Solution of the statistical bootstrap with Bose statistics*

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A brief and transparent way to introduce Bose statistics into the statistical bootstrap of Hagedorn and Frautschi is presented. The resulting bootstrap equation is solved by a cluster expansion for the grand canonical partition function. The shift of the ultimate temperature due to Bose statistics is determined through an iteration process. We discuss two-particle spectra of the decaying fireball (with given mass) as obtained from its grand microcanonical level density.

I. INTRODUCTION

The statistical description of multiparticle production in hadron-hadron collisions has a long history, starting with Fermi's proposal[1] to view the secondaries as an ideal gas of stable hadrons in a volume characterized by the Compton wavelength of a pion. Hagedorn introduced interaction into this picture by considering an ideal gas composed of fireballs.[2] He determined the fireball spectrum by a bootstrap condition that requires the level density of states of this fireball gas to be asymptotically equal to the input fireball spectrum. Frautschi[3] proposed a bootstrap equation for this spectrum in terms of phase-space integrals. An elegant solution (by Laplace-transformation methods) of the Frautschi statistical-bootstrap equation was given by Yellin.[4] While Hagedorn approximated the grand canonical partition function by the corresponding Boltzmann-statistics expression, Frautschi used Boltzmann statistics from the very beginning. The bootstrap equation becomes much more complex when quantum statistics is included. The main predictions of the model, however, i.e., the existence of an ultimate temperature and the exponential increase of the fireball mass spectrum remain unaltered, as has been discussed in Refs. 5 and 6. Because of its complexity the quantum-statistical bootstrap had not been treated in detail in the literature until Chaichian, Hagedorn, and Hayashi[7] stressed the importance of quantum statistics in particle-physics applications of the ideal gas. This motivated the present investigation on the bootstrap equation for identical bosons.

In Sec. II of this paper we present a very transparent way of obtaining the quantum-statistical bootstrap equation, that shows explicitly the underlying physical assumptions. Furthermore, we discuss the role of fugacity, which is somewhat obscure within the previous Boltzmann-statistics bootstrap. In Sec. III we describe in detail the solution of the bootstrap equation in the form of a cluster expansion, as indicated briefly in a previous letter.[8] In particular, we construct and evaluate a monotonic sequence that converges to the ultimate temperature.

Section IV contains some applications on two-particle distributions which show rather pronounced quantum-statistics effects. The size of Bose effects on the one-particle spectrum was found previously to be fairly small.[8]

II. DERIVATION OF THE BOOTSTRAP EQUATION

Since the physics behind the quantum-statistical bootstrap equation has never been expounded, we start in this section with an introduction to the bootstrap equation with Bose statistics, as it was suggested in the literature.[5,6] The easiest access is in terms of the grand canonical partition function $Z_f(\beta,z)$ that depends on the inverse temperature. For an ideal Bose gas of pions with mass m, one finds in standard textbooks

$$Z_f(\beta,z)=\exp\left[-\sum_{\alpha_i}\ln(1-ze^{-\beta\epsilon_{\alpha_i}})\right]-1\ , \qquad (1)$$

where α_i denotes the discrete one-particle states of the system with energy ϵ_{α_i}, and z is the fugacity. The exponential and the logarithm can be expanded

$$Z_f(\beta,z)=\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{\infty}\sum_{\alpha_i}\frac{z^k}{k}e^{-\beta k\epsilon_{\alpha_i}}\right)^n\ . \qquad (2)$$

Introducing the discrete Laplace transform t of the one-particle density of states

$$t(\beta,m,z)=z\sum_{\alpha_i}e^{-\beta\epsilon_{\alpha_i}}\ , \qquad (3)$$

Eq. (2) reads

$$\begin{aligned}Z_f(\beta,z)&=\exp\left[\sum_{k=1}^{\infty}\frac{t(k\beta,m,z^k)}{k}\right]-1\\&=\sum_{k=1}^{\infty}\frac{t(k\beta,m,z^k)}{k}\\&\quad+\sum_{n=2}^{\infty}\frac{1}{n!}\left[\sum_{k=1}^{\infty}\frac{t(k\beta,m,z^k)}{k}\right]^n\ . \qquad (4)\end{aligned}$$

