QUESTIONS ON SOME ISSUES IN THE PSYCHOLOGY OF MATHEMATICS INSTRUCTION

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INTRODUCTION

In his invited talk at the AERA meeting at New York City, March 1982, Robert Gagné gave rise to a renewed discussion of the cognitive phenomena involved in children's learning of mathematics, mainly focussed on arithmetic, and of implications for instruction. According to Gagné, attention has to be given to each of the three phases of a performance model as follows: (1) Translation of a verbally described situation into mathematical form; (2) Central computation phase; (3) Validation of the solution.

In particular he argued toward the maxim of an automaticity of skills which are involved in the computation phase, in order to bring about optimal performance. He ultimately stated that the cognitive capability required for students from kindergarten through highschool is a set of stored rules which are best carried out automatically, and that lots of practice should be devoted toward automatizing such rules (Gagné, Note 1).

While the decade of the 1970's was characterized by the "back to the basics" movement, and the product-versus-process question is still being debated, these statements of Gagné might prove harmful in that they are likely to cause misunderstandings, give wrong impressions to teachers, and cause a precipitate repudiation of Gagné's ideas by math educators.

The purpose of this paper is to give a more careful inquiry into Gagné's proposals, and at the same time take up the discussion of skill automaticity in a new light. On the one hand, this is done to mediate the controversy between the views of mathematics educators--best expressed in the terms "learning with understanding--and the seemingly unpopular theses of Gagné. On the other hand, this discussion shall raise Questions on Some Issues in the Psychology of Mathematics Instruction which, in the author's opinion, require further research attention.

WHAT IS A SKILL?

"A skill is what a learner should be able to do." This pragmatic answer is quoted from Suydam & Dessart (1980), where an extensive discussion of skill learning in mathematics is given. They continue, "Skills arise from concepts and principles and provide a foundation for the development of other concepts and principles. Conceptual thought is derived in part from the understanding attained as skills are developed." (p. 207) This view shows intertwined dependencies between skills and conceptual knowledge. In fact, "The relationship between comoutational skill and mathematical understanding is one of the oldest concerns in the psychology of mathematics." With these words Resnick and Ford (1981) open their outlining remarks on the concerns of a psychology of mathematics instruction, and they continue, "Instead of focusing on the interaction between computation and understanding, between practice and insight, psychologists and mathematics educators have been busy trying to demonstrate the superiority of one over the other. [...] The relationships between skill and understanding were never effectively elucidated." (p. 246; emphasis theirs). Certainly, an attempt to analyze this relationship should also include mathematical skills other than computation.

Modern cognitive psychology as for example represented in the work of Anderson (1980), tends to view knowledge categorized as declarative knowledge and procedural knowledge. Declarative knowledge comprises the facts that one knows; and procedural knowledge comprises the skills one knows how to perform (p. 22 ff). The declarative-procedural distinction graphically supports the fact that two different things are of concern in any learning process. Declarative knowledge is what is stated in propositions and can be verbally communicated, for example, $3 + 5 = 8$, or $(a + b)^2 = a^2 + 2ab + b^2$, or "To solve quadratic equations, one method is to complete the square." This type of knowledge is viewed to be stored long-term in what is called semantic or factual memory. Skills, as procedural knowledge, are also stored in long-term memory ("procedural memory"), however, rather as an ordered sequence of actions that are performed when recalled. The term cognitive skill, (as opposed to motor skill) refers to the ability to perform various intellectual procedures. For example, writing out the steps necessary to solve a quadratic equation by completing the square is a cognitive skill.

The learning-with-understanding approach of today's mathematics education has led to the view that a learner shall be put in the position to do mathematics--for instance, solve a problem--by deriving it meaningfully from declarative knowledge. However, it is the procedural, not the declarative, knowledge that governs skilled performance.

