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TEACHING

Teaching is more difficult than learning. We know that, but we rarely think about it. And why is teaching more difficult than learning? Not because the teacher must have a larger store of information, and have it always ready. Teaching is more difficult than learning because what teaching calls for is this: to let learn. The real teacher, in fact, lets nothing else be learned than — learning. His conduct, therefore, often produces the impression that we properly learn nothing from him, if by "learning" we now suddenly understand merely the procurement of useful information. The teacher is ahead of his apprentices in this alone, that he has still far more to learn than they — he has to learn to let them learn.

Martin Heidegger, *What is called thinking?* (p. 15)

Two Modes of Thinking — also Relevant for the Learning of Mathematics?*

IPKE WACHSMUTH

"2:43 p.m. — what time is it now, then?" I'll not readily forget my colleague thinking aloud that way when reading off his new digital watch. A student told me that, having looked at his old analogue watch, he'd know what time it was; if asked for the time a few minutes later he couldn't answer without having another look first. (To get back his sense of time with his new digital watch he first imagined what the hands of his old watch would show.)

What these examples could imply in general, and for the learning of mathematics in particular, will be illustrated in the following. For the moment we will continue with the above example.

What is the difference between the two sorts of watch? The digital watch gives us exact information about the time by means of a *linguistic* string. So does the analogue watch but, beyond that, it has a characteristic face, a "*gestalt*", and displays time through its "facial expression", i.e. by the position of its hands. Besides communicating the time 2:43 p.m. exactly, the analogue watch, by its face, gives us the (pleasing) feeling of having plenty of time yet until "three" (or, about a quarter of an hour later, the alarming one that it must now be high time). In what follows we shall illustrate the hypothesis of different modes of thinking being involved in each case. First we will show this more precisely in relation to theories of knowledge representation. That such modes of thinking can actually exist will then be shown by some results in modern neurology and neuro-psychology. Finally, we will indicate some implications for mathematics learning and teaching, and sketch some projects of recent and current research.

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Knowledge representation

Norman and Rumelhart [1975] propose two extreme possible forms of representating knowledge:

— a propositional system, expressing concepts by means of statements upon conceptual interdependences between the concepts involved;
— an analogous representation, preserving an accurate image of the original scene.

Their "one-system hypothesis" is based exclusively on the first possibility which presupposes knowledge to be linked with language; and their model of "active structural networks" is now of great importance in the theory of knowledge representation. It disregards, however, an essential component of thinking which could also have a correspondance in representation. The fact that images may be experienced when remembering and processing visual information is explained in their model as an occasional generation of imagery by the propositional system (which is certainly possible: the student who, after reading off his digital watch, imagines the analogous position of hands thereby translates linguistic into visual information). To support their hypothesis Norman and Rumelhart rely on experiments where subjects could "insufficiently" remember images, which they explain, in terms of their model, as arising from conceptual failures in the propositional representation.

We consider it possible to oppose their conception that, in such a one-system representation, all human knowledge is assumed to be linked to language. The common opinion that man, besides explicit knowledge which he can communicate, disposes of "tacit knowledge", suggests we should allow for the existence of knowledge not linked to language when reflecting on knowledge representation.

In developing a theory of visual perception, David Marr and his co-workers from MIT outline how images and scenes were recognized by the use of "sketches", based only on a general precognition of the scene [e.g. Crick/Marr/Poggio 1980; for further references see this report]. In a first step of processing by the visual cortex, the grey-level array given on the retina is transformed *simultaneously* into a "prime-sketch" consisting of lines which in part correspond to the contours of objects, and in part show variations in the surface shape.

Extrapolating Marr's ideas, we could imagine a representation in terms of neural networks adequate to such processes of perception. The "core" of such a representation might consist of a storage of still more reduced sketches (like pictograms) where net representations of more complex structural features of the perceived scene are linked. We could then assume a more complete image to arise from the activated subnet of the core representation through "resonance", in an act of recognition; much as a stringed instrument, when it is animated by a pure tone resonating with its body (as a "core sketch") produces a sound of some sonority (as a more complete image).

