c\} = 2\delta_{bc} \qquad \mathrm{Tr}\{\lambda_b\} = 0;$$
$$\sum_{b=1}^{N_c^2-1} \left(\frac{\lambda_b}{2}\right)_{\alpha\beta} \left(\frac{\lambda_b}{2}\right)_{\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{N_c}\delta_{\alpha\beta}\delta_{\gamma\delta}\right). \tag{2.8}$$

We shall need the value of the quadratic Casimir operator of $SU(N_c)$ in the fundamental (N_c dimensional) representation, which is

$$C_2 \equiv \sum_{b=1}^{N_c^2-1} \frac{\lambda_b}{2}\frac{\lambda_b}{2} = \frac{N_c}{2} - \frac{1}{2N_c}. \tag{2.9}$$

III. The Hilbert Space of Physical States

In this section we shall first define, following [1, 2] (see also [16]), the Hilbert space of states where the operators introduced in the previous section act. After this is done we have a well defined theory and the rest is in principle straightforward: one has to find the eigenstates (and eigenvalues) of the Hamilton operator. In practice this is, of course, a formidable task, therefore one has to find some appropriate approximation scheme. In the following sections the approximation will be to take N_c very large. What we need, therefore, is to calculate matrix elements of our operators for any value of N_c. The basic relations necessary for this will be derived in the second part of this section.

The Hilbert space of states is defined as a direct product of Hilbert spaces belonging to the individual links on the lattice. The "coordinates" of the gluon field are elements of the colour gauge group $SU(N_c)$ on every link. The wave functions depend on these "coordinates" and the scalar product is given by the invariant measure dU on $SU(N_c)$ (normalized to 1):

$$\langle \psi_2 | \psi_1 \rangle = \int_{SU(N_c)} dU_1 dU_2 \ldots \psi_2^*(U_1, U_2, \ldots) \cdot \psi_1(U_1, U_2, \ldots). \tag{3.1}$$

(Here $U_1, U_2, \ldots$ stand for the $SU(N_c)$ elements on the different links.) The string operator $U[\ell \mathbf{r} k]$ acts in this Hilbert space as multiplying the wave functions by the $SU(N_c)$ element belonging to the link $\ell \mathbf{r} k$. The string operator of the oppositely oriented link $U[\ell - \mathbf{r} k]$ acts multiplying by the adjoint (equal to the inverse of the $SU(N_c)$ element), therefore we have

$$\sum_{\omega} U[\ell \mathbf{r} k]_{\rho\omega} U[\ell - \mathbf{r} k]_{\omega\sigma} = \delta_{\rho\sigma}. \tag{3.2}$$

The "mathematical" vacuum state $|0\rangle$ (not equal to the "physical" vacuum $|v\rangle$ defined as the lowest energy state) has, by definition the wave function $\equiv 1$. The operator $E_{b,r}^{(k)}$ of the colour electric flux annihilates the mathematical vacuum:

$$E_{b,r}^{(k)} |0\rangle = 0. \tag{3.3}$$

This relation together with the commutator in (2.7) defines $E_{b,r}^{(k)}$ uniquely.

Now we are in the position to calculate matrix elements of our operators using well known properties of the invariant integration on unitary groups (see e.g. [17, 18]). A simple example is the following:

$$\langle 0 | U[\ell - \mathbf{r} k]_{\tau\sigma} U[\ell \mathbf{r} k]_{\rho\omega} | 0 \rangle \equiv T_{\rho\sigma\tau\omega}$$

$$= \int dU\, U_{\tau\sigma}^{\dagger} U_{\rho\omega} = \frac{1}{N_c} \delta_{\rho\sigma} \delta_{\tau\omega}. \tag{3.4}$$

(Here the written out integration is over the $SU(N_c)$ belonging to the link $\ell \mathbf{r} k$. The integrations over the other links give only a trivial factor 1.) This result can also be derived from $\langle 0|0\rangle = 1$ and the invariance of the measure $dU = d(U_\ell^{-1} U U_k^{-1})$ (for arbitrary U_ℓ) implying that the matrix element in (3.4) is an invariant tensor:

$$T_{\rho\sigma\tau\omega} = U_{\ell\rho\rho'} U_{k\tau\tau'}^{\dagger} T_{\rho'\sigma'\tau'\omega'} U_{\ell\sigma'\sigma}^{\dagger} U_{k\omega'\omega}. \tag{3.5}$$

In the same way we can deduce from (3.4) that

$$\begin{aligned} &\langle 0 | U[\ell - \mathbf{r} k]_{\tau_1\sigma_1} U[\ell - \mathbf{r} k]_{\tau_2\sigma_2} U[\ell \mathbf{r} k]_{\rho_1\omega_1} \\ &\cdot U[\ell \mathbf{r} k]_{\rho_2\omega_2} | 0 \rangle \\ &= \frac{1}{N_c^2 - 1} (\delta_{\rho_1\sigma_1}\delta_{\tau_1\omega_1}\delta_{\rho_2\sigma_2}\delta_{\tau_2\omega_2} + \delta_{\rho_1\sigma_2}\delta_{\tau_2\omega_1}\delta_{\rho_2\sigma_1}\delta_{\tau_1\omega_2}) \\ &- \frac{1}{N_c(N_c^2-1)} (\delta_{\rho_1\sigma_1}\delta_{\tau_2\omega_1}\delta_{\rho_2\sigma_2}\delta_{\tau_1\omega_2} \\ &+ \delta_{\rho_1\sigma_2}\delta_{\tau_1\omega_1}\delta_{\rho_2\sigma_1}\delta_{\tau_2\omega_2}). \end{aligned} \tag{3.6}$$

More generally, one can determine recursively the matrix element of a larger number of string operators. The result can be written in the following way:

$$\begin{aligned} &\langle 0 | U[\ell - \mathbf{r} k]_{\tau_1\sigma_1} \ldots U[\ell - \mathbf{r} k]_{\tau_K\sigma_K} \\ &\cdot U[\ell \mathbf{r} k]_{\rho_1\omega_1} \ldots U[\ell \mathbf{r} k]_{\rho_K\omega_K} | 0 \rangle \\ &= \sum_{\pi(!)K} \sum_{\bar{\pi}(!)K} A[P_K(\bar{\pi})] \prod_{i=1}^{K} \delta_{\rho_i\sigma_{\pi(i)}} \delta_{\tau_{\bar{\pi}(\pi(i))}\omega_i}. \end{aligned} \tag{3.7}$$

