INCONSISTENT STUDENT BEHAVIOR IN APPLICATIONAL SITUATIONS OF MATHEMATICS

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APPLICATIONAL SITUATIONS OF MATHEMATICS

Mathematics is a field that frequently requires a coordination of diverse knowledge structures which were acquired at different occasions, in order to successfully deal with a task situation. Vergnaud (1983) has pointed out the importance of obtaining insights into children's use of mathematical knowledge in applicational situations, since the knowledge to be learned has to be related to situations for which this knowledge is functional. It is not sufficient that children can demonstrate the use of certain mathematical knowledge in pure settings, if one wants them to be able to utilize this knowledge in situations that are not specifically designed for, or do not explicitly call for, application of a given rule. Even when all necessary pieces are known, lack of coordination of relevant knowledge can be a reason of failure.

INCONSISTENCY OF STUDENT BEHAVIOR

Frequently one finds that learners master the standard problems in a certain field but stumble in contexts they have not encountered before, even though the knowledge necessary to master the situation should be available. It may happen that alternative knowledge frameworks in the mind of a learner are evoked by the situation and override knowledge more adequate for the task. One effect is that the answers to the same mathematical question posed in different contexts can differ. In this sense, the behavior of a student can be inconsistent across a variety of situations.

One attempt to shed light on this phenomenon is the work of Thomas Seiler (1973). Seiler has conducted a series of experiments showing that juveniles already thinking formally (in the sense of Piaget) are not always able to use formal thinking operations in all problem solving tasks (not even in all tasks used by Piaget). Therefore Seiler considers it necessary to introduce a "situation and range-specific factor" in the developing cognitive structure which inhibits its generalization. Seiler's statements include that while traditional cognitive theories can well explain the big leaps of generalization that are certainly observable, they tend to ignore the common fact that, without further guidance, an individual never can apply a given rule in all novel situations. Formal thinking structures arise from the individual's experience with specific problems in specific situations and barely reach an unrestricted, universal generality. Further, it is very probable that, within one individual and with respect to one subject domain, different thinking structures can coexist which can become activated alternatively, depending on the symbol system primarily triggered or cued by a situation (in particular cf. Seiler, 1973, p. 268).

As a consequence, we would not expect it to be a question of "yes" or "no" whether a student has acquired a certain concept or rule but expect that s/he might be unable to apply that concept or rule in all circumstances. To discover inconsistent behavior, then, does not mean to find that a student has not gotten a concept or rule so far, but rather, to find evidence for a restriction of the range in which the student's concept or rule applies. To identify the conditions and laws of inconsistent behavior means to prepare the grounds for remediation towards further maturization of the student's concept.

In this paper we shall discuss a student's inconsistent behavior in an applicational situation in the domain of rational number learning and attempt to clarify the conditions of its origin.

CONTEXT

The notion of rational number comprises a conceptual field (Vergnaud, 1983) that involves a large number of subconcepts and subaspects. Thus it constitutes a rich domain to study children's grasp and use of mathematical ideas. In a series of studies conducted by the Rational Number Project (see acknowledgment), situations were constructed that did not expressively call for, but required a coordinated application of, several subconcepts of rational number in order to succeed. One of these studies is the "Gray Levels Study" which was also the background for an earlier discussion of structures and mechanisms of children's mathematical knowledge (Wachsmuth, 1984). After 30 weeks of experimental instruction, each child in a group of sixteen 5th-graders was presented with a complex problem solving task, embedded in a video-taped clinical interview.

The task involved a set of 12 fractions, written as symbols a/b on little cards, which were said to represent ink mixtures with a parts black ink in b parts solution. The fractions were to be ordered by size and to be associated with stages on a scale of 11 distinct gray levels arranged by increasing "grayness" from 0% (white) to 100% (black) in stages of 10%. Presented were the fractions 0/20, 1/5, 2/7, 6/20, 2/5, 4/10, 6/15, 2/4, 4/8, 4/6, 6/9, and 12/15. Although the visual information could be used as a guidance, it was necessary to use the numerical information and apply various pieces of rational number knowledge in order to perform well on the task. Thus the gray levels task was expected to elicit how children bring their rational number knowledge to function in a complex applicational situation.

