NUMBER LINE REPRESENTATIONS OF FRACTIONS[1]

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The study explored students' interpretations of representations of fractions on number lines and the effect of instruction on those interpretations. Subjects were five fourth-graders, and instruction was a four-day unit on the use of number lines. A 16-item, multiple-choice pre- and posttest was used along with videotaped interviews. Performance improved except when students had to associate a reduced fraction symbol with an equivalent, unreduced fraction representation on a number line. The difficulty at "unpartitioning" a representation suggests that care needs to be given to developing a concept of equivalent fractions. Further, translations among various models of fractions might foster better performance.

This study (a) explored ways that students might interpret, or misinterpret, the representation of fractions on number lines and (b) determined possible influences of instruction on those interpretations. The number line, which embodies the measure subconstruct of rational numbers (Kieren, 1976), was chosen for study because it is a pervasive model of number representations throughout school mathematics instruction.

As a model for representing fractions, the number line differs from other models; e.g., sets, regions; in several important ways. First, a length represents the unit, but more important the measure construct suggests both iteration of the unit as well as simultaneous subdivisions of all iterated units. That is, the number line can conceptually be treated as a ruler. For example, if one wants to measure with a ruler to the nearest one-seventh of a unit, all units would be subdivided into sevenths. Second, on a number line there is no visual separation between consecutive units. That is, the model is totally continuous. Both sets and regions as models possess visual discreteness. When regions are used, for example, space is typically left between copies of the unit.

[1]The research was supported in part by the National Science Foundation under Grant Number SED-20591. Any opinions, findings, and conclusions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Third, the number line requires the use of symbols to convey part of the intended meaning. For example, point A in Figure 1a has no numerical meaning until at least two reference points are labeled. Two possible meanings are given in Figures 1b and 1c. Figures 1d and 1e, however, do convey meaning without any accompanying symbols, though their interpretation requires some standard conventions about the nature of a unit.

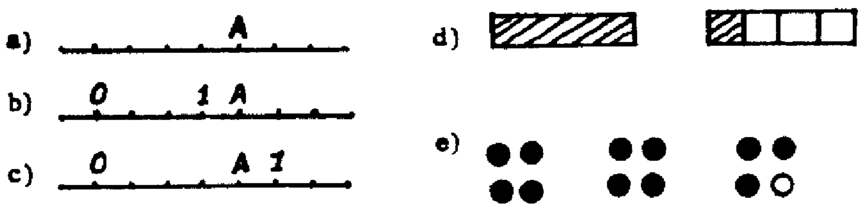

Figure 1. Representations of Fractions

The use of symbols to label points on a number line may focus a student's attention on those symbols rather than on the pictorial embodiment of the fractions. This focusing may in turn cue symbolic processes as the predominate mode of manipulation of information. Too, the necessary but not directly used marks on a number line may act as perceptual distractors (Behr, Post, Lesh, & Silver, 1982).

These differences between the number line and other models suggest that students may interpret number line representations only with some difficulty. Too, instruction may have to be carefully keyed to the critical attributes of number line representations in order to be effective.

METHODS

Subjects

Subjects were five fourth-graders in an elementary school in northern Illinois; three boys and two girls. They were selected, through teacher evaluations, to represent a cross-section of facility with arithmetic concepts and were also subjects in an 18-week teaching experiment (Behr, Lesh, & Post, Note 1).

Instruction

Instruction was a four-day lesson concerning association of fraction concepts, relations, and operations with points, comparisons, and transformations on a number line. Specific objectives were as follows:

1) To associate whole numbers, fractions, and mixed numbers with points on the number line.
2) To use the number line to help connect "improper" fraction names to "mixed number" names.
3) To use number lines to determine which of two fractions is less or whether they are equivalent.
4) To use number lines to generate equivalent fractions.

The lesson on number line representations was presented near the end of the larger teaching experiment. The fraction test of Novillis (1980) was given immediately prior to and immediately after the instruction.

Test

Novillis's 16-item, multiple choice test can be partitioned into two 8-item subscales in several ways: (a) fraction given with representation to be chosen versus representation given with fraction to be chosen, (b) number line shows 0 to 1 versus number line shows 0 to 2, and (c) representation on number line shows unreduced fraction versus representation shows reduced fraction. For each item there were five choices, one of which was "Not Given"; this choice was never the correct choice. In all cases, the fraction symbol in the correct fraction/representation pair was reduced even if the representation was for an unreduced equivalent fraction.

RESULTS

Scores on the six various possible subscales are given in Table 1. For five of the six subscales, performance uniformly increased or remained constant from pretest to posttest. The sole exception was when the representation was unreduced and the fraction symbol was reduced. As a follow-up of this subscale, scores were separated according to the other categories of items (Table 2). With the exception of Student 1, students were unable to choose a reduced fraction name when an unreduced equivalent form was represented on a number line.

To help determine what processes the students might be using, incorrect responses on the unreduced representation subscale were examined. On

Table 1: Subtest Scores[a]

Subtest	Student 1	2	3	4	5	Number of Times Posttest-Pretest >0	=0	<0
fraction given	4(8)	4(4)	2(4)	1(4)	2(4)	4	1	0
representation given	4(8)	2(4)	2(2)	1(3)	0(3)	4	1	0
number line 0-1	8(8)	5(5)	2(3)	2(4)	1(4)	3	2	0
number line 0-2	0(8)	1(3)	2(3)	0(3)	1(3)	5	0	0
reduced representation	4(8)	5(7)	3(6)	1(6)	0(7)	5	0	0
unreduced representation	4(8)	1(1)	1(0)	1(1)	2(0)	1	2	2

[a]x(y) means x = pretest score, y = posttest score. Each subtest had a maximum possible score of 8.

