ASSESSING FIFTH GRADE CHILDREN'S RATIONAL NUMBER KNOWLEDGE IN A NON-VERBAL APPLICATION CONTEXT : THE DARTS GAME

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The task of assessing children's quantitative understanding of rational numbers appears to be difficult, perhaps because a quantitative notion evolves from, and is relevant for, numerous and diverse situations. Indeed, instead of a concept of rational number, it is more adequate to speak of a conceptual field (Vergnaud, 1983), since the notion of rational number is comprised of a number of subconcepts and subaspects (Kieren, 1976). A (positive) rational number can be characterized as the property common to all fractions in an equivalence class, and assessment of children's notions of fraction equivalence can contribute insights into their quantitative conceptions of rational number. Also relevant to understanding rational number conception is the ordering of fractions (as representatives of rational numbers).

Vergnaud (1983) points out the importance of obtaining insights into children's use of mathematical knowledge through application situations, since knowledge to be learned in mathematics has to be related to situations for which this knowledge is functional. In a series of studies (Wachsmuth, Behr, & Post, 1983), situations were constructed that required a coordinated application of several subconcepts of rational number. This article focuses on children's abilities to

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demonstrate understanding of rational number concepts in a particular application situation involving ordering and the denseness property of rational numbers on a number line.

The study

The study was conducted by the Rational Number Project (Behr, Post, Silver, & Mierkiewicz, 1980) during 1982-83 and is part of a set of studies arising from an extended teaching experiment across part of the subjects' grade four and grade five school years. Starting about the middle of subjects' grade four years, the teaching experiment extended over 30 weeks. Subjects were (a) eight children from an elementary school in DeKalb, Illinois, selected to reflect a range of ability, and (b) a classroom-size group of 34 homogeneously grouped, middle-ability children from a relatively high-achieving school in Minneapolis, Minnesota.

Based on the Perceptual Variability (multiple embodiment) Principle of Dienes (1967), the instruction included the carefully planned use of various manipulative aids and representations : (a) colored fractional parts of circular and rectangular models, (b) paper folding, (c) centimeter rods, (d) a discrete model using counting chips, and (e) the number line. These were used so as to highlight an important aspect in the process of knowledge building ; namely, making translations among and between various modes of representations for mathematical ideas. Based on the Mathematical Variability Principle (Dienes, 1967), the instruction included the rational number constructs of part-whole, quotient, measure, and ratio.

Data from videotaped clinical interviews with the eight children from DeKalb and eight of the Minneapolis children form the basis of this article. The data were gathered during an assessment in the second project year (i.e., during the children's 5th grade), after about 27 weeks of instruction in the project. At the time of this assessment, the children had had extensive experience with order and equivalence of fractions in symbolical task settings. Insights about children's performance with such tasks were available from earlier work in the project (Behr, Wachsmuth, Post & Lesh, 1984 ; Post, Behr, & Wachsmuth, 1983 ; Post, Wachsmuth, Lesh & Behr, 1985).

The instruction was based on several embodiment materials, including : color-coded circular and rectangular fractional pieces (each faction 1/n, for n from 1 to 10, 12, and 15 was represented with a different color), paper folding, Cuisenaire rods, colored counting chips, the number line, and graph paper. An outline of the instructional sequence is given in figure 1. The DARTS tasks were included in the interview following lesson 17.

Fig. 1. Design of instructional materials.

Lesson	Embodiment	Activity
1	Color-coded circular pieces	Name pieces Compare sizes Observe that as size decreases, number to make whole increases Observe equivalence
2	Color-coded rectangular pieces	Name pieces Compare sizes Observe that as size decreases, number to make whole increases Observe equivalence
3	Color-coded circular and rectangular pieces	Observe similarities and differences between circular and rectangular pieces Translate between circular and rectangular pieces
4	Color-coded circular and rectangular pieces	Attach unit fraction names (one-fourth, one-fifth) to parts of whole
5	Paper folding with circles and rectangles	Attach fraction names to shaded parts of folded regions Learn names for unit and proper fractions

Lesson	Embodiment	Activity
		Associate names with color-coded parts and with shaded parts of folded regions Note similarities and differences
6	Cuisenaire .ods	Attach fraction names to display of rods Note fractions as sums of unit fractions Compare display with colored pieces and paper folding Investigate real-world problems
7	All materials from Lessons 1 to 6	Review, using all four embodiments Identify proper fractions orally, in written form, symbolically, and pictorially Translate from one mode to another
8	Chips	Review division as partitioning (18 – 3 = 6 represents 18 chips, 3 groups, 6 in each group) Represent fractions by covering equal-sized groups of chips Associate fractions with amount covered (3 of 4 groups covered is 3/4)
9	All materials from Lessons 1 to 7	Translate to any mode, using any embodiment, a fraction represented with physical objects, orally, or with a written symbol
10	Color-coded pieces Paper folding Chips	Represent improper fractions using pieces Translate between improper and mixed number notation orally and in writing Translate representation to paper folding Translate representation to chips
11	Number line	Associate whole numbers, fractions, and mixed numbers with points on number line Convert improper fractions to whole or mixed numbers Determine equivalence