Here $\sum_{\pi(!)K}$ denotes the summation over the permutations $\pi(1), \pi(2), \ldots, \pi(K)$ of the numbers $1, 2, \ldots, K$. The coefficient A depends on the partition P_K of the number K (into positive integer numbers) given by the cyclic structure of $\bar{\pi}$. The values of a few A's are (including the case $K = 2$ in (3.6)):

$K = 2$:

$$\begin{aligned} A[1,1] &= (N_c^2 - 1)^{-1} \\ A[2] &= -[N_c(N_c^2 - 1)]^{-1}; \end{aligned}$$

$K = 3$:

$$\begin{aligned} A[1,1,1] &= (N_c^2 - 1)[N_c(N_c^2 - 1)(N_c^2 - 4)]^{-1} \\ A[2,1] &= -[(N_c^2 - 1)(N_c^2 - 4)]^{-1} \\ A[3] &= 2[N_c(N_c^2 - 1)(N_c^2 - 4)]^{-1}; \end{aligned}$$

$K = 4$:

$$\begin{aligned} A[1,1,1,1] &= (N_c^4 - 8N_c^2 + 6)[N_c^2(N_c^2 - 1) \\ &\quad \cdot (N_c^2 - 4)(N_c^2 - 9)]^{-1} \\ A[2,1,1] &= -[N_c(N_c^2 - 1)(N_c^2 - 9)]^{-1} \\ A[2,2] &= (N_c^2 + 6)[N_c^2(N_c^2 - 1) \\ &\quad \cdot (N_c^2 - 4)(N_c^2 - 9)]^{-1} \end{aligned}$$

$$A[3,1] = (2N_c^2-3)[N_c^2(N_c^2-1)\cdot(N_c^2-4)(N_c^2-9)]^{-1}$$
$$A[4] = -5[N_c(N_c^2-1)\cdot(N_c^2-4)(N_c^2-9)]^{-1}. \quad (3.8)$$

These relations hold, of course, only for $N_c \geqq K$. The poles for $N_c < K$ are the sources for the "anomalies" observed by de Wit and 't Hooft[14], which appear because of the vanishing of the antisymmetric tensors of rank $K > N_c$ for $SU(N_c)$. The consequence of these anomalies is that the expansion in $1/N_c$ does not reproduce, in general, the results for $N_c = 3$ (see also below).

The antisymmetric tensor of rank N_c is a scalar in $SU(N_c)$. (This coupling is important for the construction of colour singlet baryon wave functions). This implies that there are non-vanishing matrix elements where the number of string operators in the two directions on a link differ by an integer multiple of N_c (not only equal numbers in both directions like e.g. in (3.7)). An example is, for $N_c = 3$:

$$\langle 0|U[\ell \mathbf{r} k]_{\rho_1\omega_1} U[\ell \mathbf{r} k]_{\rho_2\omega_2} U[\ell \mathbf{r} k]_{\rho_3\omega_3}|0\rangle = \tfrac{1}{6}\varepsilon_{\rho_1\rho_2\rho_3}\varepsilon_{\omega_1\omega_2\omega_3}. \quad (3.9)$$

This is, however, not independent from relations like in Eq. (3.7) (equal numbers of $U[\ell - \mathbf{r}k]$ and $U[\ell \mathbf{r} k]$). We have namely, again for $N_c = 3$:

$$U[\ell - \mathbf{r}k]_{\tau\sigma} = \tfrac{1}{2}\varepsilon_{\rho_1\rho_2\sigma}\varepsilon_{\omega_1\omega_2\tau} U[\ell \mathbf{r} k]_{\rho_1\omega_1} U[\ell \mathbf{r} k]_{\rho_2\omega_2}. \quad (3.10)$$

Using relations like (3.7), (3.9) one can calculate, in principle, every matrix element of the Hamilton operator (2.6). This way is, however, in most cases rather cumbersome because of the large number of indices involved which have to be contracted, anyway. The *physical* states in the Hilbert space of states are, namely, gauge invariant (with respect to time-independent gauge transformations in the temporal gauge used here). Every such state can be created [1, 2, 16] by the action of the gauge invariant gluon loop operators on the mathematical vacuum. That is, we have to consider only states like

$$A(i \leftarrow i)A(j \leftarrow j)\ldots A(k \leftarrow k)|0\rangle. \quad (3.11)$$

In such states, the colour indices of the string operators for links are contracted with each other, therefore, trace-like expressions appear. Useful tools for handling the group integrals of these expressions are given by the theory of group characters [17]. The character χ_R of the representation R is the trace of the representant matrices:

$$\chi_R(U) = \mathrm{Tr}\{R(U)\}. \quad (3.12)$$

(U is the group element). A fundamental relation for the characters is the following [17, 18]:

$$\int dU\, \chi_R(U)^* \chi_R(U) = \sum_I m_I^2. \quad (3.13)$$

Here m_I is the multiplicity of the irreducible representation I in the representation R.

In order to illustrate the use of (3.13) let us consider the (non-self-intersecting) curve $i \leftarrow i$ in (2.2). Let us denote the $SU(N_c)$ element belonging to the links $i_1\mathbf{r}_0 i, i_2\mathbf{r}_1 i_1, \ldots, i\mathbf{r}_n i_n$, respectively, by $U_0, U_1, \ldots, U_n$. If the oppositely oriented curve is

$$i \rightarrow i \equiv i - \mathbf{r}_n i_n - \mathbf{r}_{n-1} \ldots - \mathbf{r}_1 i_1 - \mathbf{r}_0 i, \quad (3.14)$$

then we have

$$\langle 0|A(i \rightarrow i)A(i \leftarrow i)|0\rangle = \int dU_0 \ldots dU_n\, \mathrm{Tr}(U_0 \ldots U_n)\mathrm{Tr}(U_n^\dagger \ldots U_0^\dagger) = \int dU_0\, \mathrm{Tr}(U_0)\mathrm{Tr}(U_0)^* = 1. \quad (3.15)$$

Here the invariance of the measure dU_0 and the normalization relation $\int dU_j = 1$ were used. The result in (3.15) can, of course be easily obtained also from (3.4), but in more complicated situations the use of (3.13) is much simpler. For instance, the well known methods [17, 18] of the reduction of direct product representations give (in the case $\kappa, \lambda < N_c$ and $\kappa = \lambda = N_c$):

$$\langle 0|A(i \rightarrow i)^\kappa A(i \leftarrow i)^\lambda|0\rangle = \delta_{\kappa\lambda}\kappa!. \quad (3.16)$$

In more general cases this relation does not hold. For $N_c = 3$ we have e.g.

$$\langle 0|A(i \leftarrow i)^3|0\rangle = 1;\ \langle 0|A(i \rightarrow i)A(i \leftarrow i)^4|0\rangle = 3;$$
$$\langle 0|A(i \rightarrow i)^4 A(i \leftarrow i)^4|0\rangle = 23;$$
$$\langle 0|A(i \rightarrow i)^5 A(i \leftarrow i)^5|0\rangle = 103. \quad (3.17)$$

Comparing this to (3.16) we immediately see the non-analytic behaviour as a function of N_c. These are examples of the anomalies observed by de Wit and 't Hooft [14].

