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Rational Number Learning Aids: Transfer from Continuous Models to Discrete Models

Merlyn J. Behr
Northern Illinois University

Ipke Wachsmuth
IBM Deutschland GmbH

Thomas Post
University of Minnesota

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This paper is concerned with how well children who have had instruction about rational number concepts based on a continuous manipulative aid are able to transfer their knowledge to accomplish tasks based on a manipulative aid which can be interpreted as discrete. The aid used was an egg carton which, if the subject views the holes as salient, is discrete; if, on the other hand, the subject focuses on the proportion of total area enclosed by the edges of each of the holes, then it is viewed as a continuous entity.

The Issue

The research concerning learning via manipulative materials is somewhat equivocal. The question of what characteristics of manipulative aids best facilitate learning is unanswered. For developing rational number concepts, the manipulative aids most frequently used are called **area** or **measurement** models, and **set** models; equally descriptive of the two categories are the words **continuous** and **discrete**, respectively. The work of Payne and his students (Suydam, 1978) suggests that instruction based on the set (discrete) model interferes with children's fraction understanding previously developed from instruction based on the area model. Novillis (1976) used hierarchical analysis to establish that tasks based on the set model are more difficult for children than those based on an area model. Behr, Post, Silver, and Mierkiewicz, (1980), on the other hand, suggest that what has been called **interference** might be more correctly described as **appropriate cognitive disequilibrium**. Such disequilibrium causes the child to reinterpret the concept by reconciling the differences between models while at the same time observing their similarities or isomorphic properties. This suggestion is consistent with the learning theories of Gagne (1973), Bruner (1966), and Dienes (1971), and with the developmental psychology of Piaget.

Continuous and Discrete Models, Cognitive Distinctions

From the perspective of the part-whole construct of rational number (Kieren, 1976), there are significant similarities and differences between a continuous model and a discrete model for showing rational number concepts. To represent rational number concepts, each requires: (a) The identification of a unit; for a continuous model, the unit is a set of discrete objects. (b) Partitioning of the unit into parts of equal size (i.e., equal measure); for the continuous model, each part is again a single continuous piece and contiguous to other part(s); for the discrete model, the parts may be a single discrete object, or several discrete objects, and, in general, are not contiguous with other parts. Thus, there seem to be distinctly different cognitive demands for representing fractions with discrete and continuous models.

The discrete model involves the need to perceive a set of discrete objects as a unit, **one** entity. If 15 objects serve as a unit to show fifths, the child must: first, perceive 15 objects as a single entity (i.e., perceive 15 objects as one whole); second, perceive such subset of 3 objects as a sub-entity (i.e., perceive 3 objects as one sub-part). In the majority of instances, objects as one sub-part). In the majority of instances, objects of the same shape criteria are used, although there are substantial reasons to systematically provide for variation of this. It is not required that the discrete objects which comprise the unit be the same size or even the same shape, since the appropriate measure of the unit and subunit relates solely to the cardinality of the sets. For discrete models, the number of items is crucial, not their size and shape; for continuous models, the crucial elements are size and shape. Thus, the perceptual distortion can become even greater with the discrete model than for the continuous model. This can provide for an additional level of cognitive disequilibrium, and provide further opportunity for learning.

Finally, a situation which is initally perceived to be discrete, may, in its analysis and solution, need to be interpreted from both points of view — discrete and continuous. The problem of sharing 14 cookies equally among 3 children can (and will, by some children) be solved by distributing 4 individual cookies to each of the children, the cookies being perceived as discrete objects. To complete the solution, the 2 remaining cookies are treated as continuous objects, each partitioned into thirds, and 2 of the thirds (1/3 + 1/3) allocated to each child.

The Study

The present study was conducted by the Rational Number Project during 1980-81 (Behr et al., 1980). The Rational Number Project is a multi-site effort which begin in 1979 with funding from NSF. One focus of the project continues to be to assess the impact of manipulative materials on the development of rational number concepts.