To illustrate this we give an example. The possibility of recognizing the picture shown in Figure 1 would accordingly be based on some "analogous" core representation,

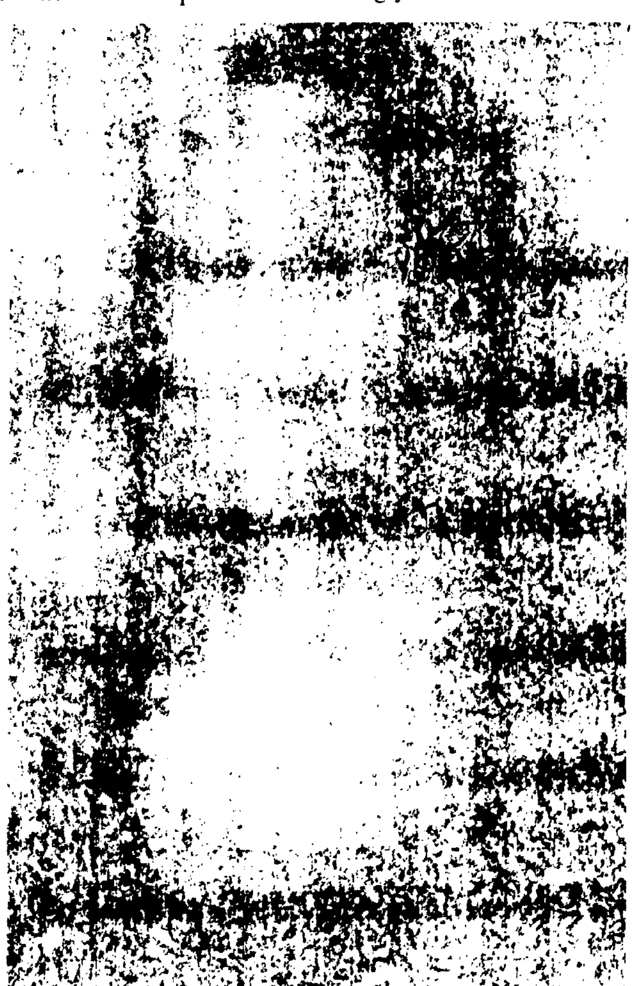


Figure 1

as in the case of a caricature featuring the same person, the degree of abstraction of which surely exceeds that of a primary sketch. What one eventually "sees" in both cases is: "It *is* the Mona Lisa!" That is to say, the *semantic* description displays the result of perception. Marr's hypothesis follows ideas of gestalt psychology in postulating a precognition which arises from the entire perception of a scene, and which evokes a mental set (an attitude) which makes it possible to interpret details of the scene.

Another example (Figure 2) indicates a phenomenon which in some respects relates to the one exemplified above. Often spontaneous recognition of a figure previously concealed is triggered by a hint given by the context, or independently (as we give now: "cat!"). The ability of the brain that is demonstrated by this power, to obtain, by way of "seeing", the insight "It is a sleeping cat!" is certainly based on an analogous representation. (By the way, frequently the main problem in geometric situations is to recognize a significant figure in a pattern of lines; then the mere "seeing" of completing auxiliary lines can yield success.)

Figure 2

Thus there are enough aspects to show the need *for propositional as well as for analogous* forms of representation (and processing) of knowledge to appear in human thinking. In particular, one of the forms may appear exclusively and *cannot* arbitrarily be transferred to the other, as is supposed by Norman and Rumelhart [1975]. Thus it seems to be quite natural to formulate a theory including both aspects, as is given by the dual code theory of Paivio, for example [1971; for further references and a broader summary of the theory see Wippich/Bredenkamp, 1979].

Allen Paivio's theory postulates two independent coding systems to represent "our knowledge of the world" and which are involved in our processing of information. A non-verbal (imaginal) and a verbal coding system are distinguished, in the following respects:

1. by the kind of preferred *information* that is represented and processed,
2. by the kind of *organizing* of information into more extensive memory units, as well as their restructuring,
3. by the kind of *processing* of encoded information.

In the imaginal system, preference is given to the processing of concrete or "imaginative" content organized in synchronous or spatial structures. In the first case one could imagine, say, musical sound patterns, or the parallel transformation of a triangle in the plane; in the other case a chess configuration, or a spatial diagonal of a cube, and, concerning the kind of processing, Rubik's Cube, or the rotation of an ellipse around one of its main axes. In all these cases *parallel* processing takes place, where in each run a number of information units is processed simultaneously.

