# CHILDREN'S PERCEPTION OF FRACTIONS AND RATIOS IN GRADE 5

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## THE PROBLEM OF BEGINNING FRACTION INSTRUCTION WITH EMBODIMENTS

Common methods of fraction instruction use models that embody the idea of fraction; for example, regional models support fractions as parts of an equipartitioned whole, or counting models view fractions as a subset of a given whole set of counting items. Advantages of such approaches are that fractional ideas are based on concrete, imaginal representations which are related to the child's reality (e.g., pizza models). The disadvantage is that the conception of fraction derived from such models relies on a specified whole unit. Rather than thinking of 2/3 as a number, that is, of a quantity like 1 or 2, children would likely think of 2/3 <u>of a whole unit</u>. Likewise, in the context of fractions children tend to speak of whole numbers as 1 <u>whole</u>, 2 <u>wholes</u>, etc. The question emerges whether an instruction what uses part-whole embodiments really can provide a foundation for a concept of (positive) rational number. Current curricula very early call for application of rational number concepts, for example, in proportional reasoning. Proportional situations require an understanding of fraction which is independent of fixed units. An instruction aimed toward developing an understanding of rational number thus needs to deal with the question of how to achieve students' independence of fixed units for fractions.

## THE EXPERIMENTAL INSTRUCTION IN THE RATIONAL NUMBER PROJECT

The Rational Number Project is a multi-site effort funded by the National Science Foundation from 1979 through 1983. Instruction in a 30-week teaching experiment in 1982-83 was based on the multiple-embodiment principle (Dienes, 1971) and included the use of several types of manipulative materials, representational modes, and rational-number constructs (Behr <u>et al</u>., 1980). An important aspect was that <u>translations</u> between different modes of representation which subjects frequently were to make during instruction would facilitate the abstraction of rational-number ideas (Lesh <u>et al</u>., 1980). One focus of the project is to investigate the development of the number concept of fraction in children. The study presented in this paper is part of a larger set of studies aimed at assessing children's quantitative notion of (positive) rational number (see also Wachsmuth <u>et al</u>., 1983).

## RATIO AND PROPORTION IN THE STUDIES OF NOELTING AND KARPLUS

The Orange Juice Tasks (Noelting, 1980) and Lemonade Puzzles (Karplus et al. 1980) were experiments aimed at assessing the development of proportional reasoning in children and early adolescents. Noelting's study differentiate developmental stages for subjects from ages 6-16 years with respect to problem types that he hypothesized to depend on the development of ideas of ratio and proportion in children. We are interested in the cited studies since a subject's ability to deal with a proportional situation might be an indicator for the quantitative concept s/he has developed of the ratios involved. The concentration of a mixture resulting from, say, 4 parts orang juice and 2 parts water in some sense embodies the quantitative aspect of th ratio 4:2 -- it is a concentration of "4/6 orangy" (i.e. 4 parts sirup per 6 parts liquid; note that only a transformation of the part-part ratio to th part-whole ratio will yield a measure for the concentration). The comparison of the concentrations, for example, of a 4:2 and a 2:1 mixture requires that children realize that these ratios, though different in quantity, have the same value (are equivalent). At Stage II A in Noelting's hierarchy (1980) children begin to realize that there is an internal relatio between the two terms of a ratio (or between numerator and denominator of a fraction) whose value is independent of the total quantity of liquid. Consequently, an assessment of children's performance on mixture tasks coulc elicit to what an extent they employ strategies that are based on the "withi relation" (Noelting) of terms, in other words, exhibit children's quantitative understanding of ratio or fraction.

## CAN MIXTURE TASKS GIVE INSIGHTS INTO SIZE PERCEPTION?

Several reasons suggest an ambiguity over whether children's performance on the quoted mixture tasks adequately documents their understanding of proportion and rational-number ideas. (1) According to Karplus et al. (1980, p. 141) there is evidence that "consistent use of proportional reasoning is not a developmental outcome, but depends instead on overcoming task related obstacles." (2) Noelting and Gagné (1980) contrasted the orange juice tasks with the "Sharing Cookies" experiment which suggests the comparison of fractions rather than ratios, and with comparisons of numerica fractions, with identical numbers from one experiment to the other. They found low correlation between subjects, and differences between these and the ratio situations which they explain "by the greater importance of 'betwe relations in ratio and of 'within' relations in fraction" (p. 132). This suggests that children's performance might depend on the problem representation. (3) Comparative items (i.e., "which of two mixtures is stronger?") dc not explicitely assess children's perception of the size of a single fraction or ratio. (4) It is unclear to what an extent a child's reasoning is based on visual perception rather than perception of the size of the involved ratios. Perhaps the failed answer of one of Noelting's subjects (1980 , p. 225): "Because there is less water in A (4:2) than in B (5:3)" was based on the right impression, namely, that there is relatively less water in A than in B (Figure 1 suggests that there are water parts for only half of the orange parts in A but for more than half of them in B).