Lesson	Embodiment	Activity
		Add fractions with same denominators
12	All materials from Lessons 1 to 11	Using pieces, chips, rods, or pictures of parts of a unit, construct the unit
13	Chips Paper folding	Represent a model for multiplication of fractions Generalize to algorithm with product of numerators and product of denominators
14	Chips Paperboard with two « set circles » marked A and B	Represent a ratio between A and B with red chips in A and blue chips in B Show equivalent ratios
15	Equipment for experiments with ratios	Make table of values and extrapolate to unknown data : length of drinking straws and a projected image, number of a given color of M & Ms to the total number, ratio of length of centimeter rods to surface area
16	Chips Paperboard with two « set circles » marked A and B	Solve missing value proportion problems Solve comparison of ratio problems
17	Graph paper	Graph data Define proportion in terms of graphs

Assessment with DARTS given after Lesson 17.

Previous structured interviews had assessed students' abilities to answer direct questions concerning writing of equivalent fractions, comparisons of equivalent and non-equivalent fractions, and ordering of non-equivalent fractions. These questions were designed to elicit a repertoire of particulars of knowledge. DARTS was selected as a situation in which direct questions were not asked but in which students would apply their repertoire of procedures and strategies. Such applications would, of course, require that students decide which particulars to apply and then coordinate those particulars to reach the DARTS goal (i.e., pop the balloons). Results from this setting complement findings about children's understanding of the number line in a purely symbolical setting (Bright, Behr, Post, & Wachsmuth, in press). In the instruction presented to the students, there had been neither prior use of these or similar tasks nor formal teaching of possible strategies for completing the tasks.

The tasks

A microcomputer video game, DARTS (Apple Computer, 1979) originally developed for instructional purposes, was used to present taks requiring the application of rational number knowledge. Each screen in the game consisted of a vertical (positive) number line segment with randomly generated beginning and ending marks and one other mark at some other point of the number line. At three random positions, balloons were pictured as attached to the number line. The task was to pop each balloon by keying in a fraction or a mixed number [i.e., a number of the form (integer, fraction)] to shoot a dart at a balloon corresponding to a location on the number line. Since the balloons were « larger » than points, a balloon could be popped by a « close » shot (altough DARTS permits a choice of « large » or « small balloons », large ones were used because none of the students had had any previous experience with DARTS). The balloons were all the same absolute size on the screen, so the « closeness » of a shot wich popped a balloon was a function of the difference between the end points of the particular number line. That is, a balloon obstructed a greater length on a number line from 1 to 8 than on a number line from 2 to 3. The students had no way of estimating the length obstructed other than observing the outcomes of their shots.

Being able to pop a balloon by a close shot was considered an advantage rather than a disadvantage because close shots (e.g., a balloon at 5/8 popped by a shot at 9/15) might reveal much about a subject's notion of rational number size. That is, being able to give estimates of rational number size is regarded as closely related to quantitative understanding of rational numbers (e.g., Behr, Wachsmuth, & Post, 1985). In addition, popping balloons by close shots is a motivational advantage in that positive feedback is provided to accurate, though not exact, estimates of rational number size. A sample task in shown in Figure 2.