Subjects

Subjects in this investigation were six children in a 4th-grade experimental group in DeKalb, Illinois, five of whom were matched with subjects in a 4th-grade comparison group of the same school. In addition, data are available from a group of six 5th-graders, which was the pilot experimental group in 1979-80, and a 5th-grade comparison group. Data from a similar group of six experimental children from the Minneapolis site were also analyzed and are included. The 4th- and 5th-grade experimental subjects from both sites were the students of a teaching experiment conducted over a period of about 18 weeks.

Selection of students to participate as experimental and control subjects of an intact class as high, middle, and low mathematics achievers, after having removed from consideration those children at both extremes as well as any with behavior problems of a sort that would inhibit them from providing useful information about children's mathematical cognition. Four children were randomly selected from each achievement category and then two in each category randomly assigned to the experimental and control groups, respectively. In this way, two groups of six children, matched on mathematics achievement was obtained.

Assessments

Interview assessments for this study were given on a one-on-one basis abut half way through the 18 weeks (mid) and another after the completion of the 18 weeks of instruction (final) as part of a more extensive interview assessment. The mid-assessment took place when children's instructional exposure had been limited to continuous models.

Instruction

The teaching experiment provided children with manipulative-oriented, theory-based instruction (Behr et al., 1980). At the time of the (mid) assessment, the children had had experience with the four different continuous manipulative aids: colored fractional parts of circular and rectangular models, paper folding and centimeter rods. Instruction was based on the multi-embodiment principle (Dienes, 1967). Students had learned to translate a representation using one of the four aids to a representation using any of the others. They had also learned to translate between different modes of representation, such as manipulative, symbolic, written natural language and oral language. They had associated fraction symbols with embodiments and with word names of fractions. A particular feature of the instruction required children to translate between a familiar and unfamiliar (new) manipulative aid, to explain how the new aid represents the same concepts as the familiar aid, and to observe the similarities and differences between them.

Immediately following the mid-assessment, instruction included activity based on the discrete model, using counting chips. Instruction was then continued on the basic rational number concept, order and equivalence, the concept of unit, and extended to mixed numbers, improper fractions, and addition and subtraction of mixed numbers and proper and improper fractions with like and unlike denominators.

Control

Children in the control groups utilized the regular textbook program which was highly symbolic and proceeded rapidly through area and set models to focus on algorithmic manipulations.

Tasks

All tasks, given during the mid- and final-assessment, were parts of larger sets of interview tasks. Two types were utilized. In the first type, children were directed to "put eggs (cubes) into the egg carton so that some fraction of the holes are filled." This direction was intended to suggest a discrete interpretation of the egg carton. The three tasks of this type — Egg Carton-Discrete — required children to show 3/4, 2/6, and 1/3 of the holes filled (see Figure 1a).

The second type of task — Egg Carton-Continuous — used precut area sections of the egg carton, sized to cover fractional parts of a whole egg carton. The child was asked, "What fraction of the egg carton is covered?" These tasks were intended to suggest a continuous interpretation of the egg carton. The fractional part and cover configuration for the four tasks of this type are shown in Figure 1b. Each subject was given each task at the mid-assessment and again at the final assessment.

Table 1.

Response Frequencies by Response Categories by Subject Groups

for Egg Carton: Discrete Tasks

	Category Label	5th Grade				4th Grade					
		Experimental (E)		Control (C)		E - DeKalb		C - DeKalb		E - Mpls.	
		Mid	Final	Mid	Final	Mid	Final	Mid	Final	Mid	Final
	G	0	0	0	3	1	2	0	0	2	3
	D	0	0	0	0	0	0	0	2	0	0
High[a]	P	9	9	0	0	5	11	0	2	10	15
	LTP	3	1	0	3	0	0	0	1	1	0
	EP	0	0	0	0	3	0	0	0	0	0
	RDU	2	2	4	0	5	0	3	4	1	0
	I	3	3	4	5	0	0	3	1	0	0
Low[b]	M	1	0	2	0	1	3	4	3	0	0
	A	0	0	0	0	0	0	4	0	0	0
	Don't Know	0	0	5	4	1	0	1	2	4	0

[a]High = Strategies leading to correct responses.

[b]Low = Strategies leading to incorrect responses.

Table 2.