The verbal system is focused on the processing of abstract information. Paivio supposes such information to be represented in sequentially styled organizing units. Thus he considers such units, as strings of symbols or of phonemes, etc., to be processed *sequentially*, as a matter of principle.

We note here that this point is seen in a different light by Wippich and Bredenkamp [1979, e.g. experiment no. 4, pp. 83, 84, 85]. Also, how can we account for the phenomenon of "integrating" abstract information, i.e. gestalt processes? (The ability of man to read diagonally could be a hint.) This will be of some importance for our thesis on problem-solving which we shall formulate later.

Paivio distinguishes three phases that appear in any processing of information:

1. The *representational phase* where the stimuli of a perception generate code-referring responses (verbal or imaginal), in the responsible system;
2. the *referential phase* which follows, where in the first instance meaning becomes attached so that — as far as possible! — associations between both systems are made (symbols give rise to imagery or names become associated with imagery);
3. the *associative phase*, where meaning becomes attached through chains of associations within the original system.

(Remember the example with the digital clock where, by reference to the analogous position of hands, meaning ("the time") becomes attached.)

How far links are constructed from one coding system to the other in the processes of *learning* depends not only on the kind of information given, but also on exterior conditions as, for example, an intention to learn, duration, and instructions which favor imagery. The meaning of concrete objects and situations, however, is essentially determined by imagery, whereas abstract information gets its meaning mainly through language processing and verbal associations.

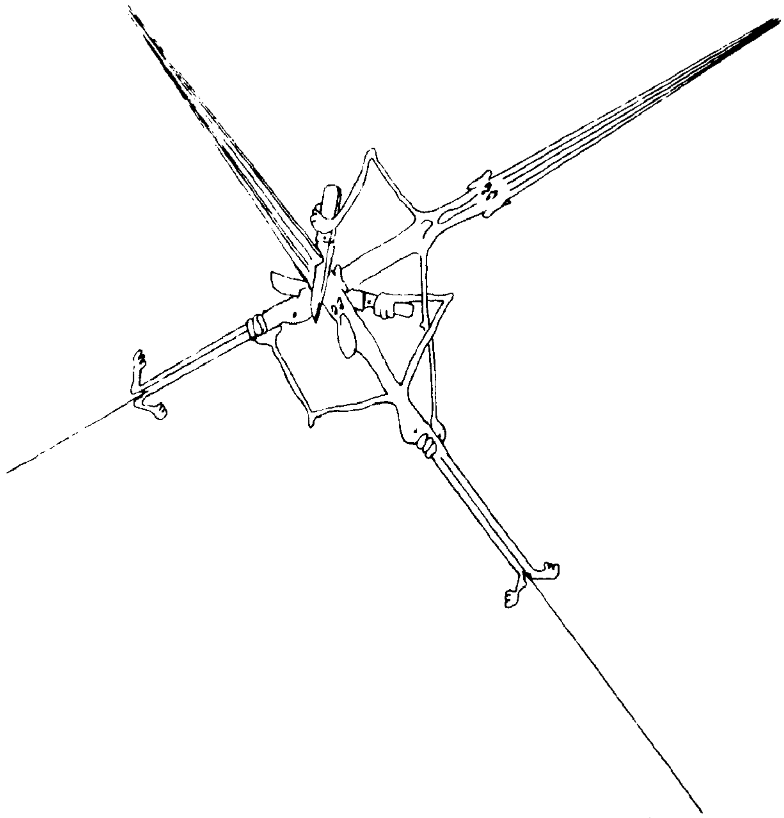
The dual coding theory of Paivio implies the following thesis: *The more concrete is the information to be processed, the better the chance for a dual coding: verbal and imaginal*. Anyone may follow this with the help of the following examples:

"Several concrete aspects"

as an example of abstract information. (Did *you* have any visual imagery?)

"Two intersecting lines"

as an example of more concrete information yielding imagery. (For additional material favoring an imaginal coding see Figure 3.) The advantages of such dual coding are



Two intersecting lines (greatly enlarged), by M. Erné
(It's a pun in German, reminding one of
"two lines cutting one another" as well.)

