TURNING THE PLATES — SIZE PERCEPTION OF RATIONAL NUMBERS AMONG 9- AND 10-YEAR OLD CHILDREN[1]

Merlyn J. Behr Ipke Wachsmuth

Northern Illinois University
DeKalb, IL 60115

The question of how children perceive the "bigness" of rational numbers is of importance. Addition of the fractions like $\frac{1}{3}$ and $\frac{3}{5}$ does not make sense to a child without a quantitative notion of the addends. Only if children perceive a fraction, given and first appearing to them as an ordered pair of whole numbers, as an entity, is it likely that they will extend their notion of number to include fractions. One measure of children's quantitative notion of rational number is their ability to perceive the relative size of pairs (or sets) of rational numbers; that is, to determine which of two given fractions is less or whether they are equal. This is discussed in (Behr, Post, Wachsmuth, 1982), based upon information acquired in the context of a 16-18 week teaching experiment.

A concrete response mode was used in the present study, conducted during the 1980/81 period of the Rational Number Project[2] still in progress (Behr, Post, Silver, and Mierkiewicz, 1980), to gain information about children's perception of the absolute size of fractions. Subjects were six children in a 4th grade experimental group in DeKalb, five of which were matched with subjects in a 4th grade comparison group of the same school. In addition, data is available from a group of six 5th graders which was the pilot experimental group in 1979/80, and a 5th grade comparison group.

To show the size of a fraction with a concrete embodiment, children were provided with a set of two intersecting plates of red and of green color. These could be turned relatively to each other in a way to show a smaller or bigger amount of the whole circular device shaded green with the restbeing of red color, thus embodying a fractional part of a circular unit. This device actually was recommended as a teaching tool by Leutzinger and Nelson (1980), and was used here to gain information about children's perception of the absolute size of a fraction; the angle measure

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[2] The term "rational number", in this paper, is used when emphasizing the number concept of fraction (as opposed to ordered pair).

serving to record the precision of the students' responses. None of the subjects had seen or used this device before; however, its perceptual proximity to the usual circular models used in instruction provided immediate understanding of its functioning.

Students were asked to use these intersecting plates to show each one of the fractions $\frac{1}{3}$, $\frac{1}{5}$, $\frac{3}{5}$, and $\frac{6}{10}$ as fractional part of the circular whole. Questions were posed in the form "Make it so that one-third is green" [as writing $\frac{1}{3}$]. Recorded was the student's behavior, the degree measure of the angle shown, and the student's oral explanation of what s/he thought to do it. Interview data is available from two assessments; the first, half way through and the second, at the end of the teaching experiment.

Children's responses on these items are categorizable into seven categories plus one additional -- don't know. Descriptions and representative responses from children to exemplify the categories follow.

Category UFI (Unit fraction iteration) Subject finds three-fifths for example, by first finding one-fifth and then doing a definite iterative behavior.
To show three-fifths "I moved to one-fifth, then I thought of another one. That would be two-fifths. Then make it one more fifth."

Category PP (Physical Partitioning) Subjects' response is based on the use of imaginary lines which trace unit fractions of the desired fractions. After partition of the whole plate is made, S moves directly to the original fraction.
To show one-third S explains "one-third is like a Y. I drew imaginary lines in my head and spaced them out. All the parts were like [the one] shaded green."

Category E (Recognized equivalence) Subject recognized that the fraction on the display was equivalent to the desired fraction and did not change the display in showing the desired fraction.

Category RU (Reference to a unit fraction, other than 1/2) Subject compares the desired fraction to a known unit fraction.
To show 1/3 the subject says "Can you do that with a quartered plate [showed one-fourth of the plate green]? Probably like that [increasing the angle from the one-quarter position]?"

Category RH (Reference to 1/2) Subject makes a relative comparison of the desired fraction to the known fraction 1/2.
To show 6/10 S explains "See if you make an imaginary line [making 2 halves], that's 5 [tenths] and then that's 6, 7, 8, 9, 10 . Well I had it one-half first and that was wrong which I know. If you make it one-half and then over 1 more, that is 6, and 7, 8, 9, 10 pieces".

Category RIM (Reference to incorrect inventive model). Reference is made to some incorrect concrete model that denotes some divisions (e.g. uses a clock; 5 o'clock = 1/5).
Subject apparently uses the 3 o'clock position to show 3/5 saying "You went like ... here's 10 o'clock, 20 o'clock and I went over to 30 o'clock [3 o'clock] for three-fifths."

Category AD (Addition by denominator difference). Subject uses an addition rule to get from a previous position to the desired position; adds differences (e.g. if previous display was at 1/3, 1/3 + 2 = 1/5, therefore move 2 times from what was 1/3).

Showing one-fifth having one-third displayed the subject explains "Well it was on 3 so I just moved it 2 more times."

Category DK (Don't Know) Subject is unable to solve or gives a random response and says he/she guessed.

A distribution of subjects responses by groups is given in Table 1.