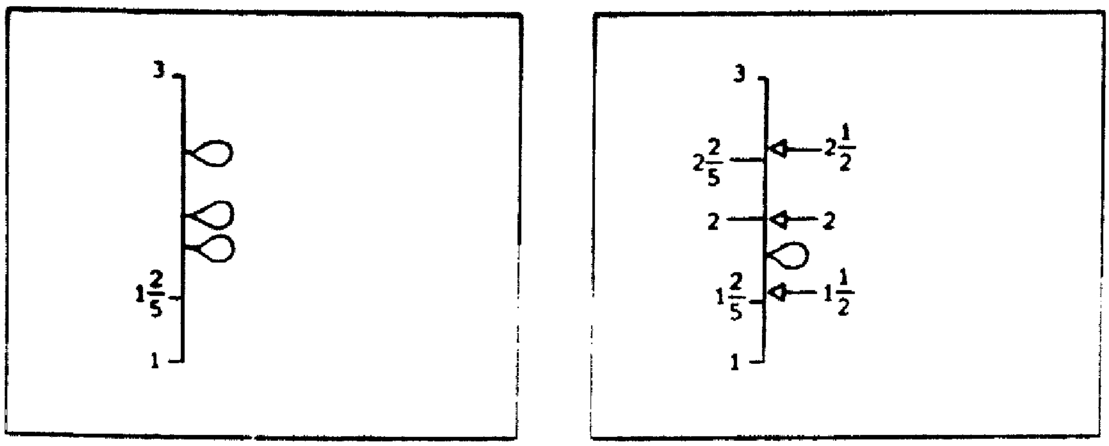

Fig. 2. DARTS screen

Fig. 2a. DARTS screen with balloons attached at $1\frac{3}{4}$, 2, and $2\frac{2}{5}$

Fig. 2b. The same screen after 3 shots have been made at $1\frac{1}{2}$, 2, and $2\frac{1}{2}$

The number line segments were randomly generated by the DARTS program and consisted of one or more units. Each subject was presented a sequence of three screens on which appeared one such segment. Since the unit size varied randomly from screen to screen, memorization of the unit size was unlikely. The subjects' performance in attempting to pop the balloons was video recorded, along with unstructured interview dialogue between an experimenter and each subject. Subjects were encouraged to « think aloud » throughout the tasks.

If a subject's attempt at popping a balloon was unsuccessful (i.e., the arrow missed the balloon), the rational number corresponding to the point where the arrow hit was displayed on the screen as a label on the number line. That is, each attempt a subject made was followed by immediate feedback. A subject's next attempt was expected to build on this feedback, so the DARTS task was expected to elicit behaviors which would provide insights into how students organized their cognitive structures for rational number knowledge. One aspect of particular importance to these tasks is the denseness property of rational numbers. In their attempts to get closer and closer to a balloon, subjects should realize and demonstrate implicit understanding of this property.

In some sense, the DARTS game exposes children to situations similar to those in the « explorer game » (« jeu de l'explorateur ») discussed by Brousseau (1981, p. 74), except the role of the adversary is taken by the computer. The DARTS tasks required the size of a rational number to be associated with a length. The difficulty frequently consisted in finding fractions in an interval between two others, in order to close in on the unknown target position.

Major portions of the Rational Number Project instruction were aimed at developing pupils' understanding of fractions (as one representation of rational numbers particulary prominent in the U.S.). We searched for a means of assessing children's understanding in which the focus was on the flexibility with which knowledge could be brought to bear in applied situations. DARTS occurred to us to be one way whereby knowledge could be elicited non-verbally and without focusing the pupil's attention with specific questions.

Results

Descriptive data

Before examining sample transcripts, it seems useful to get an overall picture of the performance of the subjects. In Table 1, the number of shots per screen (i.e., three balloons) and the average number of shots taken are presented (all names are fictitious). Data for the first three screens only are given for each subject in Tables 1-3 since some subjects completed only three screens (of course, a perfect score on a screen is three shots, though it is possible when the balloons are very close together to pop more than one balloon with a single shot).

Subject	Site[a]	Shots/Screen	Average
Kristy	D	4, 3, 3	3.3
Jeannie	D	4, 3, 7	4.7
Bert	D	6, 4, 5	5.0
Andy	M	5, 4, 6	5.0
Brett	M	6, 4, 6	5.3
Margret	M	5, 4, 7	5.3
Mack	D	3, 7, 7	5.7
Richard	M	9, 4, 5	6.0
Tricia	M	3, 4, 12	6.3
Joan	M	8, 6, 5	6.3
Ted	D	4, 9, 6	6.3
Erica	M	11, 9, 5	8.3
Jeremy	D	9, 14, 6	9.7
Till	M	8, 15, 9	10.7
Jessie	D	11, 14, -[b]	12.5
Terri	D	17, 9, 12	12.7

[a] D = DeKalb site, M = Minneapolis site
[b] No third screen was completed

Table 1 : Numbers of shots on the DARTS tasks

The variability in averages raises the possibility of potential differences in strategies ; of particular concern is the effect of the information presented by the program on the choices of fractions used by subjects. In this version of DARTS, the students' shots are written to the right of the vertical number line, while the rational numbers identifying the end points (always whole numbers), the given reference point, and the balloon positions (shown only after a balloon is popped) are written to the left of the number line (Figure 2). Thus, there is a clear, visual differentiation between the

« authoritative » information presented by the program and the guesses of the student.