AUTOMATICITY OF SKILLS

Taking his three phase model as a basis to achieve good problem solving, Gagné (Note 1, Hypothesis 2) claims that "performance is likely to be best if the phase of computation is done automatically". In illustrating this, he makes reference to the assumption of a working memory existing in the human mind, in which one is to store those items of cognitive processing where attention is directed to, and that this working memory is limited (further reference is found in Resnick & Ford, 1981). Hence, "the scarce cognitive resource of attention needs to be devoted to the most intricate and complex parts of the task, if its solution is going to be efficiently and successfully arrived at." (Note 1, p. 17). Gagné then remarks that automatization of some kinds of algebraic operations gives rise to rapid and fluent thinking in applicational situations, and that differences in speed and accuracy of computation found among students is due to a lack of automaticity of computational skills in some of them. In a problem solving situation a higher amount of attention would then have to be devoted to skill performance which would interfere with and slow down the cognitive operations devoted to solving the problem.

Anderson (op. cit. p. 226) lists three steps in which skill learning occurs: (1) The <u>cognitive stage</u> (declarative), in which a description of the procedure is learned with the aim to understand it; (2) The <u>associative stage</u>, in which the declarative information is transformed into a procedural form, that is, a method for performing the skill is worked out; (3) The <u>autonomous stage</u>, in which the skill becomes more and more rapid and automatic (by way of practice); verbal mediation in the performance of the skill often disappears at this point. It may be questioned -- and this is Gagné's concern--whether trends in curriculum development tend to ignore the importance of the third stage. Emphasis is put on the acquisition of factual (declarative) knowledge. Understanding of a task is said to enable the learner then to "figure it out". Time-and cost-effectiveness standards are applied to the use of instructional time, and "too

much" practice is avoided; see, for example, the "Agenda" of the National Council of Teachers of Mathematics (1980, pp. 6-12). Further practice of a skill is left to problem-solving situations that require application of that skill. However, this is a case where too much spacing can be counter-productive.

Stage 3 of the above listing comprises the promotion of a skill to an automatized skill which is autonomous in the sense that it does not even require a verbal correspondant in declarative form. Not only does efficiency of a skill increase as it gets more and more automatic, but also mental effort in performing the skill decreases; there is less need to monitor each action with care. Suydam and Dessart (op cit.) refer to the fact that mastered skills require relatively little reflection as they become "reflexes" which are conducted at a subconscious level. As the demands on conscious resource diminish, a reduction in mental work load is achieved.

It was demonstrated (cf. Anderson, op. cit.) that highly practiced skills cease to interfere with other ongoing behaviours; that is, different cognitive activities can be conducted simultaneously. Anderson moreover indicates that automatizing subskills before going on to higher skills contributes to learning higher skills (in computer programming; cf. p. 249 ff). The reason apparently lies in the relationship between attention and automaticity: As subcomponents in programming become more automatic, programmers can focus more attention on higher level problems.

ROTE SKILLS OR MEANINGFUL SKILLS?

Given any skill, what makes the difference between its being (a) a rote skill or (b) a meaningful one? We are to distinguish two cases, in our analysis of each. First, the skill in question is not automatized; second, it is automatized.

1a) A rote skill that has not been automatized requires the recall of a rule, or a set of rules, which is then applied mechanically, without understanding. If part of the rules, or knowledge of how to instantiate the variables, is forgotten, there is no way to reconstruct it from other knowledge. (For instance, under these circumstances this often happens with the quadratic formula.)

1b) A skill derived meaningfully and that has not been automatized makes reference to factual knowledge that a learner has, in order to apply this for a certain purpose. (For instance, a learner knowing how to represent numbers with Dienes blocks can use this to derive the subtraction algorithm, including "borrowing".) If such a skill has not been practiced to the point of recall, it can be viewed as a problem solving process (cf. Wachsmuth, 1982). Instead of dealing with symbols directly (or

"concretely", as Gagné calls it), it is dealt with symbols indirectly (or "abstractly", as Gagné calls it); namely, it is actually dealt with the referents of symbols.

2a and 2b) The idea of practicing a skill to the point of recall ("over-practicing") is the following: The sequence of actions necessary to do something is made to occur automatically; without having to derive, or reconstruct, any action from an understanding or recall of facts, purposes, and goals. In this process, one step triggers the next; and it is just the ability to perform this sequence of actions in order to achieve a certain thing which makes up the skill.

The point was raised earlier that children's maturity to fully understand why and how a skill works may appear at a later time than they are able to perform rote skills; e.g., see the discussion on fraction algorithms of Suydam (1979, pp.302/3). So the question is whether it could be of advantage if automatized rote skills precede a full understanding of such skills.