A B

Figure 1

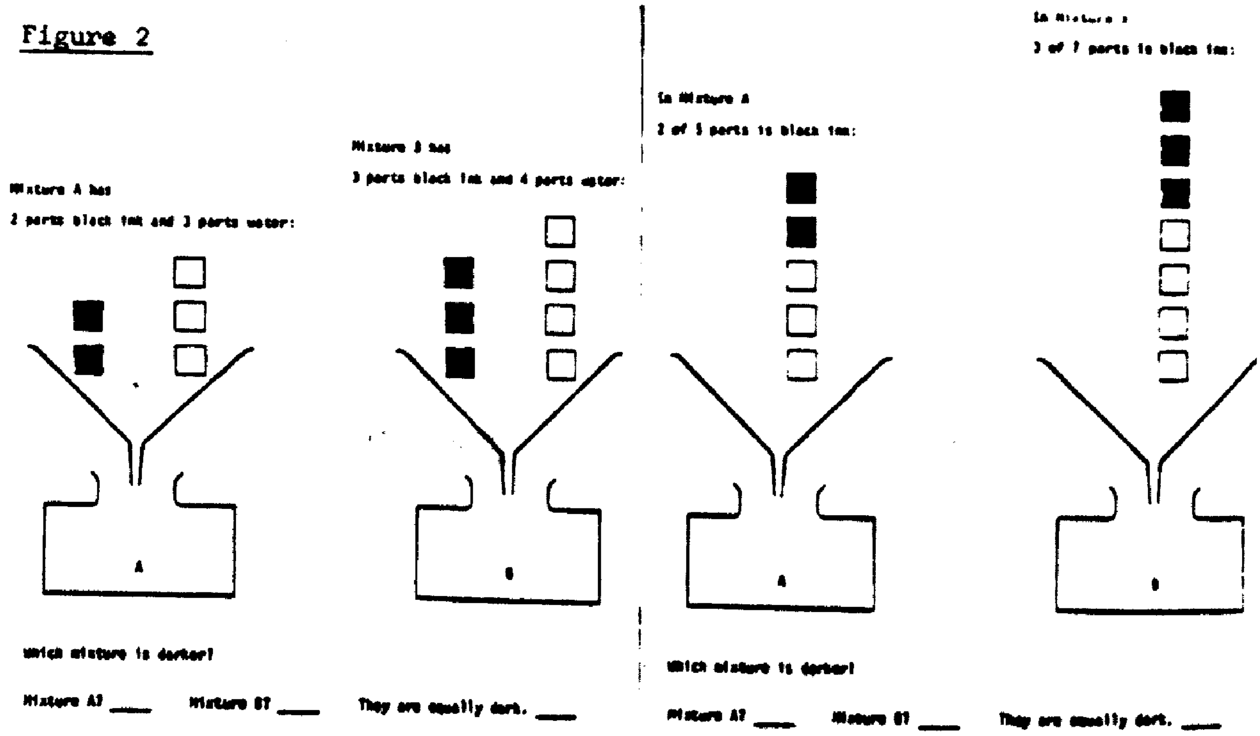
(5) Another subject's answer to the item just mentioned: "A gives two wholes and B: 1 2/3" was passed, but is based on a wrong perception of the concentrations (two wholes is not the concentration of the orange mixture but it is 4/6 of 2/3). Although mathematically correct in a comparative situation, since $a/b < c/d \Leftrightarrow a/a+b < c/c+d$, the subject's reasoning presumably was not based on this insight. (6) In general, pictures as in Figure 1 can be interpreted in various ways: as fractions (i.e., 4/6 vs 5/8), or as ratios (i.e., 4:2 vs 5:3, with the possibility of reading them as four-halves and five-thirds). Thus, we cannot be sure to what an extent observations of children's perception of the size of ratios and fractions in mixture tasks are distorted by task-related variables.

## THE INK-MIXTURE AND THE GRAY-LEVELS STUDIES

In video-taped clinical interview settings close to and at the end of the 30-week teaching experiment in the Rational Number Project, two studies were conducted to get further insights into children's quantitative understanding of ratios and fractions. Subjects were 16 fifth-graders, eight from each of two experimental groups in elementary schools in DeKalb, Illinois, and a suburb of Minneapolis, Minnesota.

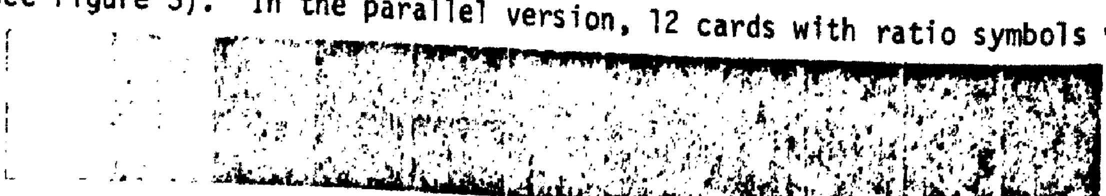
The ink-mixture study was done after 27 weeks of instruction. Each of 9 separate tasks was concerned with the comparative darkness of two ink mixtures, similar to the mixture tasks of Noelting and Karplus, but was presented twice contrasting a "ratio format" with a "fraction format": Having different pictures and wordings, one version suggested a part-to-part while the other suggested a part-whole interpretation of the same problem (see Figure 2). Before answering the questions, subjects had been shown possible results of mixing black ink and water in different ratios in form of gray-colored cards. One card was pointed out to represent a mixture where "2 parts is black ink