Figure 3

evident; Paivio can for instance show that images of objects would lead more quickly to a comparison of their properties than the names of the objects would do. Furthermore it is obvious that one image only is needed to represent "two intersecting lines", whereas presumably more than one linguistic unit is required for "several concrete aspects".

From the statements given above it seems to be sensible to consider different kinds of representation in what follows. Thus imaginal kinds are not restricted only to the visual, and the verbal kinds contain several aspects. On the one hand, the *temporal* sequence of learning content plays a significant role when it is presented verbally (which would surely not be the case when presenting a picture, which is perceived and processed in parallel, as was pointed out above.) On the other hand, it seems to be important to take notice of the *causal* sequence when dealing with verbal information "logically".

Moreover, one has to distinguish between an "exterior", *communicative* language, by means of which the individual can receive and communicate verbal information (in or from his cognitive system), and an "interior", *functional* language which plays an active role in organizing the knowledge and action of the individual; such a language is, for instance, considered as "egocentric language" by Wygotski [1977], and reappears as "personal metalanguage" in Davis and McKnight [1979, p. 109; see also pp. 96, 101].

A psychological theory probably has to take into consideration many more kinds of coding of information, as exemplified in Wippich and Bredenkamp [1979]. However, in continuing with the rough distinction "*verbal* vs. *nonverbal*" we get an essential dichotomy in human thinking and, in particular, in mathematical thinking.

Paivio's theory does not, as far as we know, include the aspect that there could exist imaginal represented knowledge which *a priori* has no verbal dual. Where knowledge is created resulting from imagery, in particular, it is often quite difficult to "grasp" it in words (as most mathematicians will have experienced), which is necessary as a basis for scrutiny by a formal proof, as well as for being able to communicate it. Frequently one succeeds in making a new mathematical concept explicit just by the fact that the first glimpse of it becomes clear and reproducible through the use of a striking word or metaphor.

> "I have forgotton the word
> I wanted to say,
> And bodiless does thought return
> To the stateroom of shadows"

Wygotski says [translated from Wygotski 1977, p. 291] at the beginning of his expositions in "Thought and Language", and thereby gives a poetic hint on the role of words (or symbols) as the embodiment of thought, yielding support to and access to meaning and imagery. Further, Wygotski's sentence contains an allusion to the richness of the inaccessible "dark room" from which originate imagination and vision, and maybe *creation*. We will turn to this point now.

L-modal and R-modal thinking

Mathematical thinking appears in different modalities. This statement was illustrated earlier and is hardly controversial. We will pick it up now in the light of the preceding. At many places in the literature, not necessarily listed here, there can be found hints to their existence; frequently examples of mathematical thinking are cited by way of complementary or "dual" pairs. It is suggested that they can be arranged into two groups following the presumptive type of their processing in the cognitive system. In one group, logical thinking and language seem always to be involved, whereas in the other we find standards of thinking that are somewhat more relaxed. In the following we will therefore speak of "L-mode" and "R-mode" for short. We list some of the issues below.

L-MODE	R-MODE
concentration upon detail	disregard of detail; rambling of thoughts
canalization of thoughts (aiming at a systematic solution)	associating (in extreme cases, "free" associating)
causal thinking (linear time)	spatial thinking (no reference to time)
understanding, reasoning; using words and symbols	evolving visions and ideas; feeling
sequential processing	parallel processing
"convergent thinking", fully conscious	"divergent thinking", partly unconscious

Not only does this list express that imagery can be involved when solving a problem, but also that, besides logical thinking, there appears to be a non-causal, unconstrained, liberal way of thinking which may also yield illogical mental links. L-modal thinking means concentration and the conscious sequentializing of trains of thought which otherwise would appear concurrently, e.g. in finding the solution of a problem systematically. R-modal thinking means relaxation and, leaving aside the detail, it favors parallel, holistic thoughts of broader range, which will perhaps yield spontaneous *insights*.