Response Frequencies by Response Categories by Subject Groups

for Egg Carton: Continuous Tasks

Category	Label	5th Grade Experimental (E) Mid	Final	Control (C) Mid	Final	4th Grade E - DeKalb Mid	Final	C - DeKalb Mid	Final	E - Mpls. Mid	Final
High[a]	G	1	2	2	4	2	7	0	3	2	11
	EF	0	1	0	0	0	0	0	0	11	6
	PLT	6	5	0	0	1	9	0	0	5	7
	PHT	17	12	18	16	17	4	16	13	4	0
Low[b]	PD	0	0	0	0	3	0	0	0	0	0
	M	0	0	0	0	4	5	4	5	0	0
	R	0	0	0	0	1	0	0	0	0	0

[a] High = Strategies leading to correct responses.

[b] Low = Strategies leading to incorrect responses.

Egg Carton-Discrete

Strategies leading to correct solutions are illustrated by student responses classified in the following categories. **Category G (Generalized)**. Responses in this category indicate that the subject abstractly solved the problem by computing n/d of 12 =_____, or used multiplication or division.

- About 3/4: "Well, three left over is one-fourth; two-fourths there, three-fourths, four-fourths (counting groups of three filled holes)."
- About 3/4: "I went by three's. Here's one-fourth; here's another, two-fourths...; three-fourths. There's twelve. Three goes into twelve, four times."

Table 3.

Proportion and Percent of Responses in High[1] Categories

for Egg Carton: Discrete, and, Egg Carton: Continuous Tasks

Type	5th Grade Experimental		5th Grade Control		4th Grade E - DeKalb		4th Grade E - Mpls.		4th Grade C - DeKalb	
Mid Assessment										
Discrete	12/18	67%	0/15	0%	9/16	56%	13/18	82%	0/15	0%
Continuous	24/24	100%	20/20	100%	20/24	83%	22/22	100%	16/20	80%
Final Assessment										
Discrete	10/15	67%	6/15	40%	13/16	81%	18/18	100%	5/15	33%
Continuous	20/20	100%	20/20	100%	20/20	100%	24/24	100%	16/20	80%

[1]High = Categories exemplified by responses exhibiting strategies leading to correct responses.

- About 1/3: "I was trying to cut into three pieces because one-third; I thought four plus four plus four, so I covered one-fourth, I mean one-third."

Category D (Discrete). Responses in this category suggest that the subject converted to an equivalent fraction with twelfths: e.g., covered 3/4 by solving 3/4 = 9/12 and putting in nine eggs.

- To show 2/6: (Looks at written 2/6 and writes
  2 ×2 = 4
  6 ×2 = 12).

Category P (Partitioning). The subject's explanation suggested that (s)he first partitioned the carton into sections, then filled holes; partitioning was observed by the subject's overt outlining of sections.

Table 4.

Subject Response Codes, Fourth Grade

Subjects	Discrete Tasks[a] 3/4 Mid	3/4 Final	2/6 Mid	2/6 Final	1/3 Mid	1/3 Final	Continuous Tasks[b] Task 1 Mid	Task 1 Final	Task 2 Mid	Task 2 Final	Task 3 Mid	Task 3 Final	Task 4 Mid	Task 4 Final
Experimental Group - DeKalb														
1	P+	C+	P+	P+	C+	P+	C+	C+	PD-	C+	C+	C+	PD-	C+
2	RDU-	C+	P+	P+	P+	P+	PHT+	PLT+	PHT+	PLT+	PLT+	PLT+	PD-	PLT+
3	EP+	P+	EP+	P+	EP+	P+	PHT+	EF+	PHT+	PLT+	PHT+	PLT+	PHT+	PLT+
4	RDU-	P+	DK-	P+	P+	P+	PHT+	PLT+	PHT+	PHT+	PHT+	PLT+	PHT+	PHT+
5	RDU-	M-	M-	M-	RDU-	M-	R-	C+	PHT+	C+	PHT+	PHT+	PHT+	PHT+
6	RDU-	d	PI-				PHT+		PHT+		PHT+		PHT+	
Control Group - DeKalb														
1	A-	DK-	A-	DK-	A-	M-	PHT+	EF+	PHT+	EF+	PHT+	PHT+	PHT+	EF+
2	A-	I-	M-	PHT+	M-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PLT+
3	RDU-	LTP+	RDU-	P+	RDU-	LTP+	PHT+	PHT+	PHT+	PHT+	PHT+	PTH+	PHT+	PHT+
4	I-	RDU-	I-	RDU-	I-	RDU-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
5	DK-	RDU-	M-	M-	M-	M-	M-	M-	M-	M-	M-	PHT+	M-	M-
Experimental Group - Minneapolis														
1	P+	P+	P+	P+	P+	P+	EF+	PLT+	EF+	PLT+	EF+	EF+	EF+	EF+
2	C+	P+	P+	P+	P+	P+	PLT+	C+	PLT+	C+	PLT+	PLT+	PLT+	C+
3	LTP+	P+	P+	P+	RDU-	P+	PLT+	EF+	EF+	C+	EF+	EF+	EF+	EF+
4	DK-	P+	DK-	P+	DK-	P+	PHT+	PLT+	PHT+	PLT+	PHT+	PLT+	PHT+	PLT+
5	DK-	P+	P+	P+	P+	P+	C+	C+		C+		C+	C+	C+
6	P+	C+	P+	C+	C+	C+	EF+	EF+	EF+	C+	EF+	C+	EF+	C+