Besides the matrix elements like in (3.16) we shall also need those containing gluon loop operators which belong to closed curves obtained by going around the curve $i \leftarrow i$ several times. Let us denote by A_α^+ the gluon loop operator of the closed curve going around the curve $i \leftarrow i$ α-times (in the positive direction). The same for negative direction is A_α^-. In particular, for $\alpha = 1, 2$ this means

$$A_1^+ = A(i \leftarrow i);\quad A_1^- = A(i \rightarrow i);$$
$$A_2^+ = A(i \leftarrow i \leftarrow i);\ A_2^- = A(i \rightarrow i \rightarrow i). \quad (3.18)$$

In this notation we have, for instance,

$$\langle 0|A_2^- A_2^+|0\rangle = \int dU_0 \ldots dU_n \mathrm{Tr}(U_0 \ldots U_n U_0 \ldots U_n)\cdot \mathrm{Tr}(U_n^\dagger \ldots U_0^\dagger U_n^\dagger \ldots U_0^\dagger) = \int dU_0\, \mathrm{Tr}(U_0 U_0)\mathrm{Tr}(U_0 U_0)^*. \quad (3.19)$$

Let us denote the character of the irreducible representation with the Young-pattern $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by $\chi_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(U)$. Using the explicit expression [17] of the character $\chi_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}$ and the rules for the direct product of representations we obtain:

$$\mathrm{Tr}(UU) = \frac{\chi_{(n+1,n-2,n-3,\ldots,1)}(U)}{\chi_{(n-1,n-2,\ldots,1)}(U)} = \chi_{(2)}(U) - \chi_{(1,1)}(U). \tag{3.20}$$

Hence from (3.19) and (3.13) we obtain:

$$\langle 0|A_2^- A_2^+|0\rangle = 2. \tag{3.21}$$

The general result for

$$\sum_i i\kappa_i \equiv \kappa \leqq N_c, \tag{3.22}$$

is the following

$$\langle 0|\prod_i A_i^{-\kappa_i} \prod_j A_j^{+\lambda_j}|0\rangle = \prod_i \delta_{\kappa_i\lambda_i} i^{\kappa_i}\kappa_i!. \tag{3.23}$$

Up to now we considered only a single curve $i \leftarrow i$. Of course, in the general case the gluon loop operators of several different curves appear in the matrix elements. For the product of the gluon loop operators of disjoint curves the matrix element is simply the product over the different curves. Moreover, if there is a link in the configuration where only a single curve is running (may be several times), then one can choose the variable of this link as U_0 and the ransformations like in (3.15), (3.19) can be performed without influencing the other curves. As a consequence, the contribution of this curve can be factored out. This procedure does not help, however, in a configuration like e.g. in Fig. 2. In this case the curves $i \leftarrow i$ and $j \leftarrow j$ have a common segment: $i_\gamma = j_\gamma$ for $\alpha \leqq \gamma \leqq \beta$ and we are considering ($\leftarrow$ means always a sequence in increasing order and $\rightarrow$ in decreasing order):

$$\begin{aligned}
&\langle 0|A(i\leftarrow i)A(j\leftarrow j)A(i_\beta = j_\beta \rightarrow j_\alpha = i_\alpha \leftarrow i_\beta)|0\rangle \\
&= \int dU_0 \ldots dU_n dV_0 \ldots dV_{\alpha-1} dV_\beta \ldots dV_m \\
&\cdot \mathrm{Tr}(U_n^\dagger \ldots U_0^\dagger) \\
&\cdot \mathrm{Tr}(V_0 \ldots V_{\alpha-1} U_\alpha \ldots U_{\beta-1} V_\beta \ldots V_m) \\
&\cdot \mathrm{Tr}(U_\beta \ldots U_n U_0 \ldots U_{\alpha-1} V_{\alpha\ 1}^\dagger \ldots V_0^\dagger V_m^\dagger \ldots V_\beta^\dagger) \\
&= \int dU dV \mathrm{Tr}(U^\dagger)\mathrm{Tr}(V)\mathrm{Tr}(UV^\dagger) = \frac{1}{N_c}.
\end{aligned} \tag{3.24}$$

The last equality follows from

$$\begin{aligned}
\mathrm{Tr}(UV^\dagger) &= \frac{1}{N_c}\mathrm{Tr}(U)\mathrm{Tr}(V^\dagger) \\
&+ \frac{1}{2}\sum_{b=1}^{N_c^2-1} \mathrm{Tr}(U\lambda_b)\mathrm{Tr}(\lambda_b V^\dagger);
\end{aligned}$$

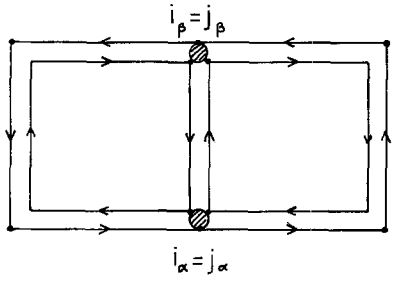

Fig.2. A non-factorizable configuration with three curves

$$\int dU\, \mathrm{Tr}(U)^* \mathrm{Tr}(U\lambda_b) = 0. \tag{3.25}$$

These relations are sufficient to calculate all the matrix elements we need in the following sections.

IV. The Vacuum State

In order to diagonalize the Hamiltonian H in (2.6) and to find the lowest energy state (physical vacuum) at least approximately one has to find some expansion scheme allowing for a perturbative calculation. In the present paper we consider the limit when the number of colours (N_c) is large.

First we have to determine the action of the Hamiltonian on the Hilbert space of states. The general form of the physical states is given by Eq. (3.11): they are created from the "mathematical" vacuum $|0\rangle$ by an arbitrary number of gluon loop operators $A(i \leftarrow i)$ along arbitrary (closed) curves $i \leftarrow i$. The action of the colour magnetic part H_A of the Hamiltonian is, therefore, rather simple: it adds gluon loops along the plaquette boundaries in the lattice.