The cooperation of these modes could in particular involve the interplay of creative and productive thinking important in mathematics. Already Descartes, in his *Regulae and directionem ingenii* [1974], stated "chains of logical inferences" to be in opposition to *intuition*, and that he understood intuition to be the sudden perception of links and interrelations between different appearances. And Henri Poincaré, in *Mathematical creation* [1956], has given some quite clear remarks about how two such modes would collaborate. There he says that the ideas which arise in phases of inspiration have to be verified in phases of conscious work. He makes a clear distinction between discipline, awareness, will, and thus consciousness is involved in the latter case, and freedom and the absence of discipline in the first; and he considers the subconscious to play an essential role in mathematical invention.

The hypothesis which we illustrate here, that two modes of thinking can exist simultaneously, is supported by hints on substantial differences between the special functions of the two sides (hemispheres) of the human brain, which we can only sketch here [for more details see Ornstein, 1972, and Sinz, 1978]. These hemispheres, which appear mirror-like from their surface structure, are interconnected by the "corpus callosum", a cord of about 2. 10^8 nerve fibres between which billions of impulses per second carry on rapid communication. Moreover by a part cross-over of the optic nerves, information coming from each eye reaches each of the differently organized sides of the brain, and similarly with the ears. This can be of some importance for perception and learning. Finally, an indirect lateral transmission of information is assumed to take place through the brainstem. the cooperating sides of the brain control the two sides of the body diagonally, and are said to be, for most right-handed people, specialized in their functions of thinking as follows:

left brain	right brain
operating with discrete elements in series	holistic mental activities
processing of language almost exclusively here	production and reception of music; almost no language
mode of operation mainly linear	spatial orientation; sense for images and patterns
analytic	synthetic
oriented toward details	recognition of gestalt

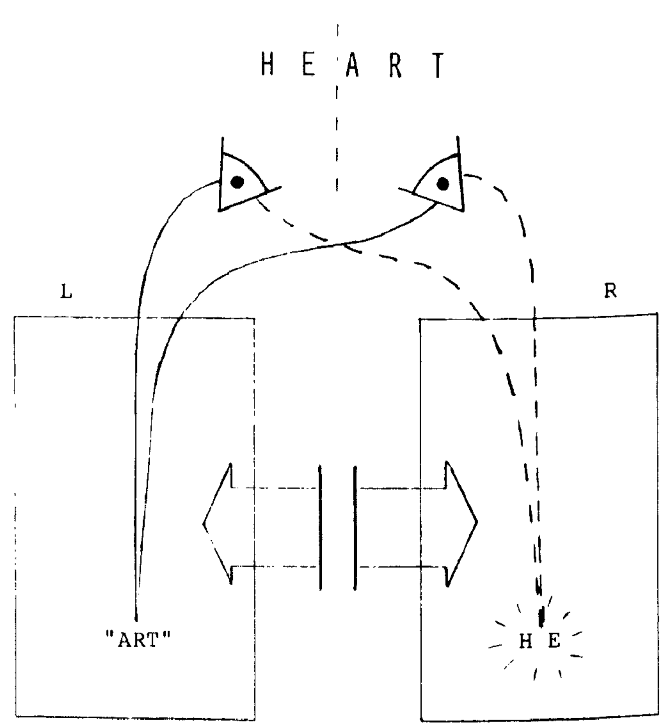


Figure 4

Indications of a hemispheric specialization came in the sixties from clinical neurology and neuro-surgery, in particular from Sperry and Bogen of the California Institute of Technology. In cases of heavy epilepsy they achieved some success with therapeutic sections of the corpus callosum. (In cases of epileptic seizure patients can control at least one side of the body.) These so-called split-brain patients have been the subject of many subtle tests, one of which is sketched in Figure 4 [Ornstein, 1972; for further references see there].

In an experimental arrangement, a patient was shown the word HEART in such a way that he could "see" the whole word, but HE in his right brain only, and ART only in his left brain (Figure 4). When questioned what he could see he answered ART; being asked to select between signboards with HE, ART, and HEART with his *left* hand (being controlled by the right hemisphere), he pointed to HE; that is to say, his left hemisphere specialized in language and his non-verbal hemisphere gave different answers!

A lot of these results have been verified in healthy people. In experimental investigations the superiority of the right brain for simultaneous information processing has been established. Verbal-analytic and spatial/visual mental activities are distinguished by eye movements (the "motoric component of imagery") and, also by individual records of the EEG of each hemisphere: a unilateral appearance of alpharhythm (signalizing relaxation) seems to indicate that a hemisphere not being addressed by the information presented is temporarily disconnected.