[a] See Figure 1a for task descriptions.

[b] See Figure 1b for task descriptions.

[c] + denotes, response is correct; -, incorrect.

[d] A blank space indicates no data available

- To show 2/6: "(Fills four holes) I was thinking that you'd have six-sixths and take two of them. And each two holes is one-sixth."
- To show 2/6: "Because here's one, two, three, four, five, six (pointing to six groups of two holes) and I covered two of them."

Category LTP (Lowest Terms Partition). Responses in this category suggest that the subject made groups of holes with d in each group, then showed n/d as n of d parts in each group of d.

Table 5.

Subject Response Codes, Fifth Grade

Rational Number

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	Discrete Tasks[a]						Continuous Tasks[b]							
	3/4		2/6		1/3		Task 1		Task 2		Task 3		Task 4	
Subject	Mid	Final	Mid	Final	Mid	Final	Mid	Final	Mid	Final	Mid	Final	Mid	Final
Experimental Group														
1	P+	P+	P+	P+	P+	P+	PLT+	PLT+	PLT+	PLT+	PLT+	PLT+	PLT+	PLT+
2	LTP+		LTP+		LTP+		PHT+		PHT+		PHT+		PHT+	
3	P+	P+	P+	P+	P+	P+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
4	I-	I-	I-	I-	I-	I-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
5	RDU-	RDU-	RDU-	RDU-	M-	LTP+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
6	P+	P+	P+	P+	P+	P+	PLT+	PLT+	PLT+	C+	C+	C+	PHT+	EF+
Control Group														
1	M-	G+	DK-	G+	DK-	G+	PHT+	C+	C+	C+	PHT+	C+	C+	C+
2	RDU-	DK-	I-	I-	M-	I-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
3	DK-	DK-	DK-	DK-	DK-	DK-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
4	RDU-	LTP+	RDU-	LTP+	RDU-	LTP+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+
5	I-	I-	I-	I-	I-	I-	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+	PHT+

[a] See Figure 1a for task descriptions.

[b] See Figure 1b for task descriptions.

- To show 3/4: "(Arranged eggs in holes as:

■ ■ | ■ ■ | ■ ■ I put them into fours and then I
■ ○ | ■ ○ | ■ ○ took three out of each section."

- To show 1/3: "(Arranged eggs in holes as:

■ ○ ○ | ■ ○ ○ I make three eggs for the whole
○ ■ ○ | ○ ■ ○ (points to groups of three) and then I put one egg in the hole (one hole out of three)."

Category EP (External Partitioning). The subject first solved the task outside the egg carton and then transferred the finished solution to the egg carton.

Table 6.