The action of the colour electric part H_E follows from the commutation relation (2.7) and from the fact that according to (3.3) the colour electric flux operator annihilates the "mathematical" vacuum. Using the identity

$$[E^2, U] = [E,[E,U]] + 2[E,U]E, \tag{4.1}$$

and the commutators

$$\begin{aligned}
&\sum_{b,s}[E_{b,s}^{(\ell)}, [E_{b,s}^{(\ell)}, U[\mathbf{r}'\ell]_{\alpha\beta}]] = \frac{1}{2}C_2 U[\mathbf{r}'\ell]_{\alpha\beta}; \\
&\sum_{b,s}[E_{b,s}^{(\ell)}, [E_{b,s}^{(\ell)}, U[\ell\mathbf{r}]_{\alpha\beta}]] = \frac{1}{2}C_2 U[\ell\mathbf{r}]_{\alpha\beta}; \\
&\sum_{b,s}[E_{b,s}^{(\ell)}, [E_{b,s}^{(\ell)}, U[\mathbf{r}'\ell]U[\ell\mathbf{r}]]] \\
&= (1+\delta_{rr'})C_2 \cdot U[\mathbf{r}'\ell]U[\ell\mathbf{r}].
\end{aligned} \tag{4.2}$$

one can obtain the following rule: the action of H_E on a state with some gluon loop configuration gives

$$\frac{g^2}{8a}\left\{4C_2\Sigma\lambda(\text{o.c.}) + \Sigma\delta\left(\text{s.c.} - \frac{1}{N_c}\text{o.c.}\right)\right\}. \tag{4.3}$$

Here C_2 is the value of the quadratic Casimir operator given in (2.9). $\Sigma\lambda$ is the sum of the "length of arc" of the curves in the configuration defined for the curve $i \leftarrow i$ in (2.2) like

$$\lambda = \frac{1}{2}\sum_{j=0}^{n}(1+\delta_{r_j r_{j+1}}). \tag{4.4}$$

(o.c.) in (4.3) means the "original configuration" of the curves of the gluon loop operators. The second term in (4.3) is present only if there are points on the lattice belonging to more than one curve (or to one

curve more than once). The sum in the second term is over such points and over the different pairs of curves in every such point (if there are more than two curves). The "direction factor" δ depends on the directions of the two curves in question before and after the point in question. If before the point the directions on the two curves are, respectively, r_1 and r_2 and after the point r'_1 and r'_2, then we have:

$$\delta = \delta_{r_1 r_2} - \delta_{-r_1 r_2} + \delta_{r'_1 r_2} - \delta_{-r'_1 r_2} + \delta_{r_1 r'_2} - \delta_{-r_1 r'_2} + \delta_{r'_1 r'_2} - \delta_{-r'_1 r'_2}. \tag{4.5}$$

(s.c.) in (4.3) stands for the "switched over" configuration when instead of r_1, r'_1 and r_2, r'_2 the two curves in the point are r_1, r'_2 and r_2, r'_1. This is illustrated in a simple case by Fig. 3.

It is interesting to note that the "length of arc" (in lattice units) λ defined in (4.4) is not simply the number of links on the curve. It depends also on the direction of the links. This is the consequence of our choice for associating the electric flux operator to the points (rather than to links). More specifically, this follows from the commutation relation (2.7).

In order to have a well defined eigenvalue problem we have to impose also some boundary conditions. As usual, we take a cube of sides aN (N links) and identify each face with its opposite ("periodic boundary conditions"). In this cube there are altogether

$$P = 6N^3 = \sum_{k[rs]} 1 \tag{4.6}$$

different plaquette boundaries (if the two possible orientations are counted separately). This periodicity is, of course, completely unphysical and it serves only to define the mathematical problem properly. The physical quantities (like e.g. energy density etc.) cannot depend on P.

It can be seen from (4.3) that the colour electric energy of a gluon string configuration is proportional to the Casimir operator C_2. (Actually C_2 in (2.9) is the value of the Casimir operator in the fundamental representation, but using the results of the previous section it can be shown that for curves like in (3.18) running around some closed curve several times (4.3) reproduces the value of the Casimir operator in the higher representations). According to (2.9) the value

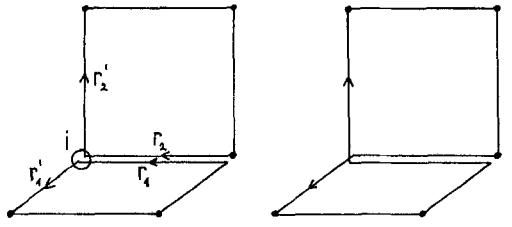

Fig. 3. An example of the "original" (a) and "switched over" (b) configuration in the point i with "direction factor" $\delta = 1$ (because of $r_1 = r_2$)

of C_2 is proportional to N_c, therefore in the limit $N_c \to \infty$ (g fixed) the colour electric energy H_E dominates: $H_E \gg H_A$. (The value of P in (4.6) is also fixed when $N_c \to \infty$). It can be seen from (4.3) that the dominant part of H_E proportional to N_c is very simple. Choosing it as the unperturbed Hamiltonian (H_0) the rest of H can be considered for $N_c \to \infty$ as a perturbation (V). According to (2.6), (4.3) V consists of two pieces: the second part of the colour electric energy operator in (4.3) "switching over" the string operators which have common points and the colour magnetic piece producing gluon loops around the plaquettes. This defines the $1/N_c$ expansion we are considering here. Note that H_A is proportional to g^{-2}, therefore in the limit [6] $N_c g^2 =$ fixed, $N_c \to \infty$ H_A is dominating. Our $N_c \to \infty$ limit is closer to the one taken in [8] where both N_c and $g^2 N_c$ are assumed to be large. The colour electric energy H_E is dominating also in the "strong coupling" limit [1, 2] $g \to \infty$ ($N_c = 3$). Our unperturbed Hamiltonian H_0 is, however, different from H_E and, in addition, the structure of the whole Hilbert space of states is simpler because of $N_c \to \infty$ (see the previous section).

The Hamiltonian dynamics of the gauge field formulated in terms of the string operators [15] becomes also simple in the $N_c \to \infty$ limit. To the leading order (taking only H_0 instead of H) the theory is equivalent to the theory of free, non-interacting strings. The interaction given by V consists of a "switching over" of strings with common points and the creation of additional gluon loops along the plaquettes.