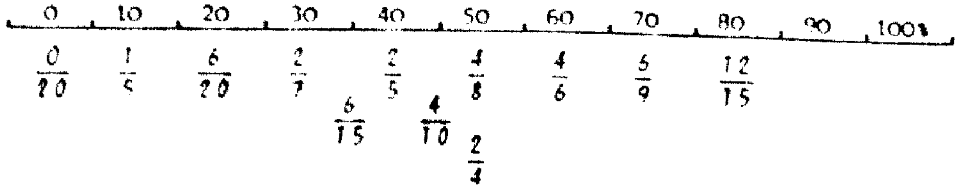
Three of the sixteen children were so successful that the average deviation in their card displacement was less than half a stage off the correct location. Other children were much less successful. The differences are assumed to be due to different ability in activating relevant domains of fraction knowledge and coordinating it on the task. A low performer, Terri, was discussed earlier (Wachsmuth, 1984). Here, a much better but still not perfect performer, Bert (not his real name), is the subject to generate further hypotheses about factors that impede optimal functioning of a developing cognitive structure.

A CASE OF INCONSISTENT BEHAVIOR

In his performance on the gray levels task, Bert, in general a relatively high-achieving subject, exhibits inconsistent behavior in the following respect. In the beginning Bert recognizes the equivalence of the fractions 2/4 and 4/8, and also of 4/6 and 6/9. At this time, he is able to infer that equivalent fractions should be associated with the same gray value. However, in the course of working the problem, Bert associates the fractions 4/6 and 6/9 which he had previously regarded as equal with different but adjacent gray levels at about the correct location. That is, independently of his knowledge about fraction equivalence, Bert exhibits a good perception of the size of these fractions. When he is reminded at his previous statement in the follow-up interview, he realizes his mistake and corrects it. The interview begins with Bert's initial observations on some of the fractions.

0. BERT: (Early-on, sorts the cards and puts 2/4 and 4/8 together on table.)
1. INTERVIEWER: You put two-fourths and four-eighths together?
2. BERT: (picks them up) They're equal.
3. INTERVIEWER: I see... you would put them on the same card (i.e., gray level)?
4. BERT: Yeah... (now puts 6/9 together with 4/6) These two are equal...

In his final placement of all cards at the scale, Bert has put 4/6 at the 60% level and 6/9 at the 70% level. That is, with respect to placement on the scale, Bert rates these fractions as very close but has lost sight of their equivalence. Similarly, he has put 2/5 at 40%, 4/10 at 45%, and 6/15 at 35% (see figure below; the percent marks were not present on the gray level scale).



5. INTERVIEWER: (after the whole task has been completed) You put four-tenths left of four-eighths, why did you do that?
6. BERT: Because four-tenths is less,... less than half a unit.
7. INTERVIEWER: And you put six-ninths right of four-sixths, why did you do that?
8. BERT: Because four-ninths-and-a-half would be half a unit...
9. INTERVIEWER: ... Before, you mentioned that they are equal ... four-sixths and six-ninths ...
10. BERT: Oh yeah, they are! (picks up 6/9 and 4/6) I think they'd be right there (puts both cards on 60%).
11. INTERVIEWER: Why did you put twelve-fifteenths over there? (points to 80%)
12. BERT: Because that's only three-fifteenths away from a whole.
13. INTERVIEWER: Why did you put six-fifteenths over here? (points to 35%)
14. BERT: Because that's... that's less than half a unit... (thinks)... umm, seven and a-half a unit would be... seven and a-half would be...
15. INTERVIEWER: What did you think about when you put six-twentieths? (points at 20%)
16. BERT: Because six-twentieths is greater than one-fifth; one-fifth equals four-twentieths.
17. INTERVIEWER: You put one-fifth right here (points to 10%) and two-fifths here (points to 40%)...
18. BERT: (points to 20% and 30%) Well see, there could be fractions between there.
19. INTERVIEWER: You put two-fifths there (40%) and four-tenths there (45%). What was your thinking?
20. BERT: Well, four-tenths would probably be ... well ... they're equal! (laughs, puts 4/10 over 2/5 on 40%) I didn't notice this.

DISCUSSION

With respect to Piaget's stages of cognitive development, Bert (age 10;11;24) could be considered transitional from the concrete to the formal-operational stage. In an earlier interview assessing children's ability to compare pairs of fractions and pairs of ratios presented in a symbolical form (cf. Wachsmuth, Behr, & Post, 1983), Bert had mastered each of 18 (2 x 9) tasks of varying difficulty. Thus, the above document seems suited to illuminate some critical aspects about Bert's developing cognitive structure with respect to the range specificity of his rational number knowledge. In the following we attempt an explanation for the inconsistent behavior exhibited by Bert.

1. Bert has a repertory of rules that he can use when the task is to make a judgment about equivalence or non-equivalence of fractions. One of his rules might be

formulated as follows: "Two fractions are equal, if one can be transformed into the other by multiplying its numerator and denominator by a common factor." For example, in the same interview session Bert found 4/6 and 20/30 (presented as symbols) to be equal, explaining "six times five is thirty and four times five is twenty." He also has a rule to determine the equivalence of two fractions which could match for 4/6 and 6/9 (i.e. where the corresponding terms are not multiples of one another), like: "Two fractions are equal, if a lower-terms fraction of one of them can be found from which the other one can be generated as a higher-terms fraction." For example, on a different item in the same interview, Bert stated that 8/10 and 20/25 are equal, explaining that "four-fifths is lower-terms, so four times five is twenty, and five times five is twenty-five."