Frequencies of Same-Item Mid-to-Final Response Pairs, with Final Response Correct, that Dropped, Stayed the Same, or Rose in Category Level

Group	Egg Carton: Discrete				Egg Carton: Continuous			
	Dropped	Same	Rose	n[a]	Dropped	Same	Rose	n
DeKalb (E)[c]	1	4	7(47)[b]	15	0	6	14(70)	20
Mpls. (E)	1	9	8(44)	18	2	4	14(70)	20
4th (C)	0	0	5(33)	15	0	13	4(20)	20
5th (C)	0	0	6(40)	15	0	18	2(10)	20

[a] n = Total number of responses in given category.

[b] ( ) = percent of n.

[c] E - denotes experimental group, C - control.

- To show 1/3: "(took twelve cubes, made three equalized groups, put one group into the carton) I made three equal-sized parts and put in one of them."

Strategies leading to incorrect solutions are illustrated by student responses classified into the following categories.

Category RDU (Redefines Unit). Subject centered on only a section of the egg carton with holes equal in number to the denominator and filled holes equal in number to the numerator, in that section.

- To show 3/4: (Identified four holes, filled three of them).

Category I (Isomorphic Representation). Subject made a physical n/d representation in the carton.

- To show 2/6: "(Fills holes as ■ ■ ○ ○ ○ ○
  ■ ■ ■ ■ ■ ■
  It's the same thing...two-sixths; two cubes here and cubes here. There are two cubes on top and six cubes on the bottom."

Category M (Multiplication). The subject's explanation suggests that (s)he covered holes according to the formula: n ×d equals the number of holes to fill.

- To show 2/6: "Here's one six (filling six holes) and here's another six (filling remaining six holes)."
- To show 1/3: "(Fills three holes) That's one-third because there is one group of three."

Category A (Addition). The subject's explanation suggests that (s)he covered holes according to the formula: n + d equals the number of holes to fill.

- To show 3/4: (Fills seven holes) Well, three-fourths, put in three and then four."

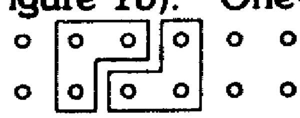
Category DK (Don't Know).

Egg Carton-Continuous

Strategies leading to correct solutions are illustrated by student responses classified into categories as follows:

Category C (Continuous). The subject's response suggests that (s)he treated the one or two covering sections as a single entity (area) and mentally generated the lowest terms fraction. No reference was made to the unreduced fraction name.

- About Task 4 (See Figure 1b): "One-half, put like this (moves coverings to



Category EF (Equivalent Fraction). Responses in this category suggest that the subject first generated the answer in twelfths and then converted it to an equivalent lower-terms fraction.

- About Task 1 (Figure 1b): "Two-twelfths or one-third (explains how to take the two holes at the end together with two more, separated from those by the cover section)."
- About Task 3 (Figure 1b): "It could be four-twelfths or it could be two-sixths or...it could be one-third."

Category PLT (Partitioned Lower Terms). Responses in this category suggest the subject treated the covering section(s) separately or in subsections (other than size one) and generated a fraction reflecting that partitioning. If the subject verbalized n/12 at any time, a lower-terms fraction was given as well.

- About Task 3 (Figure 1b): "Four-twelfths; or you could look at it like, you count that (points to groups of two holes), and that (another two holes), and that's one. So that would be...three groups of two and that would be one-sixth, two-sixths..It's hard to look at it that way, but it is possible...it's kind of hard to