Due to the extreme simplicity of our "unperturbed" Hamiltonian H_0 the eigenstates and eigenvalues of H_0 are trivially known. Any state with a given gluon loop configuration is an eigenstate of H_0 and the eigenvalue is proportional to the sum of "length of arc" λ. The lowest eigenvalue is 0 and it belongs to the "mathematical" vacuum $|0\rangle$. The physical vacuum $|v\rangle$ is the state obtained from $|0\rangle$ by taking into account the $1/N_c$ perturbations. The vacuum has to be translationally and rotationally invariant, therefore in order to determine $|v\rangle$ it is enough to consider the translation- and rotation-invariant states only. The simplest such state is obtained from $|0\rangle$ by the application of the colour magnetic energy operator H_A:

$$|1\rangle = \sum_{k[rs]} A(k[rs])|0\rangle = ag^2 H_A|0\rangle. \tag{4.7}$$

The next state would be

$$|2\rangle = \sum_{k[rs]} \sum_{k'[r's']} A(k[rs])A(k'[r's'])|0\rangle = a^2 g^4 H_A^2|0\rangle, \tag{4.8}$$

but this is split up into several components by the action of the non-leading part of H_E considered as a perturbation. The decomposition is according to the number and δ-value of the common points of the two

plaquettes. (Note that this is again a translation- and rotation-invariant characterization.) Let us denote by $|0x0\rangle$ the part of the state $|2\rangle$ where the two plaquettes do not touch at all. Furthermore, let us define the states with twice the same plaquette (in the same and in the opposite orientation) like

$$|4\times 2\rangle = \sum_{k[rs]} A(k[rs])A(k[rs])|0\rangle;$$
$$|4x-2\rangle = \sum_{k[rs]} A(k[rs])A(k[sr])|0\rangle. \quad (4.9)$$

In the notation $|\mu x\nu\rangle\mu$ gives the number of common points and ν the corresponding value of δ in (4.5). (In these states the δ-values of all the common points are the same.) It is easy to see that the other possible values of $\mu x\nu$ are:

$$\mu x\nu = 2x\pm 0;\ 2x\pm 1;\ 1x\pm 2;\ 1x\pm 1. \quad (4.10)$$

In this notation the action of H_A on the states $|1\rangle$ is the following:

$$H_A|1\rangle = \frac{1}{ag^2}\sum_{\mu,\nu}|\mu x\nu\rangle. \quad (4.11)$$

The action of the operator H_E on the state $|0\rangle, |1\rangle$ and $|\mu x\nu\rangle$ follows from (4.3):

$$H_E|0\rangle = 0;$$
$$H_E|1\rangle = \frac{g^2}{a}C_2|1\rangle;$$
$$(\mu x\nu \neq 4x-2):\ H_E|\mu x\nu\rangle$$
$$= \frac{g^2}{a}\left[\left(2C_2 - \frac{\mu\nu}{8N_c}\right)|\mu x\nu\rangle + \frac{\mu\nu}{8}|\underline{\mu x\nu}\rangle\right];$$
$$H_E|4x-2\rangle$$
$$= \frac{g^2}{a}\left[\left(2C_2+\frac{1}{N_c}\right)|4x-2\rangle - PN_c|0\rangle\right]. \quad (4.12)$$

Here in the third line the state $|\underline{\mu x\nu}\rangle$ is the "switched over" configuration coming from $|\mu x\nu\rangle$. (The case $\mu x\nu = 2x1$ is illustrated in Fig. 3). The case $\mu x\nu = 4x-2$ is an exception because switching over two oppositely oriented coinciding loops gives $\mathrm{Tr}(1) = N_c$ times the vacuum.

The state $|\underline{\mu x\nu}\rangle$ is, however, not orthogonal to $|\mu x\nu\rangle$ as from (3.24) we have

$$(\mu x\nu \neq 0x0,\ 4x\pm 2):\ \langle \mu x\nu|\underline{\mu x\nu}\rangle = \frac{\xi(\mu,\nu)}{N_c}P, \quad (4.13)$$

where the numbers $\xi(\mu,\nu)$ are given by

$$\xi(2,\pm 0) = \xi(2,\pm 1) = 8;$$
$$\xi(2,\pm 1) = 16;\ \xi(1,\pm 1) = 32. \quad (4.14)$$

In addition to (4.13) we have

$$(\mu x\nu \neq 0x0,\ 4x\pm 2):\ \langle \mu x\nu|\mu x\nu\rangle$$
$$= \langle \underline{\mu x\nu}|\underline{\mu x\nu}\rangle = \xi(\mu,\nu)P. \quad (4.15)$$

Therefore, appropriate orthonormal states for the perturbative expansion are:

$$|0) = |0\rangle;$$
$$|1) = \frac{1}{\sqrt{P}}|1\rangle;$$
$$|0x0) = \frac{1}{\sqrt{2P^2-132P}}|0x0\rangle;$$
$$|4x2) = \frac{1}{\sqrt{2P}}|4x2\rangle;$$
$$|\underline{4x2}) = \frac{1}{\sqrt{2P}}|\underline{4x2}\rangle;$$
$$|4x-2) = \frac{1}{\sqrt{2P}}\{|4x-2\rangle - P|0\rangle\};$$
$$(\mu x\nu \neq 0x0,\ 4x\pm 2):$$
$$|\underline{\mu x\nu}) = \frac{1}{\sqrt{\xi(\mu,\nu)P}}|\underline{\mu x\nu}\rangle;$$
$$|\mu x\nu) = \frac{1}{\sqrt{\xi(\mu,\nu)P(1-N_c^{-2})}}\left\{|\mu x\nu\rangle - \frac{1}{N_c}|\underline{\mu x\nu}\rangle\right\}. \quad (4.16)$$

These are the states containing the gluon loop operators of at most two plaquettes which occur up to the fifth order in the $1/N_c$ expansion. Going to higher orders requires the introduction of the three plaquette states. The construction of states goes in complete analogy to the two plaquette case. The only practical problem is that there is a large number of possibilities how the three plaquettes can touch each other. This results in a large number of states, therefore we do not consider higher orders in the present paper.

The standard time independent perturbation theory [19] gives for the vacuum energy density ε_v up to the 5th order:

$$\varepsilon_v = -12a^{-4}\left(\frac{1}{N_c g^6} + \frac{1}{N_c^3 g^6} + \frac{1}{N_c^5 g^6} - \frac{8{\cdot}43}{45N_c^5 g^{14}} + \ldots\right). \quad (4.17)$$

Deriving this result (and the ones below) we used the program "Schoonship" for algebraic manipulations. It is important that the energy density ε_v does not depend on the number of plaquettes in the volume of periodicity (P) as expected for physical quantities. It can be seen that the expansion goes, in fact, according to the powers of $1/N_c^2$. This is because of the extra powers of $1/N_c$ in the matrix elements of the "interaction operator" V. Note that the eigenvalues of H_0 are proportional to N_c, therefore the last two terms are actually already 6th order coming from the 5th order term in the perturbation series. (These terms can still be modified in the 6th order of the perturbation

theory involving matrix elements with three plaquette states).