One measure of the influence of the information displayed by the program is the similarity of students' choices of denominators to those denominators presented by DARTS. Five categories were used to classify the student choices of number value for a balloon's location :

S = same denominator as one already used by DARTS,
M = multiple of a denominator used by DARTS,
F = factor of a denominator used by DARTS,
W = whole number in whole number form,
O = all other choices.

When a whole number was typed as a fraction with 1 as the denominator, that choice was classified as « O ». When only whole numbers had been presented by DARTS and the student typed a whole number in whole number form, that voice was classified as « S ». The data generated by this classification scheme are presented in Table 2.

Two patterns seem to emerge from comparison of first and third tasks. First, the better scorers seemed to shift from using the same denominators to using multiples of the denominators presented by DARTS. Second, the poorer scorers seemed to start out by trying to use multiples, but tended to abandon that attempt. In addition, the poorer scorers seemed to make more « O » choices than the better scorers.

As a follow-up to these data, counts were made of the number of times students repeated a previously made shot ; that is, chose a fraction either equivalent to or identical to a fraction already appearing on the screen. These data are presented in Table 3. The pattern of « wasted » shots by the poorer scorers is probably not surprising.

In order to investigate the skills of subjects to narrow in on a target (that is, to use the denseness property for rational numbers), we consider it necessary for subjects to have taken at least three shots aimed at the same target, so that subjects'

Task				
Subject	1	2	3	Totals
Kristy	2,0,0,2,0[a]	0,0,0,1,2	0,3,0,0,0	2,3,0,3,2
Jeannie	2,0,0,2,0	2,0,0,0,1	1,3,0,1,2	5,3,0,3,3
Bert	3,0,0,0,3	3,0,0,1,0	1,1,0,1,2	7,1,0,2,5
Andy	3,0,0,0,2	1,0,1,2,0	1,0,0,2,3	5,0,1,4,5
Brett	2,0,0,1,3	2,0,0,1,1	1,1,1,2,1	5,1,1,4,5
Margret	1,2,0,2,0	2,0,0,0,2	0,5,0,0,2	3,7,0,2,4
Mack	0,0,0,1,2	3,0,1,2,1	2,2,0,0,3	5,2,1,3,6
Richard	1,6,0,0,2	0,0,0,0,4	4,0,0,1,0	5,6,0,1,6
Tricia	1,0,1,1,0	1,0,0,2,1	4,0,0,0,8	6,0,1,3,9
Joan	1,7,0,0,0	2,0,0,1,3	1,4,0,0,0	4,11,0,1,3
Ted	0,0,1,1,2	2,1,0,0,6	5,1,0,0,0	7,2,1,1,8
Erica	3,5,0,1,2	3,2,1,0,3	3,0,0,1,1	9,7,1,2,6
Jeremy	3,2,0,0,4	0,8,0,0,6	2,2,0,0,2	5,12,0,0,12
Till	0,0,0,3,6	3,7,0,1,4	1,2,0,0,6	4,9,0,4,16
Jessie	1,4,0,0,6	1,4,0,1,8		2,8,0,1,14
Terri	1,2,2,0,12	0,0,2,3,4	4,2,2,0,4	5,4,6,3,20

[a] The five numbers q, r, s, t, u represent the following :
q = number of S choices
r = number of M choices
s = number of F choices
t = number of W choices
u = number of O choices
(The explanations of choices are given in the text.)

Table 2 : Classification of Moves

We wish to thank the very careful reviewer who discovered the discrepancies between numbers in Tables 1 and 2. In order to resolve these discrepancies, we went back to the original data and discovered an error in coding. We have corrected this and, in turn, have corrected the entries in Table 1 and Table 2.

success at placing fractions in successively smaller intervals can be judged (of course, the shot that pops a balloon is closer than any prior shot, so inclusion of last shots in the analysis did not seem productive). In such cases, successive differences between shots and targets can be analyzed for decreasing or increasing absolute values. An arbitrary deci-

Subject	Frequency
Kristy	0
Jeannie	0
Bert	0
Andy	0
Brett	0
Margret	2
Mack	0
Richard	4
Tricia	4
Joan	3
Ted	1
Erica	1
Jeremy	4
Till	3
Jessie	2
Terri	6

Table 3 : Use of Repeated Shots

sion was made to require at least five shots for a target rather than the minimal three, in order that consistency of understanding be demonstrated. That is, the data in Table 4 are for those subjects who were not able to close in on a balloon within four shots.

In 8 of the 15 cases, the number of decreasing differences outnumbered the number of increasing differences, in 4 cases the order of these numbers was reversed, and in 3 cases these numbers were equal. Not surprisingly, 11 of the 15 cases were generated by the five poorest scoring students, so these results are difficult to interpret.