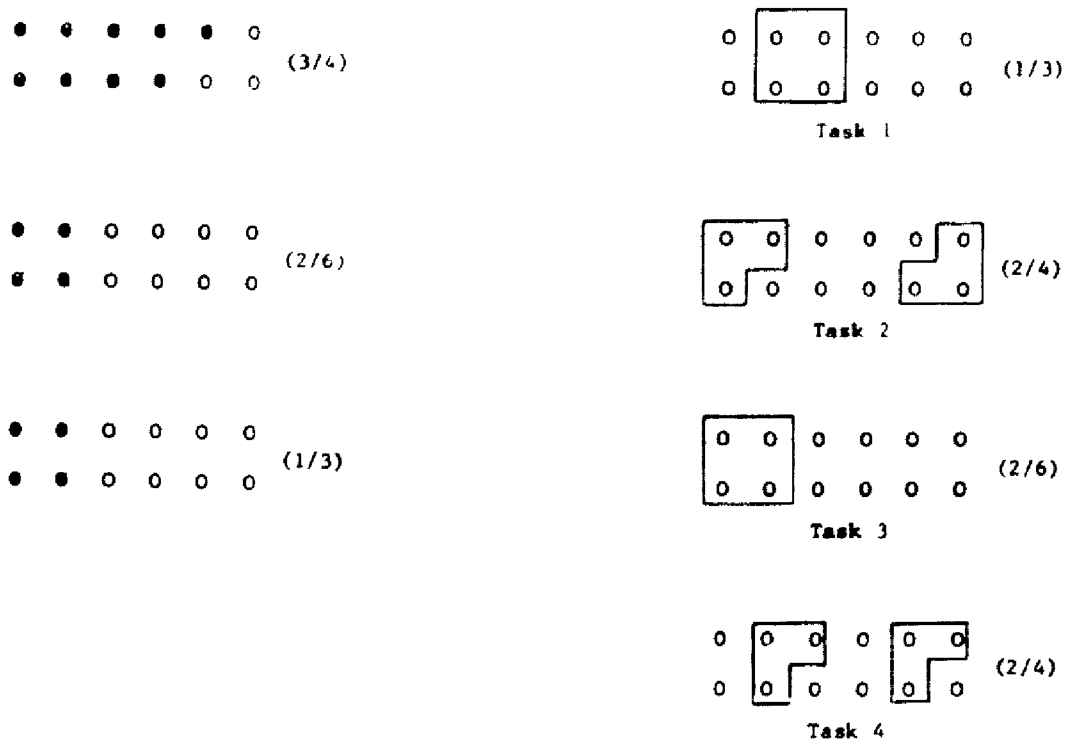


Figure 1a
Discrete Interpretation

Possible discrete interpretations for indicated fractions, Xs imply indiviual counters.

Figure 1b
Continuous Interpretation

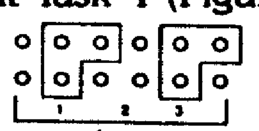
Encircled portions imply use of cardboard piece(s) covering indicated area

Figure 1

Two Interpretations of Egg Carton Tasks

picture two holes as one (entity)."

- About Task 2 (Figure 1b): "...(Puts hand on three holes.) It's two-fourths. It's two-fourths 'cause, see, there's one-fourth and one-fourth and one-fourth and one-fourth (pointing in turn to distinct sets of three holes.)"
- About Task 4 (Figure 1b): "...(Counts four groups as indicated:



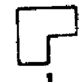
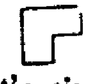
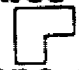
Four groups and two covered; two-fourths."

Category PHT (Partitioned Highest Terms). Responses in this category suggest the subject generated the x/12 form and then explained in terms of 12 holes with x holes filled.

- About Task 3 (Figure 1b): "Four-twelfths, there are twelve holes in the carton and then that four (pointing to the covering section)."

Strategies leading to incorrect solutions are illustrated by student responses classified into categories as follows.

Category (Partial Discrete). Response in this category suggest that the subject incorrectly generated the x/12 form in terms of 12 holes with x sections filled.

- About Task 4 (Figure 1b): "Six-twelfths or, let's see, two-twelfths...I'll pretend that those (covers  and ) are all one and that six, three here and three here, that's six. And then that (points to ) equals one and that (points to other ) equals one, so that would be two; two-twelfths."

  (Note: In further questioning, this child showed no apparent concern about the same display representing both 6/12 and 2/12.)

Category M (Multiplication). Responses in this category indicate that the subject gave a whole-number interpretation that n/d is n sets of d.

- About Item 4 (Figure 1b): "Two-twelfths, no, two-thirds, because there are two red (covering sections) things of three things. There's three of them in it."

Category R (Ratio). Responses in this category suggest the subject used the ratio of the number of covered parts to the number of uncovered parts.

- About Item 1 (Figure 1b): "Four-eighths, there's eight pieces (Counting those not covered) and there's four in there (pointing to four holes in covering section).