The vacuum energy density (4.17) is strictly negative up to the term of order $1/N_c^3$. The energy is measured here relative to the "mathematical" vacuum $|0\rangle$, therefore it is not clear whether ε_v is measurable or not. It would be roughly measurable as the "bag constant" if inside the "hadronic bags" [20] the gluonic state would be $|0\rangle$.

The really measurable quantity is the vacuum expectation value of the gluon field squared (or Lagrangian density) given in (1.1). Up to 4th order in the perturbation theory we have

$$\frac{1}{(aN)^3}\langle v|H_E+H_A|v\rangle=\frac{36}{a^4N_c\,g^6}+\frac{36}{a^4N_c^3g^6}+\ldots. \tag{4.18}$$

The quantity $(aN)^{-3}(H_E+H_A)$ can almost be considered as the Lagrangian density of the gluon field. The only point we have to remember is that the magnetic energy operator H_M contains an additional constant terms besides $-H_A$ [1, 2]:

$$H_M=\frac{PN_c}{ag^2}-H_A. \tag{4.19}$$

The Lagrangian density is then

$$L_{YM}(0)=\frac{1}{(aN)^3}(H_E-H_M), \tag{4.20}$$

therefore (4.18–20) and (4.6) give

$$\begin{aligned}\langle v|L_{YM}(0)|v\rangle&=-\frac{6N_c}{a^4g^2}+\frac{36}{a^4g^6N_c}+\frac{36}{a^4g^6N_c^3}+\ldots\\&=-\frac{18}{a^4g^2}+\frac{12}{a^4g^6}+\frac{4}{3a^4g^6}+\ldots.\end{aligned} \tag{4.21}$$

Here in the second line we have put $N_c=3$. This expression will be compared to the slope of the $q\bar{q}$-potential and to the experimental value in the next section.

V. The Quark–Antiquark and Gluon–Gluon Potential

In order to calculate the non-relativistic potential between slowly moving heavy quarks one has to put external quark and antiquark sources in the pure gluonic quantum field considered up to now. The potential is the difference between the energy of the lowest states in the two cases (with and without external sources).

The gauge invariant states with an external quark and antiquark source contain, in addition to the string operators of closed loops, the string operator of some open curve connecting the quark and antiquark positions. Let us denote the positions of the quark and antiquark on the lattice by k and $\bar{k}$, respectively, and the open curve connecting them by $\bar{k}\underset{G}{\longleftarrow}k$ (G stands for the arbitrary shape of the curve). The corresponding string operator is $U[\bar{k}\underset{G}{\leftarrow}k]_{\alpha\beta}$ and the state with an arbitrary number of additional gluon loop operators is

$$U[\bar{k}\underset{G}{\leftarrow}k]_{\alpha\beta}A(i\leftarrow i)A(j\leftarrow j)\ldots A(\ell\leftarrow\ell)|0\rangle. \tag{5.1}$$

The colour independent scalar product between two such states is defined by averaging over the colour indices of the sources, that is

$$\begin{aligned}&\langle\bar{k}\underset{G'}{\longrightarrow}k|A(\ell'\to\ell')\ldots A(i'\to i')A(i\leftarrow i)\\&\ldots A(\ell\leftarrow\ell)|\bar{k}\underset{G}{\longleftarrow}k|\rangle\equiv\frac{1}{N_c}\langle0|A(\ell'\to\ell')\\&\ldots A(i'\to i')A(\bar{k}\underset{G'}{\longrightarrow}k,\bar{k}\underset{G}{\longleftarrow}k)A(i\leftarrow i)\ldots A(\ell\leftarrow\ell)|0\rangle.\end{aligned} \tag{5.2}$$

Note that following the previous convention $\underset{G}{\longleftarrow}$ means the curve G in positive direction and $\underset{G}{\longrightarrow}$ the same curve in the opposite, negative direction, therefore $\bar{k}\underset{G'}{\longrightarrow}k,\bar{k}\underset{G}{\longleftarrow}k$ is a closed curve consisting of G in positive and (after it) G' in negative direction (both $\underset{G}{\longleftarrow}$ and $\underset{G'}{\longleftarrow}$ start from k and end at $\bar{k}$). If $G=G'$ then, of course $A(\bar{k}\underset{G}{\longrightarrow}k,\bar{k}\underset{G}{\longrightarrow}k)=\mathrm{Tr}(1)=N_c$.

The action of an operator, say H, is defined as follows

$$\begin{aligned}&\langle\bar{k}\underset{G'}{\longrightarrow}k|A(\ell'\to\ell')\ldots A(i'\to i')\\&\cdot HA(i\leftarrow i)\ldots A(\ell\leftarrow\ell)|\bar{k}\underset{G}{\longleftarrow}k\rangle\\&\equiv\frac{1}{N_c}\sum_{\alpha\beta}\langle0|A(\ell'\to\ell')\ldots A(i'\to i')U[\bar{k}\underset{G'}{\longrightarrow}k]_{\alpha\beta}\\&\cdot HU[\bar{k}\underset{G}{\longleftarrow}k]_{\beta\alpha}A(i\leftarrow i)\ldots A(\ell\leftarrow\ell)|0\rangle.\end{aligned} \tag{5.3}$$

This means that H_A acts on $A(i\to i)\ldots A(\ell\to\ell)|\bar{k}\underset{G}{\longleftarrow}k\rangle$ also here by adding gluon loop string operators along the plaquettes, whereas the action H_E can again be summarized symbolically by (4.3) where the curve $\bar{k}\underset{G}{\longleftarrow}k$ is now, of course, also included in the configuration of curves.

Let us put the quark and antiquark source on the diagonal line, say in the x–y plane (see Fig. 4). According to (4.4) the curve of minimal length of arc between the sources is unique in this case: it is the diagonal curve G with $\lambda=L$ depicted in the figure. In the case of large N_c the state of minimal energy contains no gluon loops besides the string along G and it is therefore non-degenerate. Consequently we can use non-degenerate perturbation theory. This is an advantage compared to other situations when there are more curves with the same minimal length of arc, therefore one has to use degenerate perturbation theory. An example for the degenerate case is given in Fig. 5a where the sources are put in the x-direction. It

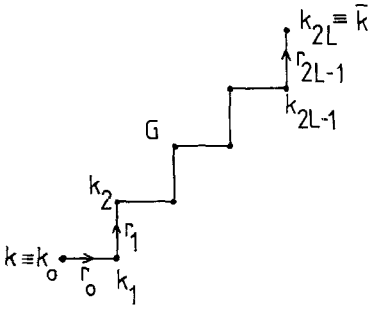

Fig. 4. The quark and antiquark source, respectively, at k and $\bar{k}$ and the string G of minimal length $\lambda = L$ connecting them

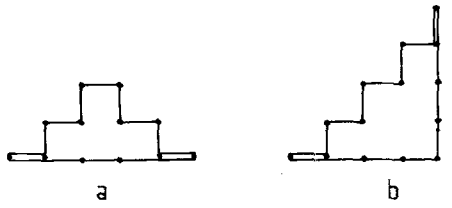

Fig. 5a and b. Curves with the same length of arc λ on the lattice if **a** λ is defined by (4.4) and if **b** λ is the number of links

is interesting to note that if the "length of arc" is defined on the lattice simply by the number of links on the curve, then there is a unique curve of minimal length of arc in the x-direction but along the x–y diagonal there are several "shortest" curves (Fig. 5b). Rotation invariance requires the potential between external sources to be independent of the direction with respect to the lattice. A general proof of this seems to be complicated, nevertheless, we checked that putting the sources along the x–y–z diagonal gives (for large distances) the same result as the x–y diagonal.