Protocal data

Examination of segments of episodes in which children responded to the DARTS tasks and to the interviewer's questions will further illuminate understandings. The episodes were selected to exemplify different levels of children's thinking in the context of rational number order and fraction

Subject	Number of balloons with ⩾ 5 shots	Performance[a]			
Kristy	0				
Jeannie	1	3,1,0			
Bert	0				
Andy	0				
Brett	0				
Margret	0				
Mack	0				
Richard	1	2,1,0			
Tricia	1	3,3,1			
Joan	0				
Ted	1	2,1,0			
Erica	2	3,0,1	2,0,2		
Jeremy	2	1,3,0	4,4,0		
Till	1	3,2,0			
Jessie	2	1,2,0	5,1,0		
Terri	4	2,3,0	2,0,1	4,4,1	3,4,0

[a] In each sequence of three numbers x, y, z :
x = number of times a difference decreased, as compared to the previous difference
y = number of times a difference increased, as compared to the previous difference
z = number of times the difference did not change, as compared to the previous difference

Table 4 : Skill at closing in on a target

equivalence. Except for one case (Jessie), episodes were selected from screens other than the initial one ; that is, when the children had gotten organized on the task and patterns in their behaviors were more easily recognized.

The first episode involves Kristy after she was presented with the number line : (<u>5</u>, 5-1/9, <u>5-1/3</u>, 5-5/8, 5-4/5, <u>6</u>), that is, a 5-6 number line segment with a further label at 5-1/3, and balloons attached at (non-labeled) points 5-1/9, 5-5/8, and 5-4/5 (this screen was the fifth one Kristy had seen so it is not included in the data of Tables 1, 2 and 3). Especially notable about Kristy's thinking in this excerpt is the flexibility with, and « automatic » reference to, equivalent fractions. It appears that Kristy is aware of an unlimited set of equivalent fractions and is able to think about a number of them automatically. In some cases she gives evidence, especially with pauses in the explanations, that she is using some computation to generate an equivalent fraction. She appears to be completely comfortable with using different fraction names for the same point on the number line.

Kristy : Oh boy, that's one-third (iterates the distance from 5 to 5-1/3 along the number line) and that (pointing to the balloon at 5-5/8) would be five and two-thirds. (The dart is projected and misses)... (Taking aim at the same balloon)... about 5-3/6.

Interviewer : How did you think to come up with five and three-sixths ?

Kristy : ...Well, I thought it (pointing to 5-2/3 on the number line) would be equal to four-sixths ; and then, you want it to be lower (but) I didn't want to take a third lower (darts misses)... OK, five and two-thirds is equal to... six-ninths... I'm going to take it (i.e., 5-7/12 for the next shot) because, that's a little bit <u>less</u> (than 2/3) (shot misses). OK, two (-thirds) is equal to... ten... ten-fifteenths and so nine-fifteenths (i.e., for the next shot) (shot hits balloon at 5-5/8).

At this point, 5-3/6 is one of several fractions marked on the number line.

Kristy : (Takes aim at the balloon at 5-1/9 by iterating the distance from 5 to 5-1/9 up to 5-3/6). That (pointing to the balloon at 5-1/9) will be five and one-eighth... because that (pointing to 5-3/6) was one-half and that took about four (i.e., iterations of the distance from 5 to 5-1/9) to get there, so that would be eight all across.

The next episode involves Bert at screen three (<u>1</u>, 1-1/3, <u>1-1/2</u>, 1-3/5, 1-3/4, <u>2</u>). [The notation (x, y) means shot y taken at balloon at point x, with a popped balloon indicated by *]. Bert had made shots (1-3/5, 2), (1-3/4, 1-2/3), (1-1/3, 1-2/6)*. Taking aim the balloon at 1-3/5, Bert explains :

Bert : One and two-thirds is more than one and two-sixths... (points to the balloon at 1-3/5). What's between a half (using the fixed point 1-1/2) and two-thirds ?... It'd be one and three-fifths.

Bert gives no overt indication of how he arrived at the fact that 1-3/5 is between 1-1/2 and 1-2/3. Since he earlier referred to 2/6, one conjecture is that he thought of 1/2 as 3/6 and then chose 3/5 because it is greater than 3/6.

In the very next episode we get a feel for Bert's sense of rational number size. Presented with screen four (1, 1-1/5, 1-1/3, 1-1/2, 1-3/4, 2), he has made the following shots : (1-1/5, 1-1/6)*, (1-1/2, 1-3/5). Like Kristy, Bert makes spontaneous use of equivalent fractions ; he also displays a good application of rational number order to the number line. He orders three fractions after indicating that one of them is « just over » the least of the three, while another one is « more » (i.e., more than « just over »). We observe Bert's strong imaginative base for his concept of fraction size which is evident by his language.