Table 1. Response Frequencies within Categories by Subject Groups

Response Category	5th Grade Exp.	5th Grade Comparison	4th Grade Exp.	4th Grade Comparison
E.	1(2.3)[a]	2(5.0)	0	1(2.5)
UFI	10(23.2)	4(10.0)	11(25.6)	2(5.0)
PP	30(69.8)	10(25.0)	28(65.1)	18(45.0)
RU	0	3(7.5)	0	1(2.5)
RH	2(4.7)	4(10.0)	4(9.3)	0
RIM	0	5(12.5)	0	2(5.0)
AD	0	6(15.0)	0	2(5.0)
DK	0	6(15.0)	0	14(35.0)
Totals	43	40	43	40

[a] percent of responses in this category as a percent of the total number of responses in the given group.

The sequence of fractions given was furthermore chosen to investigate whether the student would make a connection to the fraction preciously shown (i.e. $\frac{1}{5}$ must be represented by a smaller angle than $\frac{1}{3}$, $\frac{3}{5}$ can be perceived as $\frac{1}{5}$ and $\frac{1}{5}$ and $\frac{1}{5}$, $\frac{3}{5} = \frac{6}{10}$ so the same display can be used), which would indicate whether s/he already interlinks rational numbers. Observations noted a child's initial direction to show $\frac{1}{5}$ when $\frac{1}{3}$ was showing, for example, and it was asked, how did you know whether to make the green part bigger or smaller? Responses and observable behavior were categorized in three categories:

C^+: subject makes a connection with correct intention;
C^-: subject makes a connection with an incorrect intention;
N: no connection observable.

Subjects' responses and categorizings are shown in Table 2 and Table 3.

Observations and Conclusions

From Table 2 and 3 we gain some insights into the effectiveness of the experimental treatment with respect to the criterion variable. According to within group mean deviations, the groups were ordered by performance such that the 5th and 4th grade experimental groups were higher than either of the matched comparison groups, with one exception: in the 6/10 task of the final interview, the 5th grade comparison group outperformed the 4th grade experimental group. In most cases, the difference in favor of the experimental groups was substantial. These results

Table 2. Subject Response Data: 5th grade 1979/80 experimental and comparison groups

Subject	Mid-Experiment Assessment				Post-Experiment Assessment			
	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)
Experimental Group								
1(H)[a]	70[b] (-50)[c] PP -	68 (-4) PP C+	88 (-128) UFI C+	202 (-14) UFI N	100 (-20) PP -	50 (-22) PP C+	158 (-58) UFI N	225 (+9) PP N
2(H)	106 (-14) PP -	64 (-8) PP C+	246 (30) PP N	207 (-9) RH N				
3(M)	97 (-23) PP -	82 (+10) PP C+	115 (-101) PP N	109 (-107) PP N	105 (-15) PP -	72 (0) PP C+	210 (-6) PP C+	200 (-16) E N
4(M)	118 (-2) PP -	62 (-10) PP C+	182 (-34) UFI C+	163 (-53) UFI N	100 (-20) PP -	60 (-12) PP C+	220 (+4) UFI C+	205 (-11) RH N
5(L)	107 (-13) PP -	PP C+	142 (-74) PP C+	202 (-14) PP C-	115 (-5) PP -	60 (-12) PP C+	215 (-1) UFI C+	150 (-66) PP N
6(L)	120 (0) PP -	74 (+2) PP C+	149 (-67) UFI C+	131 (-85) UFI N	128 (+8) PP -	70 (-2) PP C+	228 (+12) - -	240 (+24) UFI N
$\bar{X}$	(17.0)[d]	(6.8) [100][e]	(72.3) [67]	(47.0) [0]	(13.6)	(9.6) [100]	(16.2) [60]	(25.2) [0]
Comparison Group								
1(H)	110 (-10) RIM -	260 (+188) AD C-	279 (+63) RH C+	358 (+142) DK C-	210 (+90) AD -	220 (+148) AD C-	300 (+84) RH N	300 (+84) E C+
2(H)	80 (-40) RU -	100 (+28) RU N	290 (+74) RU N	305 (+89) DK N	110 (-10) PP -	80 (+8) PP C+	255 (+39) PP N	255 (39) E C+
3(M)	104 (-16) PP -	62 (-10) PP N	250 (+34) UFI C+	220 (+4) UFI N	139 (+19) PP -	65 (-7) PP C+	246 (+30) UFI N	228 (+12) PP N
4(M)	85 (-35) DK -	159 (+87) AD C-	64 (-152) DK C-	189 (-27) DK N	20 (-100) RIM -	35 (-37) AD C-	150 (-66) PP N	152 (-64) PP N
5(L)	37 (-83) RH -	162 (+90) RIM C-	75 (-141) RIM N	179 (-37) RIM N	22 (-98) RIM -	34 (-38) AD C-	99 (-117) UFI C+	169 (-47) DK N
$\bar{X}$	(36.8)	(80.6) [0]	(92.8) [40]	(59.8) [0]	(63.4)	(47.6) [40]	(67.2) [20]	(49.2) [40]

[a] Denotes mathematics achievement level: High, middle, low.
[b] Degree measure of subjects trial.
[c] Deviation of subjects trial from true measure.
[d] Mean deviation.
[e] Percent of subjects who made positive connection.