Table 2: Refined Scores for Unreduced Representation Subscale[a]

Subcategory	Student 1	2	3	4	5
fraction given with					
0 to 1 number line	2(2)	1(1)	1(0)	1(0)	1(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(1)	1(0)
representation given with					
0 to 1 number line	2(2)	0(0)	0(0)	0(0)	0(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(0)	0(0)

[a]x(y) means x = pretest score, y = posttest score. Each subcategory had a maximum possible score of 2.

the pretest, 10 of the 31 incorrect responses were "Not Given"; two were blanks. On the posttest, however, 28 of the 30 incorrect responses were "Not Given"; none were blanks.

Additional information was available from videotaped interviews. In three interview tasks, students were to find equivalent fractions, $\frac{5}{3} = \frac{\square}{12}$, $\frac{8}{6} = \frac{\square}{3}$, and $\frac{8}{6} = \frac{\square}{12}$. Student 1 solved these problems symbolically and used the number line only to verify the solutions. Student 4 used counting strategies but solved all three correctly. Student 2 combined number line and symbolic algorithms and solved only the first and third tasks involving larger denominators. Students 3 and 5 used addition and subtraction strategies and solved the last two tasks correctly, but possibly only because of the 2:1 ratio of the demoninators.

DISCUSSION

In many ways the instruction seems to have been quite effective. Except for Student 1, there was no improvement when the number line representation was of an unreduced, but equivalent, fraction. In particular, when the representation was given and the reduced fraction was to be selected, only Student 1 gave any correct responses.

Only Student 1 spontaneously used symbolic algorithms to generate equivalent fractions in the interview tasks. Clear access to those algorithms may indicate a well developed concept of equivalent fractions which allows spontaneous reducing of fractions as well as recognition of unreduced fractions. Students who did not have access to those algorithms may not have recognized when a fraction was not reduced.

In the instruction on equivalent fractions two approaches were used: (a) given two fraction symbols, determine if they name equivalent fractions and (b) use a set of number lines already divided into halves, thirds, fourths, etc. (essentially, they were fraction bars) to write fractions equivalent to a given fraction. There were no examples of writing a fraction symbol for a given representation and then rewriting the symbol in an equivalent form, though there were some examples of changing the representation to match a given fraction; e.g., drawing a representation for $\frac{3}{4}$ and then changing the representation to $\frac{6}{8}$. The students apparently did not see the reversibility of the associations and were not able to transfer the process.

OBSERVATIONS AND IMPLICATIONS

Students difficulty at adding partitioning points to generate higher term fractions or mentally "removing" partitioning points to generate lower term fractions is not unique to the number line model (Behr, Post, Lesh, & Silver, 1982; Payne, 1976). Moreover, the greater difficulty children have generating lower term fractions by "unpartitioning" pervades children's dealings with both continuous and discrete models.

A similar phenomenon is observed with symbolic equivalent-fractions-tasks: Generating higher-term fractions seems to be easier for the children than generating lower-term fractions. At the symbolic level, this difference in difficulty may be due to children's greater facility with multiplication than division.

With manipulative-aid tasks, however, the children seem to rely heavily on the visual representation of a fraction; flexibility in the perception of equivalent fractions independent of the given representations is not yet achieved. Children not only seem distracted by "extra" lines, but also seem to question "the rules of the game." That is, some children have been observed to add partitioning lines but when faced with the "removal" of lines these same children hesitate and may query the teacher or interviewer about whether it is "alright to take out lines." Other children, however, have been found totally unable to perceive lower-term fractions in the presence of extra lines. More generally, this partitioning/unpartitioning phenomenon seems to pervade many children's work with most models for rational numbers.

A major hypothesis of the research project, of which this study is one part, is that it is translations between and within modes of representation which facilitates learning (Behr, Post, & Lesh, Note 1). As noted earlier the instruction provided models of translations of three types: (a) symbol → number line, (b) symbol → number line → number line, and (c) number line → symbol → symbol. Inclusion of translations such as symbol → number line → number line → symbol might have helped children make symbol → symbol translations in generating equivalent fractions (see Figure 2).

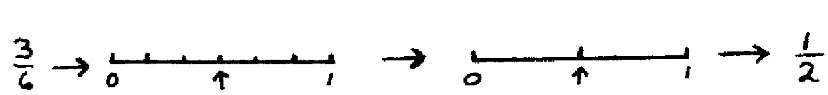

FIGURE 2.

Translations between different kinds of models might also have been helpful. That is, children might have used a region model to generate equivalent fractions; e.g., $\frac{2}{3} = \frac{\square}{9}$; and then translated their work to the number line to accomplish a comparable solution. Seeing the relationship between models might have facilitated their ability to use the number line.

Finally, knowledge of equivalent fractions seems to be important to the full utilization of number line representations. Knowledge that is developed only through symbolic algorithms may be isolated and not called upon in the context of manipulative tasks. More reasonably, work with the number line during instruction on equivalent fractions would probably be called for. For example, partitioning units of a number line first into halves, then fourths, etc., would illustrate the notion that to every point on the line there is associated many equivalent fractions. Before using the number line (e.g., to model addition and subtraction, especially of unlike fractions) more skill with equivalent fractions in the context of the number line is essential. Automatic generation of equivalent fraction representations, through further partitioning or unpartitioning of the number line "in the mind's eye", could facilitate flexibility in perception. Such flexibility seems to enhance students' performance.

Reference Note

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