Discussion

Much research in mathematics education has been either experimental or survey (status study) research, inferential or descriptive, respectively. Experimental studies are designed to test previously formulated hypotheses. Status studies describe the state of the art in terms of student, teacher, or content variables; they are not designed to assess the impact of instructional variables on students' conceptual development.

Unlike experimental- and status-study research, teaching experiments such as this are designed to generate hypotheses and to describe student behavior following carefully fomulated instruction. This is in contrast to the status study which describes behavior with little or no knowledge of its instructional precursors.

Since the data was taken from a small sample, the results should be taken as indicating trends and suggesting hypotheses, rather than giving confirming evidence. The **egg carton-discrete** items were production tasks, whereas, the egg carton-continuous items were interpretative. Inferences made from any differences in performance between the types of tasks must take account of this.

Based on the percentages of responses within categories exemplified by strategies leading to correct answers, substantial differences between experimental and control groups were observed. (See Table 3.) This was true at both the fourth- and fifth-grade levels. These differences were especially apparent for the **egg carton-discrete** tasks. Two-thirds (67%) of fifth-grade experimental children used strategies leading to correct solutions for discrete tasks at the mid-assessment, while none (0%) of the control group did. The corresponding percentages for the final assessment at the fifth-grade level were 67 and 40. Similarly, for the fourth-grade, the mid-assessment percentages correct were 56 (DeKalb, experimental); 82 (Minneapolis, experimental); and 0 (control); at the time of the final assessment, they were 81 (DeKalb, experimental); 100 (Minneapolis, experimental); and 33 (control).

Differences of this magnitude were not observed for the continuous tasks. At the fifth-grade level, both experimental and control students correctly answered all tasks in both the mid- and final-assessments. At the fourth-grade level, minor differences were found between the two experimental and control groups. These differences favored the experimental groups (see Table 3).

These observations of the data lead to several suggestions.

1. Fourth-grade children who are given instruction on rational-number concepts based on a systematic use of continuous embodiments will be more successful with rational-number learning and performance tasks based on a discrete embodiment than children whose instruction on rational number concepts is not based on a systematic use of such embodiments; i.e., instruction using continuous embodiments facilitates learning from discrete embodiments.
2. Fourth-grade children who are given instruction on rational number concepts based on systematic use of both continuous and discrete embodiments will be more successful with rational-number tasks based on a discrete embodiment, and moderately more successful on rational-number tasks based on a continuous embodiment, than students whose instruction is not based on a systematic use of embodiments.
3. The difference, stated in 2 above, of tasks based on discrete embodiments will persist through the fifth grade, although the magnitude of the differences will be less.

At the time of the mid-assessment, the fourth-grade experimental groups had received no instruction based on discrete embodiments, this leads to the following suggestions.

4. Knowledge of basic rational-number concepts gained from instruction employing continuous models transfers as knowledge for performance on basic rational number tasks based on discrete models.

The percentage of correct responses in the experimental groups (see Table 3) increased or remained constant from mid-assessment to final-assessment on the **egg carton-continuous tasks.** In fact, the performance of all individual children in the experimental group increased or remained consistent (see Tables 4 and 5). Since these children received instruction between assessments based on both discrete and continuous models, this information lessens the strength of Payne's (1976) claim that instruction based on discrete models interferes with children's performance on tasks based on continuous models.

5. There is no long-term detrimental effect on children's knowledge of rational numbers derived from instruction based on continuous models when this is followed with instruction based on a combination of discrete and continuous models.

Table 6 summarizes information from Tables 4 and 5 and gives the frequencies of pairs of responses on the same item that dropped, stayed the same, or rose in category level from the mid-assessment to final-assessment. These data only include the frequencies for which responses on the final assessment were correct.

Among the discrete tasks, the data suggests a slight trend favoring the experimental groups; and, among the continuous tasks, there appears to be a sharp difference favoring the experimental groups. This, together with the observation (see Table 3) that the experimental treatment apparently results in more correct responses suggests:

6. Instruction for rational-number concepts arising from instruction incorporating a systematic use of continuous and discrete manipulative aids will result in a higher percentage of correct responses and a higher quality of responses on rational number tasks than a "traditional program" among fourth-grade children.