Table 3. Subject Response Data: 4th grade 1980/81 experimental and comparison groups

Subject	Mid-Experiment Assessment				Post-Experiment Assessment			
	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)	$\frac{1}{3}$(120°)	$\frac{1}{5}$(72°)	$\frac{3}{5}$(216°)	$\frac{6}{10}$(216°)
Experimental Group								
1(H)	89 (−31) PP −	90 (+18) PP C+	115 (−101) PP N	120 (−96) PP N	103 (−17) PP −	77 (+5) PP N	142 (−74) UFI C+	118 (−98) UFI N
2(H)	130 (+10) PP −	68 (−4) RH N	142 (−74) UFI N	173 (−43) UFI N	110 (−10) PP −	70 (−2) PP C+	154 (−62) RH C+	N
3(M)	115 (−5) PP −	55 (−17) PP C+	145 (−71) UFI C+	245 (+29) UFI N	120 (0) PP −	70 (−2) PP N	120 (−96) UFI C+	125 (−91) PP N
4(M)	110 (−10) PP −	80 (+8) PP N	120 (−96) UFI N	185 (−31) PP N	150 (+30) PP −	92 (+20) PP C+	215 (−1) UFI C+	186 (−30) UFI N
5(L)	104 (−16) PP −	55 (−17) PP C+	122 (−94) PP N	178 (−38) RH N	119 (−1) PP −	62 (−10) PP C+	200 (−16) UFI C+	148 (−68) RH N
6(L)	56 (−64) PP −	67 (−5) PP N	145 (−71) PP N	121 (−95) PP N				
$\overline{X}$	(22.7)	(11.5) [50]	(84.5) [17]	(55.3) [0]	(11.6)	(7.8) [60]	(49.8) [100]	(71.8) [0]
Comparison Group								
1(H)	60 (−60) RIM −	80 (+8) AD C−	85 (−131) DK N	− DK N	74 (−46) PP −	83 (+11) PP C+	138 (−78) UFI N	169 (−47) PP N
2(H)	50 (−70) PP −	45 (−27) PP C+	70 (−146) DK C−	80 (−136) RIM N	96 (−24) PP −	54 (−18) PP C+	148 (−68) PP N	153 (−63) UFI N
3(M)	119 (−1) RU −	142 (+70) AD C−	230 (+14) DK N	64 (−152) DK N	80 (−40) PP −	68 (−4) PP C+	70 (−146) PP N	70 (−146) E C+
4(M)	63 (−57) DK −	37 (−35) PP N	155 (−61) PP N	122 (−94) PP N	93 (−27) PP −	67 (−5) PP C+	193 (−23) PP N	90 (−126) PP N
5(L)	68 (−52) DK −	180 (+108) DK C−	65 (−151) DK C−	142 (−74) DK N	55 (−65) DK −	180 (+108) DK C−	315 (+99) DK C+	305 (+89) DK C−
$\overline{X}$	(48.0)	(49.6) [20]	(100.6) [0]	(114.0) [0]	(40.4)	(29.2) [80]	(82.8) [20]	(94.2) [20]

suggest a positive effect on children's quantitative perception of rational number due to the experimental treatment.

From Table 1 we observe that both the 4th and 5th grade experimental groups used fewer and higher level strategies. Over 65% of the subjects in both experimental groups used the physical partitioning (PP) strategy compared to 25% and 45% in the 5th and 4th grade comparison groups. Worthy to note is the comparatively high percentage of experimental subjects who used the strategy of unit fraction iteration (UFI). The UFI strategy presupposes an understanding of the concept of unit fraction, which can be provided by the more primary PP strategy. Physical partitioning requires the perception of some physical object as a whole. The UFI strategy , on the other hand, does not require making reference to an object or a quantity as a whole. It, in reverse, takes the unit fraction 1/5 as an entity (as opposed to an ordered pair), and from this is built non-unit fractions, the whole, and fractions greater than 1. Instead of abstracting 3/5, for example, directly from physical part-whole embodiments, UFI, as with 3/5 = 1/5 and 1/5 and 1/5 (i.e. 3 one-fifths) already interlinks rational numbers at a higher order of abstraction. That is, UFI achieves the building of conceptual structures by making connections within the domain of fractions, and thus leads more directly to an abstract thinking of rational number.

Moreover, UFI provides a basic understanding of addition of fractions (with same denominator) which is close to counting-on strategies with whole numbers -- to process 3+4=N, count 3; 4, 5, 6, 7; to process $\frac{3}{5} + \frac{4}{5} = N$, "count" $\frac{3}{5}$; $\frac{4}{5}, \frac{5}{5}, \frac{6}{6}, \frac{7}{5}$ (exceeding 1 is not a problem with this strategy).

In particular the combination of PP/UFI strategies seems to facilitate knowledge building toward a quantitative understanding of fractions which becomes independent of part-whole (ordered pair) notions, by embedding single concepts in a structure of other concepts, and consolidating prior concepts. This is supported by the superior performance of the two experimental groups.

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