In Steiner [1978] it is reported that subjects could, in activating verbal processes, solve arithmetical tasks even faster when having to keep track of visual distractor information given to the "other channel" by way of a film with meaningless sequences of patterns or numerals. This completes further experiments reported in Ornstein [1972]. Performance of the hemisphere responsible for the required mode of processing seems to increase when the other one is diverted.

To sum up we could say that the two sides of the brain may embody "instances" of L-mode and R-mode processing. This lateral specialization seems to originate from the evolution of language and to be unique to man, though it is found to be of different intensity in individual cases. (For left-handed people, conversely, it has not been verified to be so discriminating.) One supposes some kind of a non-verbal, and thus concealed, consciousness to exist in the right brain — which has moreover been verified as the seat of dreams. R-modal thinking, though of lesser logic and clarity, appears to be of great importance in creativity: Einstein has spoken of a "combinatorial latitude" that exists in thought. (As is well known, dreams cannot be controlled either, and so they combine the contents of consciousness in free play.)

As has been mentioned, it appears to be difficult to transform imagination into language in order to make it accessible and communicable; for "That which appears simultaneously in thought unrolls successively in speech." [Translated from Wygotski 1977, p. 353] One may succeed by using appropriate metaphors (Bruner has emphasized this); and Ornstein considers the "other" side of science to lie in the field of creating paradigms.

At the psycho-physiological level dual *representation* is considered to be valid [Sinz 1978, p. 197]. But how little is known about the physiological foundations was recently shown by British brain investigators ["Is Your Brain Really Necessary?" *Science* Vol. 210, No. 4475, Dec. 1980]. We will start from the point that the establishment of multiple representations is one of the most important organizational principles, and that information so represented is processed by a complementary interplay of L-mode and R-mode.

In any theoretical approach to *learning* the problem of the representation of knowledge has first to be considered. We have dealt with verbal and non-verbal kinds of representation at the beginning, and then presented a suitable model of processing. Now we want to relate this to problem-solving and mathematics learning, where, as a main point of this paper, R-modal kinds of processing will be considered in particular.

A thesis on the role of L-mode and R-mode in problem-solving

As has been shown by more recent investigations, normal learning speed, creativity, and transformation into language, depend on the interplay of both modes of thinking, especially when dealing with more difficult problems. Moreover sleep contains indispensable periods of processing of information and of *reorganization* of memory [see Sinz, 1978, ch. V]. This phenomenon, well-known as "sleeping on a problem", presumably relates in the same way to problem-solving as does an alternation of phases of intensive reasoning with phases of relaxation in the manner illustrated above.

When dealing with a problem concentratedly and intensively (in the L-mode), we may suppose many single pieces of a mosaic to be introduced into the brain, e.g. facts and separate factual interdependences in a criminal case, or particular revolutions and sequences of revolutions of Rubik's Cube observed in their effects, or fragments of proof arguments taken on trial, etc.

In periods of relaxation (in the R-mode), the brain operates on such pieces beyond the rigid control of consciousness; they are tentatively combined in a mosaic, not being controlled (and narrowed?) by language (or "metalanguage") organizing the problem-solving process. (Note: The pieces themselves could nevertheless be in a language, even in abstract language, conceived as written or spoken units.) In doing this, combinations may be tried which are not near at hand, or seem illogical, or accidental. By these means it can appear that a superordinate whole, a significant "gestalt", is recognized in the mosaic; or perhaps just an "isle" of gestalt which can be accomplished in conscious work (in the L-mode). Frequently one does not realize thoughts have been "revolving around the point" all the time until such spontaneous discovery "reaches the consciousness", whether by accident or when attempting the problem (consciously) again later.

This R-modal kind of thinking apparently contains an *organizational principle*, too, which, however, is not controlled by the conscious, and can be realized by its results only. As a suitable category for this principle we suggest Köhler's conception of "*silent organization*", which he considers in opposition to "manifest organization" where "not only the result is experienced, but also very much of its "why" and "how" is felt." [Köhler, cited by Koffka, 1935, p. 383.]