Bert : (Taking aim at the balloon at 1-1/2)... something between one-third (i.e., 1-1/3) and one and three-fifths... one and three-fifths is just over one and one-half and two-thirds is more than a half, so uh... one and three-sixths, same as one and one-half.

Next Bert measures on the number line by using his fingers. Since 1-1/6 has been marked, he iterates the distance from 1 to 1-1/6 up the number line and finds that 1-5/6 takes him above the target.

Bert : It couldn't be one and five-sixths ; one and five-sevenths.

Interviewer : Tell me how you chose one and five-sevenths.

Bert : ...Since the pieces are smaller... one and five-sevenths would be a little more down (i.e., less than 1-5/6).

In the following episode, Mack, a middle scorer, displays his ability to apply concepts of rational number order in relation to a quantitative concept of rational number. The task deals with screen five (1, 1-2/5, 1-2/3, 1-8/9, 2-1/3, 3) ; the following shots have already been made : (1-2/3, 1-4/5), (1-2/3, 1-3/5)*, (1-8/9, 2), (1-8/9, 1-3/4).

Mack : No, I'm short (of balloon at 1-8/9)...

Interviewer : See that's below 1-4/5 (a previous shot)... You have to get closer to two.

Mack : I know, ...wait... one and... seven-eights (1-8/9, 1-7/8)*.

Interviewer : Why did you say...

Mack : ...I thought, it's got to be one away (i.e., one fractional part away from 2)... one thing away from something... I thought it was small pieces (i.e., the eighths are small pieces).

It seems that Mack is closing in on the balloon at 1-8/9 by consecutive shots at 2, then at 1-3/4, and then at 1-7/8. That is, having realized that the target position is below 2, he tried to shoot so as to be one fractional part away from 2.

In the next episode, Mack exhibits considerable knowledge about the number line structure and rational number order. The task is screen six (7, 7-1/3, 7-1/2, 7-4/7, 7-5/7, 8.

Mack : (Measures the line with his fingers) Holy smokes ! For the top one (i.e., the balloon at 7-5/7) it'd be seven and five-sevenths... (measures line between 7-5/7 and 7-4/7). That's got to be one-seventh so go up (measures number line from the bottom). Seven and three-sevenths (points to the balloon at 7-1/3) seven and three-sevenths... (shot misses) at least that gets me somewhere.

Mack's next shot gives evidence for his using the fact that the balloon at 7-1/2 is now bracketed by marks at 7-3/7 and 7-4/7. The shot is (7-1/2, 7-1/2)* which is midway between 7-3/7 and 7-4/7. It might be possible that Mack has used fraction equivalences in thinking of one-half as three and one-half-sevenths, or in thinking about 3/7 and 4/7 as 6/14 and 8/14, respectively, and then chooses 1/2 as 7/14.

In the interview with Jessie, who is a low scorer, we see a level of functioning with the concept of fraction equivalence which might be called latent, though in her case, the knowledge base seems weak. (Jessie completed only two tasks). That is, her use, generation, and recognition of equivalent fractions occurs only after she is directly prompted by some external source such as the interviewer or the computer screen to consider equivalent fractions. The following protocols deals with screen one (3, 3-1/9, 3-3/8, 3-1/2, 3-3/5, 4), though similar behavior was also observed on screen two.

Jessie : [Aims at the balloon at 3-1/9, shoots (3-1/9, 3-3/6)].

Interviewer : (After the 3-3/6 dart hits and Jessie can observe that 3-3/6 hits the same point as 3-1/2.) What can you say about this (3-3/6) ?

Jessie : It's equal to three and one-half... [Indicates shot (3-1/9, 3-2/3)]...

Interviewer : (Points to 3-1/3 and balloon at 3-1/9) Can you name a mixed number less than three and one-third ?

Jessie : Three and... three and two-fourths.

Interviewer : That would be below three and one-third ?

Jessie : Wait... wait, three and one-fourth... wait three and one-seventh (laughs).

Interviewer : Why do you say three and one-seventh ?

Jessie : Because the pieces are smaller. [Shot (3-1/9, 3-1/7) misses above the target] [Indicates (3-1/9, 3-2/4) as the next shot].

Interviewer : Where do you think it will go ?

Jessie : (Points to the balloon at 3-1/9) Right here. (Shot misses and records 3-2/4 at same point with 3-1/2 and 3-3/6.)