To recognize the equivalence of fractions, however, requires that this rule is "activated", i.e. is attempted to be used in the situation. This might explain why Bert has stated that 4/6 and 6/9 are equal right in the beginning, but has "lost sight" of this fact when placing the fractions. We conclude that both of the following are true: (1) The fact "4/6 = 6/9" is no longer present in Bert's short-term memory when he places these fractions at the scale, and (2) the rule that could establish the equivalence of the two fractions is no longer "active". But when the interviewer reminds Bert at his earlier statement (line 9 of the transcript), Bert immediately adjusts his solution.

2. Bert has a repertory of rules that he can use to determine the sequence (in magnitude) of non-equivalent fractions. In several instances, Bert's explanations indicate a successful attempt to estimate the size of a fraction by using 1/2, 1, or some other fraction as a point of reference. Bert used 1/2 in his placement of 4/10 (line 6 of the interview transcript). He used 1 as a point of reference to place 12/15 (line 12), and 1/5 as reference point in placing 6/20 (line 16). Using this strategy may include the generation (not: recognition!) of equivalents of some fraction (e.g., 1/5 is transformed into 4/20 in line 16). A characteristic in his placement originating from this process of estimation is the little "slack" in placing the fractions.

Even without making use of all equivalences, Bert's placement of equivalent fractions was considerably close to correct. His explanations suggest that he has again employed his estimation strategy which can generate an approximate size judgment (estimate) independently of recognition of equivalences. For example, he apparently has used 1/2 (transformed to 4 1/2 ninths) in his placement of 6/9 (line 8), and (as 7 1/2 fifteenths) in placing 6/15 (line 14), and possibly also in placing 2/5 and 4/10. What results again is a little "slack" in placing fractions that are actually equal in size such as 4/6 and 6/9. (Had he noticed that 4/6 and 6/9 are equal amounts away from 1/2, namely by 1/6 and 1 1/2 ninths, respectively, application of this strategy would have been absolutely successful.)

This analysis leads us to the following conclusion: To find that two rationals are equal, there are two totally different ways that involve different repertories of rules: (a) find the fraction representatives to be equivalent, thus representing the same number; (b) make a judgment on the size of each fraction separately and find them to be equal in size. While the former is based on a procedure of algorithmic nature, it is natural that the latter, as it pertains to the field of estimation, depends on less precise arguments. It is heavily supported by Bert's explanations that in the course of working the problem he has switched to the second rule repertory and that this is the reason for his close-to-correct placement of 4/6 and 6/9, and of 2/5, 4/10, and 6/15. The rule that could have established equivalence of 4/6 and 6/9 as representatives of the same number was no more active at this point. But when the interviewer calls Bert's attention to 2/5 and 4/10 (line 19), he does recognize their equivalence.

3. From the above discussion results that at least two different sets of rules are part of Bert's cognitive structure. The fact that they are not always coordinated presumably has given rise to the inconsistent behavior observed. Two points that absolutely need discussion here are the following:
- Is that what happened a sole event, or does it indicate an important restriction in Bert's developing cognitive structure?
- Is that what happened unique to Bert, or could it be an interindividual phenomenon that is also relevant to other children?

Concerning the first point the following is notable. In a task-based interview conducted about one-half hour later under a different format (ratio symbols were used in place of fractions, e.g., 2:3 in place of 2/5, etc.), Bert displayed similar behavior: There he put 2:3, 4:6, and 6:9 at different but adjacent gray levels. Concerning the second point it is noted that four other subjects placed 4/6 and 6/9 at different but adjacent gray levels close to the correct position. So the observation apparently does not concern a factor totally unique to Bert.

In conclusion: At least, these observations suggest to further explore the idea that a disparity of the rule repertories a learner possesses can restrict the range in which the rules can be applied. The effect is a lack in the coordination of rules such that in some cases relevant rules are not applied because one set of rules has "taken over" control of the learner's doing. For a "good conception" of rational number size it would certainly be necessary that a learner obtains an awareness of the different sets of rules that are available to him/her to make judgments and that s/he is able to coordinate these. Exactly here is the point where remedial work can be invested to help the learner progress towards more stable performance also in complex applicational situations. For example, if Bert could be made to always check for equivalences before making a size judgment for one in a set of rationals, his performance should become very close to optimal.

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