Also well-known is the phenomenon that such a recognition of gestalt frequently originates from an exterior happening, or from a word or a picture flowing into the stream of thought. This seems to us more than merely giving a starting point. As is assumed in gestalt psychology, recognition of a gestalt is possible when some precognition of the (concrete or abstract) "scene" is given. The impulse coming from the outside possibly gives a hint that is decisive, the "frame" by which one succeeds in recognizing a gestalt in the part of the mosaic already laid (remember the picture of the sleeping cat, Figure 2).

We see a difference between the logical-deductive (L-modal) solving of a problem and an "intuitive" one, as described above, in the following respect:

> The intuitive solution of a problem through a sudden realization of interdependencies does not *a priori* include "full understanding", which may mean, say, to be immediately able to write down a proof. Nevertheless, one already knows the shape ("gestalt") of the solution. *It is much easier to validate, by logical-deductive proceeding, a form already seen, than to have to find a solution the shape of which is still unknown, while forming "chains of logical inferences"!*

On the other hand one knows that intuition and imagery can lead along false paths, which is unlikely when proceeding in a formal-exact way. The full depth of the processes appearing in mathematical problem-solving, however, seems only be obtained by the interplay of these two modes of thinking which, in principle were already distinguished by Descartes.

In any case, we suppose two things to be involved in the process of problem-solving in this manner:

(L) Attempting the problem in intensive, conscious efforts for a longer time, so that e.g. particular parts of proof arguments become so fluent that one is able to place them separately in the front of the mind's eye and let them run together ("internalizing of pieces").

(R) Standing back from further conscious attempts so as to release the problem to the more relaxed environment of the R-mode (allowing mosaic (p)laying — silent organization).

(As long as "full understanding" has not been attained, one is to keep on with this alternation.)

An awareness of the problem, automatically directing one's attention to possible spontaneous insights, is then given, according to experience. Having appropriately passed through the first L-phase, one "cannot get rid of the problem".

The R-mode in mathematics learning

The role of L-mode and R-mode processing in the learning of mathematics has already been sketched. By a concrete, graphic preparation of the content to be learned, dual coding is favored which not only depends on language (verbalization) but also makes possible a more economical representation by images (as with "two intersecting lines") from which propositions can be derived as needed. In this way, the content becomes suitable for both L- and R-modal processing; in brief: it is the basis for more profound understanding. Not in every mathematical area, as is well known, is this possible without further effort.

The basis for an understanding of content to be learned is that the learner already has a *precognition* of the things to be learned. He needs categories which are already established in his cognitive structure, enabling him to interpret the things offered — if not entirely, yet connecting with them, possibly by modification or extension of his categories ("adaptation").

In particular in fields of learning which are still foreign, an "intuitive precognition" can be helpful. This may already be latent and become activated, for example, by an impulse from the teacher, or it may have first to be established in the learner (through a modification of his kind of sight): this would require more effort, the more abstract and strange were the actual field of learning. What about these considerations?

If, as we suppose, an intuitive precognition makes it possible to realize (or, more cautiously, to guess) the nature of the global interrelations between the objects of the field still unknown, learning is thus facilitated in the sense that *what is complicated in its detail may become easy when the whole of it is familiar in shape (gestalt)*. Gestalt can thereby be communicated by using "mosaic pieces" of a totally different kind, for perception of gestalt does not depend on the structure of the pieces (*example*: the Mona Lisa from Figure 1)! This is provided by the holistic type of processing, disregarding detail, in the R-mode.

In a remarkable way, this repeatedly appears in Douglas Hofstadter's book "Gödel, Escher, Bach" [1979], when for instance [on pp. 67-73] the theoretical introduction of recursively enumerable and recursive sets is preceded by a "theory" of "cursively drawable" and "recursive figures". Here the objects (e.g. the paintings of Escher) are concrete and imaginable, and the "theorems" (in the present context) are of similar shape and impact, as later are the theorems on the abstract objects. By the use of figure-ground relations, imagery and a "feeling" for what such statements "mean" are generated which transfer to the abstract field and make it possible for the theorems obtained there *soon* to accumulate to an "image" of the theory. (The virtuoso performance is left to the book cited, where a detour via Bach and Chopin is still scheduled ...)