The orthonormal states with external sources are constructed in exactly the same way as in the case of the purely gluonic states. There is, of course, the additional string operator along $\bar{k} \xleftarrow{G} k$, hence the lowest energy eigenstate of H_0 is $|G\rangle \equiv |k \xleftarrow{G} k\rangle$. The states with an extra gluon loop operator along the plaquettes can be characterized as before by the number and δ-value of the points common to the plaquette and to the curve G. The states are denoted by $|G\mu xv\rangle$ or $|G\mu_1 xv_1, \mu_2 xv_2\rangle$ where the possible value of μxv (or $\mu_1 xv_1, \mu_2 xv_2$) are

$$\mu xv = 2x \pm 0,\ 1x \pm 2;\ 2x \pm 1;\ 1x \pm 2;\ 1x \mp 1. \quad (5.4)$$

We consider only effects proportional to the distance L, therefore plaquettes touching the end points k and $\bar{k}$ are not taken into account here. The "switched over" configuration to $|G\mu xv\rangle$ can be denoted by $|\underline{G}\mu xv\rangle$. The appropriate orthonormal configurations are constructed also here like in (4.16). We have, namely

$$\langle G\mu xv|G\mu xv\rangle = \langle \underline{G}\mu xv|\underline{G}\mu xv\rangle = \xi(\mu, v);$$
$$\langle G\mu xv|\underline{G}\mu xv\rangle = \frac{\xi(\mu, v)}{N_c}, \quad (5.5)$$

where the numbers $\xi(\mu, v)$ are now:

$$\xi(2x \pm 0, 1x \pm 2) = 2L - 3; \quad \xi(1x \pm 2) = 2L - 1;$$
$$\xi(2x \pm 1) = 4L - 4; \quad \xi(1x \pm 1) = 8L - 4. \quad (5.6)$$

The states $|G\mu xv\rangle, |\underline{G}\mu xv\rangle$ are sufficient for the calculation of the energy up to the third order in the $1/N_c$ perturbation series. (Higher orders require states with two additional plaquettes besides the curve G.) The result for the quark–antiquark potential energy for large distance $r = aL (L \gg 1)$ is up to this order

$$V_q(r) = r\frac{g^2 N_c}{4a^2}\left(1 - \frac{1}{N_c^2}\right) - r\frac{112}{15a^2 g^6 N_c^3} + \ldots$$
$$= \frac{2}{3} r\frac{g^2}{a^2} - \frac{112}{405} r\frac{1}{a^2 g^6} + \ldots. \quad (5.7)$$

In the second line we have put $N_c = 3$. The expansion goes also here according to the powers of $1/N_c^2$, therefore the third order gives only $1/N_c^4$ terms which we included. (Such terms can still come from the 4th order perturbation terms.)

According to (5.7) the slope of the potential is (up to the $1/N_c$ term):

$$\frac{1}{A^2} = \frac{2g^2}{3a^2}. \quad (5.8)$$

The experimental value of A is known from the charmonium spectrum [21]:

$$A = 2.34\ \text{GeV}^{-1}. \quad (5.9)$$

This and (1.1) give in (5.8) and (4.21) (the latter one taken to the same order in $1/N_c$ as (5.8), i.e. the first two terms in the second line) for g and a the values

$$g \cong 1.8\left(\alpha_s = \frac{g^2}{4\pi} \cong 0.25\right), \quad a \cong 0.7\ \text{Fermi}. \quad (5.10)$$

These values are consistent within our approximation scheme as $g^2 N_c$ for $N_c = 3$ is reasonably large, therefore the omitted next order corrections are presumably small enough. (In fact, there exists also another pair of roots for (5.8) and (4.21) with a smaller value of g, namely, $g \cong 0.9$ and $a \cong 0.35$ Fermi where the higher order corrections are expected to be larger).

The potential energy can also be determined between other kinds of sources, for instance between sources of octet colour ("gluon sources"). In such a case (if the position of the gluon sources on the lattice is denoted by k and ℓ) there are two strings in the state: one from k to ℓ and the other from ℓ to k denoted

like $\ell \underset{G_1}{\longleftarrow} k$ and $\ell \underset{G_2}{\longrightarrow} k$. The suitably normalized scalar product is, instead of (5.2), the following:

$$\langle \ell \underset{G_1'}{\longrightarrow} k, \ell \underset{G_2'}{\longleftarrow} k | A(j' \to j') \dots A(i' \to i')$$
$$\cdot A(i \leftarrow i) \dots A(j \leftarrow j) | \ell \underset{G_1}{\longleftarrow} k, \ell \underset{G_2}{\longrightarrow} k \rangle$$
$$\equiv \frac{4}{N_c^2 - 1} \sum_{b,d=1}^{N_c^2-1} \langle 0 | \mathrm{Tr} \left\{ \frac{\lambda_b}{2} U[\ell \underset{G_1'}{\longrightarrow} k] \frac{\lambda_d}{2} U[\ell \underset{G_2'}{\longleftarrow} k] \right\}$$
$$\cdot A(j' \to j') \dots A(i' \to i') A(i \leftarrow i) \dots A(j \leftarrow j)$$
$$\cdot \mathrm{Tr} \left\{ \frac{\lambda_d}{2} U[\ell \underset{G_1}{\longrightarrow} k] \frac{\lambda_b}{2} U[\ell \underset{G_2}{\longrightarrow} k] \right\} | 0 \rangle . \qquad (5.11)$$