When an automatic skill is called upon, action is not derived from understanding how the procedure works. It is rather guided by its own action scheme. To be very clear: This is like the difference between understanding how bike-riding works and being able to ride a bike. The common fact that practicing of motor skills should not be interfered with by thinking about it seems to have a correspondant in the acquisition of procedural mathematic knowledge. Gagné's Hypothesis 1 (Note 1, p. 16) about the teaching of computational skills (omitting reinforcement of incorrect performance plus teaching correct rules again) is further supported by findings of E.G. Davis (1978): During drill sessions, it is best to emphasize remembering; don't explain. Understanding seems to be unnecessary for the execution of a procedure.

It is still controversial whether better understanding of a procedure gives rise to better performance; for the case of fraction algorithms, cf. Suydam (1978, p.302). Understanding, however, seems to be important for embedding a procedure in the context of a problem-solving task correctly. It might well make sense to teach skills both rotely and meaningfully, maybe in a spiral fashion. In summary:

1. Do we have to think about procedural and declarative knowledge as two related, however separate issues of mathematical instruction?

2. Do there exist optimal mental ages for automatizing skills in children, and for teaching children to fully understand what they do?

3. If the optimal age for automatizing skills is earlier than the optimal age for understanding a skill, how do we proceed in instruction?

VALIDATING THE SOLUTION

Under this heading, Gagné (Note 1) makes reference to the fact that students may trust in a result obtained by means of an algorithm, without inspecting the solution meaningfully which could reveal a faulty computation. The important topic adressed here concerns the different levels at which a person can deal with mathematical symbols. First, the syntactic level, at which one is to manipulate the symbols (here numerals) directly-- as "concrete" objects of thought -- according to certain rules, and totally removed from their meanings (as numbers). This is the normal situation when an algorithm is employed routinely. Second, the semantic level -- which Gagné calls "abstract" --, at which one deals with the symbols by referring back to their meanings. For example, in making use of the fact that 19 × 20 equals 19 × 2 × 10 one needs to understand how numbers are denoted by means of numerals, in the place value system.

What is really addressed with the topic of validating is the interaction between these two levels, which must be a well-functioning one if mathematics is to be employed meaningfully. It is the question of "The Influence of Semantic Content on Algorithmic Behavior", a topic investigated by Davis and McKnight (1980). The study reports a specific phenomenon encountered with third and fourth graders who, in general, could use the standard subtraction algorithm correctly, but were almost certain to make the error of "skipping over intermediate zeroes" in certain subtraction problems.

These children could clearly use Dienes blocks to represent numbers and explain trading, upon request. But this latent knowledge did not alert them in the algorithmic situation. The two levels at which one can deal with this problem -- syntactical and semantical -- appeared to coexist unconnectedly in the minds of the subjects.

A major question raised by the Davis/McKnight phenomenon is whether this is "essentially inevitable for most children of this age, or whether it is a consequence of an excessively 'algorithmic' (and meaningless) program of instruction" (p. 75). Gagné's position about ways of verifying solutions is that "it would be desirable if they [such ways] were deliberately taught" (Note 1, p. 12). But the actual problem is:

4. How can children be taught ways of verifying solutions, in the sense that their algorithmic behavior in situations that are governed by syntactical rules is monitored by their semantic knowledge?
5. Can such an interaction of syntactical and semantical levels of information processing be taught at all to children of certain grade levels?

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As one of the ten areas of basic mathematical skills listed in the NSCM position papers on Basic Mathematical Skills (Note 2), the "Alertness to the Reasonableness of Results" is viewed an essential skill. Apparantly, alertness comprises the automatic generation of warnings or critics in the course of a task performance. So, if a curriculum developing both understanding and skilled performance is the objective, research efforts will have to be devoted to the question:

6. How can warnings be automatized so as to achieve a reliable, alert interplay between skill and understanding in mathematics?

Apparantly, Gagné's view that the cognitive capability called upon for the students in grades K-12 be a set of stored rules which are best carried out automatically is only part of the story. It is also required that they can be carried out meaningfully, and that alertness skills permit to "switch to the meaningful mode" when necessary.

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