So much for the role of R-modal thinking in the learning also of abstract mathematics. In conclusion, some research projects (which are by no means representative) will be commented upon which deal in particular with functional specializations in schoolchildren.

Research projects

In Davidson [1979] a four year investigation is reported with 300 children, ages 5-18, having global learning disabilities which could not be further specified. This project was conducted at the Children's Hospital Medical Center in Boston. An "activity approach" was used to elicit the intuitions, skills, and strategies of the children in various mathematical situations. As a main point of the project a distinction between two general styles of mathematical learning was verified; the alternatives seem to relate to preferences for L-modal or R-modal processing ("left hemispheric preferring"/"right hemispheric preferring"), and can be further particularized: for instance, one type per-forms better on extending the table of ordered pairs of a function than the other, a feature which appears in the reverse manner when the task is to name the general rule. The other type may give a correct answer spontaneously without knowing *why* it is so (which would be "a why-presupposing manifest organization" [Koffka, 1935, p. 383]).

Moreover the abilities of the children to evoke compensatory strategies in learning were assessed. It is assumed that children are able to compensate more readily than adults, possibly originating from the fact that lateralization in children is not as clear-cut; furthermore, the hemispheres do not mature at the same rate, and differences may appear between boys and girls. Finally the "interface" between the two hemispheres (corpus callosum) does not develop to full functioning until age 10 (± 2 years). (*Question*: Does the development of certain levels of formal thinking relate to these matters?) Also in the potential for developing compensatory strategies a distinction was found in the preferred modes of processing. The evaluations were issued by an interdisciplinary team of specialists, compiling an overall description for each child in which his individual learning style in mathematics and a personalized set of pedagogical suggestions were formulated.

Access to any kind of a strict, functional language such as is required in mathematics is usually obtained by the roundabout method of *written* language. Already in early learning-to-read processes [Bakker 1981] there appear L/R-aspects in so far as the pure linguistic decoding of letters depends on the (R-modal) ability to discriminate written characters, so that an instantaneous processing of meaning does not become possible until the discrimination process has become subconscious. In early reading instruction, graphemes first appear as new kinds of visual signs which later have to become intimately associated with the "spoken" language previously learned. Apparently such associations are easier to make within than between hemispheres, and early reading proficiency seems to go along with a right hemispheric representation of language, which later shifts to the left after graphemes have become linguistic units. (This would happen about the second/third grade, when differences between boys and girls could also appear.) Investigations on these matters were conducted at the Department of Developmental and Educational Neuropsychology in Amsterdam, partly in collaboration with Northern Illinois University and the University of Leyden. When learning formal notations, as used in mathematics, such mechanisms would certainly be involved too. An instantaneous association and processing of meaning would not be possible until symbols have no longer to be translated into linguistic units consciously.

In what respect could reading and language be further involved in mathematics learning? Above, the role of language has been discussed as being exterior/communicative in one aspect, and interior and functional in the other. Whether such an inner language, organizing the individual's orientation to the solving of problems, does in principle depend on the disposition of natural language, is still an open question.

In this context, we indicate finally a project of research currently being conducted at Osnabrück University, which poses such questions in a particular form. Deaf children who, because of their handicap, would not normally get beyond elementary arithmetic are the subject of these investigations. By use of a formal description of algorithmic problems which does not depend on natural language but renders "intuitive" understanding of the problems posed, they can be confronted with mathematical problems of more depth and can communicate their solutions, as is shown in a first study [Cohors-Fresenborg/Strüber 1981].

Apparently an access to the children's imagery is given through this non-verbal descriptive "language"; how far they make use of it in the sense of a "functional language" organizing their actions, and whether they solve problems by intuition (R-modal), or consciously by making plans, is still uncertain. Further clarification would presumably elucidate the role of an inner language, as well as the interrelations between communicative and egocentric language in cognitive processes.

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Figure 1: taken from *Scientific American* 229, No. 5 (1973), p. 70

Figure 2: taken from "Attneave's sleeping cat", in M. A. Arbib: *The metaphorical brain*, New York 1972, p. 40

Figure 3: "Zwei sich schneidende Geraden (stark vergrößert)", by Marcel Erné