Interviewer : Why do you thing it hit the same point as three and one-half ?

Jessie : 'cause they are equal.

Jeremy also shows behavior which suggests a level of thinking on fraction equivalence that could be named latent. An instance of this can be found in the next episode. Moreover, we observe in Jeremy's protocol a weak concept of fraction order ; he is very doubtful about the order of 1/2 and 4/13. The display is screen two (0̲, 2/5, 2/3, 4/5, 1̲, 2̲). The episode begins after Jeremy was successful in popping two balloons with three shots (2/5, 2/5)*, (2/3, 3/10), (2/3, 9/15)* and has tried to hit the balloon at 4/5 with several attempts, including 1/2.

Jeremy : (Again takes aim at balloon at 4/5) Six-twelfths.

Interviewer : (After shot hits and marks the same spot as 1/2) Why will six-twelfths go through the one-half ?

Jeremy : One-half, two-fourths, six... (twelfths)... [shoots (4/5, 6/12), (4/5, 4/9), (4/5, 6/18), (4/5, 1/2), then suggests (4/5, 4/13)].

Interviewer : Would this (4/13) be more or less than one-half ?

Jeremy : It would be a little bit more, I have the feeling, I hope, wait... wait, wait, wait ; five-thirteenths (shot misses).

Observations and discussion

Different levels of children's thinking seem characterized not so much by the presence or absence of knowledge of order and equivalence, but rather by the degree of coherence and ease of application of such knowledge. In both Bert's and Kristy's behavior the spontaneous use of equivalent fractions is apparent. In particular, Kristy's flexibility with, and automatic command of, equivalent fractions is notable. The sample interviews, however, were taken from late screens ; and as the data of Table 2 indicate, shifts in strategies in later screens to take advantage of equivalent fractions seem common for the good students. Too, both Bert and Kristy have excellent command of rational number order as is demonstrated, for example, in their successive refinement of fraction size in approaching a target. Though it was never thematized in instruction, the denseness property of rational numbers seems to be realized in these subject's thinking.

Mack's behavior, as an example of middle level scoring, suggests considerable knowledge of the number line and of rational number order which he is able to relate to the task situation ; for example, he iterates unit fractions to measure on the number line. Drawbacks in his performance seem to be due to his size perception which is still too rough.

The performance of both low subjects is characterized both by weak and unreliable knowledge about rational number order and by the possession of latent knowledge about fraction equivalence which only becomes activated by exterior prompts. Perhaps this diagnosis of latency of relevant knowledge is most apparent in the context of low performance at DARTS. Repeating fractions already on the screen suggests that these students may have begun to develop effective strategies (i.e., they recognize the need to use equivalent fractions) but they do not know how to complete the strategy. Perhaps memory overload is a problem for these students ; that is, they may mentally partition the number line, but the need to manipulate the partition exceeds available short-term memory capacity.

The DARTS task elicits somewhat different behaviors than more standard number line tasks (e.g., Bright, Behr, Post, & Wachsmuth, in press). For example, in DARTS it is not possible to count partition marks ; students must generate mentally any partitioning they wish to use. Such a partitioning is not visible on the screen. On standard number line tasks, we observed a similar difficulty at dealing with equivalence of fractions. It appears to be difficult for (fifth-grade) children to impose more partitions on the diagram than are directly suggested by the denominators of the fractions displayed on the screen.

Conclusions

As with many cognitive skills, there seem to be differences between the ways that more able students approached DARTS (an applied situation) as opposed to the less able students. The more able students first seemed to use the denominators of fractions provided by DARTS itself. Then as they became familiar with the task, they shifted toward making use of their knowledge of equivalent fractions. The less able students seemed to assume that use of equivalent fractions was required, but they did not seem to know how to use that knowledge. They behaved more as if merely identifying equivalent fractions was sufficient.

A natural concern, then, is whether the less able students would ever « catch on » to the proper application of equivalent fractions. More specifically, would explicit instruction be needed to allow less able students to succeed with DARTS ? This study does not provide sufficient information to answer these concerns, but a reasonable guess might be that considerable instruction would be required.

One procedure that might be used is to insist that students systematically predict where their shots will appear on the screen. For instance, students might say that the shot will be between fractions X and Y, or it will be below or above fraction Z. The program itself could be modified to monitor these predictions and to keep records of their accuracy.