Implications

The results obtained further reinforce the earlier work of Payne (1976) and Novillis (1976) which suggest that discrete-embodiment tasks are more difficult for children than continuous-embodiment tasks. This, perhaps, should not be surprising, since the intended unit (and appropriate sub-units) from which any fraction derives its meaning is more easily ascertained in a continuous embodiment. In the discrete context, children may have difficulty with the fact that each sub-unit may be composed of several discrete objects. For example, 1/5 of 15 things is one set of 3 things. Each one-fifth is itself a set of 3 separate things. As one student said when describing a related task, egg carton task, "It's hard to look at it that way, but it is possible...it's kind of hard to picture three holes as one." With a continuous embodiment, one-fifth is one piece of the whole unless a fraction equivalent to one-fifth is represented and the task is to interpret the representation as 1/5. The perceptual cues and the associated verbiage is different for the two types of embodiments; the discrete context contains additional variables which must be mentally coordinated by the student. With this additional required mental coordination, it can be expected that students will experience increased difficulty.

Our findings suggest several things about teaching children fraction concepts using continuous and discrete manipulative aids. While children do have difficulty interpreting representations of fraction with a discrete embodiment, this difficulty has a positive learning effect in the long term. The difficulty in interpretation causes the children to rethink the meaning of fraction as part of a whole. This rethinking, or reconceptualization, deepens the child's understanding of fractions and rational numbers. When interpreting a fraction representation with a familiar manipulative aid becomes automatic for a child, it is unlikely that new learning takes place. Yet this automatically does not necessarily suggest that the child has full understanding of the fraction concept. By introducing a new manipulative aid into the learning environment which has different perceptual features, such as changing from continuous to discrete embodiments, this automatically is interrupted and new thinking takes place and new insights and understandings can occur.

The use of multiple embodiments, and especially ones that differ in perceptual features, is believed to enhance learning and understanding. This claim was originally made by Z.P. Dienes (1971) in his Perceptual Variability Principle which suggests that variation in the perceptual features of manipulative aids is necessary in order for children to observe and abstract the critical attribute represented in the aids; in this case, the attribute of equal parts of a whole.

From the research reported in this paper, and from some of our other work (e.g., Behr, Wachsmuth, Post, & Lesh, 1984) we have been able to formulate a procedure for introducing new manipulative aids into the teaching environment in such a way that the learning from multiple aids substantially increases learning over the use of a single aid. The procedure is to work with an aid until children become quite familiar with it and are able to use it to represent the concept of interest without much difficulty. A new aid is introduced in the following way: The teacher demostrates a procedure for representing a rational number, or other concept, with the new, or the familiar, aid; after each step in the demonstration, the teacher asks the children to (a) describe orally what the teacher did, (b) accomplish, on their own, the same thing with the other aid, and finally (c) discuss with the children what is the same and different about the two displays.

This activity challenges the learners to do two types of thinking. Bottom up thinking to interpret the teacher's demonstration, and top down thinking to reconceptualize the concept represented to make it "fit" with the new aid. An example will help to clarify. Suppose the children are familiar with using paper folding to represent fractions. The teacher may wish to introduce centimeter rods as another manipulative aid to show fractions. A first step could be for the teacher (choosing to work with the fraction 3/4) to fold the paper into 4 equal parts. Then the teacher would ask the children to describe what was done — took a unit and partitioned it into 4 equal parts. Next the teacher would ask the children to do the same with the centimeter rods. This would challenge the children to (a) find a rod to use as a unit, (b) recognize that partitioning could be accomplished by putting smaller rods on top of or beside the unit rod (in some cases the teacher will have to suggest a particular rod to use as a unit) and finally, (c) to discuss with the children what is alike and different about the two types of displays, so far. This procedure is repeated until the children complete a representation of 3/4 with the rods. Sometimes it works better for the teacher to demonstrate with the new aid and have children interpret using the familiar aid. The important point is that children are given the opportunity to interpret a representation for a concept and then do a cognitive reorganization of this concept to make it fit another representational devise (i.e., another manipulative aid). This change between top down and bottom up thinking will deepen the child's understandings of the concept. Additional continuous manipulative aids, and ultimately discrete aids, can be introduced in a similar manner.

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