Figure 2




and 2 parts is water" (or, "2 of 4 parts is black ink," respectively) and named "half-way between clear and black." Subjects were then to rate three ratios (fractions) in their darkness values, requiring discrimination between five different gray levels. More than 90% of all responses were correct, showing that subjects (1) understood the problem setting and (2) had a rough size notion of the ratios (fractions) involved. In both versions, the numerical relationships between the ratio (or fraction) components in the 9 tasks covered the full range of Noelting's developmental stages, with three cases where both ratio and fraction version were in the highest stage, III.

The <u>gray-levels study</u> was done after completing the 30 weeks of experimental instruction. The aim was to obtain a more fine-tuned record of subjects' perception of fraction and ratio size than the one given with the gray-color-cards experiment preceding the ink mixture study. Embedded in an ink-mixing situation, gray-color cards were to be associated with the <u>values</u> of fractions with a scale of 11 distinct gray levels increasing in darkness from 0% (white) to 100% (black) in stages of 10%. Twelve cards with fraction symbols representing ink mixtures were to be ordered from lightest to darkest and, based on their darkness values, to be associated with gray levels in the scale (see Figure 3). In the parallel version, 12 cards with ratio symbols were



Figure

used representing ink mixtures of the same darkness values as the fraction cards (i.e., 4:2 in place of 4/6, etc.).

## RESULTS OF THE INK MIXTURE STUDY

(a) Assessment of ratio comparison

In the presentation of results, items that meet Noelting's stage III (i.e., general non-equivalent ratios) are distinguished from those below Stage III (i.e., equivalent ratios and ratios that either have equal first or equal second components). Response explanations were categorized to discriminate between perceptually-based responses, responses which used ratios and proportional reasoning, and responses which used fraction thinking on a ratio task. In no case was the comparison of ratios based on the corresponding part-whole fractions expressing the concentration of ink in water (e.g., 4/6 for 4:2, etc.).

From the items that represented stages below III,72% (69) of all (96) responses were passed; among these: 46% (32) of all passed responses were based on consideration of ratios, 32% (22) of all passed responses were based on visual perception, 22% (15) of all passed responses were based on ratios read as fractions (e.g., 4/2 for 4:2, i.e. did no longer deal with the actual concentrations).

From the items represented Stage III, 33% (16) of all (48) responses were passed; among these: 25%(4) of all passed responses were based on consideration of ratios, 44% (7) of all passed responses were based on visual perception, 31% (5) of all passed responses were based on ratios read as fractions. Besides, 28% (9) of all failed responses (19% of all responses) to Stage-III items were rated as "perceptually based, right answer, wrong explanation."

(b) Assessment of fraction comparison

From the items representing stages below III, 58% (46) of all (80) responses were passed; among these: 61% (28) of all passed responses were based on consideration of fractions, 24% (11) of all passed responses were based on visual perception, 15% (7) of all passed responses were based on the corresponding part-part ratio (e.g., 4:2 for 4/6). In one case an item was failed in the category "perceptually based, right answer, wrong explanation."

From the items representing Stage III, 30% (19) of all (64) responses were passed; among these: 68% (13) of all passed responses were based on consideration of fractions, 21% (4) of all passed responses were based on visual perception, 11% (2) of all passed responses were based on the corresponding part-part ratio. Besides, 24% (11) of all failed responses (17% of all responses) to Stage-III items were rated as "perceptually based, right answer, wrong explanation."

## Observations

Overall, performance on ratio comparisons was better than on fraction comparisons (72% vs 58% success frequency for below-Stage-III items and 33% vs 30% for Stage-III items). In both formats, about twice as many responses of the (fifth-grade) subjects were passed on below-Stage-III items than on Stage-III items. In both formats, about one-fourth of all failed responses with right answers had wrong explanations which indicated that the answer was based on perception. This suggests that the pictorial presentation of mixture items helps subjects to give a right answer which they might not have obtained in a purely symbolical problem setting. In the fraction format, more than 60% of passed responses on items of all stages were based on fractic reasoning. This is in contrast to the ratio fromat: In particular for Stage-III items, only one-fourth of passed responses were based on ratios representing the true darkness value of ink mixtures while for nearly half of all responses an explanation was given which was based on visual perception. About one-third of all passed responses on Stage-III ratio items was based on fractions that did not represent the true ink concentration (or, the true value of the original ratio). Though mathematically legitimate for comparison situations these answers do not contribute to insights about children's quantitative understanding of ratio.

Results of the gray-levels study, as well as further comments on the ink mixture study, will be presented in the session.

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