Another area of concern raised by the results is how information provided by DARTS was processed differently by the students at different ability levels. Further probing of the ways that students interpreted the feedback might illuminate the processing skills of students. A parallel researchable question would be how different kinds of feedback might influence the ways that students chose shots for the game. For example, if the program suggested equivalent fractions either for those it generated or for the shots the students made, or if the program refused to allow repeats of shots, the behavior and performance of the students might be considerably different. It would also be possible to adapt DARTS so that students could temporarily impose a partitioning on the number line. (Students should also have control over which partition to use and how long the partitioning remained on the screen.) The selections made by students might reveal much about their understandings of equivalence.

The level of flexibility and spontaneity of use of knowledge of equivalent fractions is clearly different among the students interviewed. This study points out that such flexibility is a great advantage for students ; those that had this flexibility were noticeably more accurate at hitting the targets. However, the ways that instruction (or general knowledge or memory, processing capabilities, skills, etc.) influences the development of such flexibility is still in need of further research. The use of multiple embodiments of fraction concepts in the teaching experiment of the Rational Number Project is probably one step in the direction of helping students develop this flexibility, but it clearly did not succeed with all subjects. That is, multiple embodiments may be a necessary but not sufficient condition for developing flexibility.

Although the data from the DARTS interviews yield some information on children's views of the denseness property, further research seems called for. Students seemed generally able to decrease the distance between a shot and the target aimed for, but it is not clear from the interviews whether the techniques they used were based mainly in a symbol system or in conceptualizations of fractions as a distance or length on the number line. The conceptual base seems particulalry important if further generalization is to be expected from children. DARTS was used in this study as a vehicle for allowing students to demonstrate their skill in applying their knowledge ; determining their conceptual base was secondary for the interviews.

Didactical consequences

The Rational Number Project (of which this study is a part) has been focused on developing students' understanding of fractions. DARTS was selected as a means of eliciting students' applications of their knowledge of fractions rather than as a means for providing more instruction. Hence, the comments that follow represent our best inferences about how DARTS might be used in instruction. These inferences follow from the conclusions of the previous section.

One might expect that the use of DARTS will become more popular with the increasing availability of microcomputers in schools. From the base of data provided in this study, it seems that teachers would be well advised to take an active role in the incorporation of such computer programs in instruction. It is probably not sufficient simply to let students play the game on their own and to use only those strategies that they first attempt to apply. In particular, when equivalent fraction knowledge is latent, teacher intervention seems essential. The first recommendation, then, is to encourage students to verbalize about the ways they think about the DARTS situation. In this way, sharing of strategies and of insights can occur, and students will have a base of mutual experiences on which to build more sophisticated strategies.

A second recommendation is that teachers may want to demonstrate their own thinking, or the thinking of their better students, so that all students have ressources available to use to modify their cognitive structures. In particular, appropriate ways of using knowledge of equivalent fractions would seem to be very important in such modeling. It is not enough, for example, merely to know that 1/2, 2/4, 3/6, 4/8, etc., are all equivalent ; visualization of these fractions on the number line may also be required in order to know how to use this information to select a fraction « slightly » bigger (or smaller) in order to get closer to a target.

Related to this use of equivalent fractions is knowing that on a number line, a fraction represents a distance from the origin. The label is attached to a point, but the fraction represents a distance. That is, the fraction must be viewed in two contexts simultaneously. Compounding the application of this knowledge is that typically in DARTS the origin does not appear on the screen. There is no evidence from the study that students visualized where the origin would be (physically) relative to the number line on the monitor screen. No student mentioned any such visualization, and none of the language used by any of the interviewers suggested that such visualization might be useful. However, encouraging this spatial location of the origin might have been a useful technique for some of the subjects. Demonstrating the process of visualization might be part of the modeling that would accompany the use of DARTS in the classroom.

Most classroom instruction involves a verbal communication between teacher and students where usually the teacher sets the tone of the nature of verbal statements that convey about knowledge and meanings. Different from that, the DARTS game offers a « reactive environment » that communicates task and feedback nonverbally. However, as is well-exemplified in the above interview segments, the DARTS setting often brings about spontaneous verbal utterings of children. Using their own words, they form propositions that link their domain-specific knowledge to applied situations. The reactive environment of DARTS serves to reinforce, or refute, their assumptions about rational number size and may well be a medium which enhances flexibility of their utilization of relevant knowledge structures.

Incorporating application situations like DARTS into regular instruction on fractions and rational numbers would seem to be to the benefit of most students. Encouraging flexibility by providing practice with applications ought to provide a necessary base for generalization. Yet, too much instruction on a particular situation runs the risk of reducing it from an application setting to a drill and practice setting. Probably at that point it would lose some of its effectiveness at stimulating flexibility, and new application situations would be required. Relating the instances of fraction concepts among various application situations will remain as one important task for teachers.

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