The ground state of the system for large N_c is when both G_1 and G_2 are equal to G in Fig. 4. Therefore, if some plaquette touches $\mathbf{G}_1 = \mathbf{G}$ in some point with the "direction factor" δ then at the same time it touches $\mathbf{G}_2 = \mathbf{G}$ with the "direction factor" $-\delta$. The states with one additional gluon loop along a plaquette can be labelled here by $|GG\mu\nu x\rangle$ where the possible values of $\mu\nu x$ are the same as in (5.4). There are now two sorts of "switched over" states, namely, between the plaquette and either $\ell \underset{G}{\longleftarrow} k$ or $\ell \underset{G}{\longrightarrow} k$. These can be denoted, respectively, by $|\underline{G}G\mu\nu x\rangle$ and $|G\underline{G}\mu\nu x\rangle$. The scalar products of these states are, with (5.6):

$$\langle GG\mu\nu x | GG\mu\nu x \rangle = \xi(\mu, \nu);$$
$$\langle \underline{G}G\mu\nu x | \underline{G}G\mu\nu x \rangle$$
$$= \langle G\underline{G}\mu\nu x | G\underline{G}\mu\nu x \rangle = \left(1 - \frac{1}{N_c^2}\right) \xi(\mu, \nu);$$
$$\langle \underline{G}G\mu\nu x | G\underline{G}\mu\nu x \rangle = -\frac{1}{N_c^2} \xi(\mu, \nu);$$
$$\langle GG\mu\nu x | \underline{G}G\mu\nu x \rangle = \langle GG\mu\nu x | G\underline{G}\mu\nu x \rangle = \frac{\xi(\mu, \nu)}{N_c}. \qquad (5.12)$$

The orthonormal combinations appropriate for the perturbative calculation can be chosen now like

$$|\underline{G}G\mu\nu x) \equiv \left[\xi(\mu, \nu) \left(1 - \frac{1}{N_c^2}\right) \right]^{-1/2} |\underline{G}G\mu\nu x\rangle;$$
$$|G\underline{G}\mu\nu x) \equiv \left[\xi(\mu, \nu) \left(1 - \frac{1}{N_c^2 - 1}\right) \right]^{-1/2}$$
$$\cdot \left[|G\underline{G}\mu\nu x\rangle + \frac{1}{N_c^2 - 1} |\underline{G}G\mu\nu x\rangle \right];$$
$$\cdot |GG\mu\nu x) \equiv \left[\xi(\mu, \nu) \left(1 - \frac{2}{N_c^2 - 2}\right) \right]^{-1/2} \left[|GG\mu\nu x\rangle \right.$$
$$\left. - \frac{N_c}{N_c^2 - 2} |\underline{G}G\mu\nu x\rangle - \frac{N_c}{N_c^2 - 2} |G\underline{G}\mu\nu x\rangle \right]. \qquad (5.13)$$

(or with the role of $\underline{G}G$ and $G\underline{G}$ interchanged).

The gluon–gluon potential up to third order, corresponding to (5.7) in the quark–antiquark case, is the following:

$$V_g(r) = r \frac{g^2 N_c}{2a^2} - r \frac{224}{15 a^2 g^6 N_c^3} + \dots$$
$$= \frac{3}{2} r \frac{g^2}{a^2} - \frac{224}{405} r \frac{1}{a^2 g^6} + \dots . \qquad (5.14)$$

It can be seen from here that up to the order $1/N_c$ the slope of the gluon–gluon potential (for $N_c = 3$) is 9/4 times the slope of the $q\bar{q}$-potential.

VI. Summary and Conclusions

The limit considered in this paper is N_c (number of colours) $\to \infty$, g (bare coupling constant on the lattice) fixed. This implies that in the Hamilton operator the colour electric energy is dominating as it is also the case in the "strong coupling limit" $g \to \infty$, ($N_c = 3$) [1, 2]. This expansion can be called "strong coupling $1/N_c$ expansion" in distinction to the expansion in the $N_c \to \infty, N_c g^2 =$ fixed limit [6]. The advantage of making N_c large is a considerable simplification in the structure of Hilbert-space of states. It can be shown that the physical quantities like in (4.17), (4.18), (5.7), (5.14) have in general the form

$$\sum_{i,j>0} C_{ij} (g^4 N_c)^{-i+a} N_c^{-j+b} \qquad (6.1)$$

where C_{ij} are constant coefficients and a, b are constants. These series are well converging if both N_c and $g^4 N_c$ (and therefore N_c and $g^2 N_c$) are large. Actually, in the quantities we calculated i, j are even, therefore only every second power of $1/N_c$ appears. This means that a few terms give at $N_c = 3$ already a good approximation if g is not too small.

Concerning the imposed periodic boundary conditions let us remark that the size P in (4.6) of the cube of periodicity is an unphysical variable which has to disappear from any measurable quantity. This can be achieved in the $1/N_c$ expansion, see e.g. (4.17), and then the obtained physical quantity can be continued back to $N_c = 3$.

We determined the vacuum energy density (4.17) and the vacuum expectation value of the gluon field squared (4.21) up to the fifth order in the $1/N_c$ expansion. The quark–antiquark potential (5.7) and gluon–gluon potential (5.14) between external sources were calculated up to the third order. Higher orders can be calculated along exactly the same lines but require a set of states larger than considered here. We hope to be able in the future to include the next more complicated states allowing for the calculation of the next two orders.

This would be particularly interesting for the quark–antiquark potential as then it would be possible to determine g and $a(g)$ (the lattice spacing belonging to g) in the next order of $1/N^2$ by the value of the slope of the $q\bar{q}$ potential and the already known higher

order result (4.21) for the vacuum expectation value of the gluon field squared. One would expect that both g and a would come out smaller in the higher order than our present values in (5.10).

The gluon–gluon potential determined in section V may be interesting for the study of hadronic final states in e^+e^- annihilation, more precisely for the three jet events. In a simple model for the three jet final state [22, 23] the field tubes spanned out by the colour charges of the produced three partons have a star-like shape [22] if the slope of the gluon–gluon potential is less than twice the slope of the quark–antiquark potential and they form a triangle [23] if this ratio is larger than 2. The configuration of the field tubes influences the hadron distributions and can be looked for experimentally. The value of the ratio given by (5.7), (5.14) is 9/4. It would be also here very interesting to determine what happens in the next order in $1/N_c^2$. Our expectation is a decrease of this ratio, especially for smaller g, because of the increasing importance of the magnetic energy term.

An interesting application of the vacuum structure obtained in the $1/N_c$ expansion is to investigate its influence on the dynamics in the string operator formulation of QCD [15]. The hope is that one can approximate the hadronic wave functions by a configuration consisting of a small number of strings immersed in the physical vacuum. In conclusion: we think that the simplicity of the picture of the vacuum emerging from the $1/N_c$ expansion may help in the understanding of many of the confinement aspects of QCD.

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