We may interpret this equation by observing that the partition function Z_f that describes the total system is built up by a one-pion term, $t(\beta,m,z)$, together with its Bose corrections, $\sum_{k=2}^{\infty}[t(k\beta,m,z^k)/k]$, plus the more-particle contributions with $n\geq 2$ and their Bose corrections. With the usual box (of volume V) quantization and continuous counting of states

$$\sum_{\alpha_i} e^{-\beta\epsilon_{\alpha_i}} \rightarrow \left(\frac{V}{h^3}\right)^{2/3}\int\frac{d^3p}{2p_0}\exp[-\beta(p^2+m^2)^{1/2}] \ , \qquad (5)$$

it can easily be seen that the Bose corrections in Eq. (4) amount to clusters of k pions, $k\geq 2$, that we call k clusters,

$$t(k\beta,m,z^k)=B\frac{z^k}{k^2}\int d^4Q\exp(-\beta_\mu Q^\mu)\delta(Q^2-(km)^2) \ , \qquad (6)$$

with $B=(V/h^3)^{2/3}$, $\beta_\mu=(\beta,\vec{0})$.

At this stage we introduce interaction by assuming the system (=fireball) to be an ideal gas, this time not of pions, but of fireballs. This is achieved by substituting the "pion" function t by the "fireball" function Z consistently in all multipion terms on the right-hand side of Eq. (4). Thus the one-pion term is the only one remaining unchanged. The physical picture behind this procedure is to allow for interaction wherever it is expected. In particular, we expect interaction within a k cluster; at this point we differ from Hagedorn.[9] As a result we obtain the bootstrap equation

$$2Z(\beta,z)=t(\beta,m,z)+\exp\left[\sum_{k=1}^{\infty}\frac{Z(k\beta,z^k)}{k}\right]-1 \ . \qquad (7)$$

As we shall show in Sec. III, the solution is given in terms of a cluster expansion

$$Z(\beta,z)=\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k=0}^{\infty}h(n_1,\ldots,n_k)z^n \times\prod_{j=1}^{k}t^{n_j}(\beta,jm,z=1) \ , \qquad (8)$$

where

$$n=\sum_{j=1}^{k}jn_j \ .$$

From Eq. (6), one recognizes that $t(\beta,jm,z=1)$ is B times the Laplace transform of a single-particle phase space, with particle mass jm. The coefficient $h(n_1,\ldots,n_k)$ therefore refers to a configuration of clusters, where for $j=1,\ldots,k$, there are $n_j j$ clusters containing j pions each. Thus, n is the total number of pions in the configuration $\{n_1,\ldots,n_k\}$.

An equivalent, but much less transparent introduction of the bootstrap can be given in its grand microcanonical formulation in terms of an equation for the fireball level density $\tau(Q^2,z)$, where Q is the fireball four-momentum. Since the level density is the inverse Laplace transform of $Z(\beta,z)$,

$$Z(\beta,z)=B\int d^4Q\exp(-\beta_\mu Q^\mu)\tau(Q^2,z)\theta(Q_0) \ , \qquad (9)$$

this grand microcanonical bootstrap equation can immediately be obtained from Eq. (7). Its solution is then directly related to the grand canonical solution Eq. (8). Thus we find

$$B\tau(Q^2,z)=\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k=0}^{\infty}h(n_1,\ldots,n_k)B^l z^n\Omega^l(Q^2;m,\ldots,m,\ldots,km,\ldots,km) \qquad (10)$$

in terms of Lorentz-invariant phase-space volumes Ω^l for l clusters, $l=\sum_{j=1}^{k}n_j$, containing a total of n pions. (In the argument of Ω^l, there are n_1 variables equal to $m,\ldots,n_k$ equal to km.)

In macroscopic thermodynamics, the fugacity z is chosen to reproduce a given average particle number and is thus freely adjustable within a certain range. On the other hand the number of produced particles in elementary-particle collisions is determined by dynamics. For a given volume, a statistical description of such processes must necessarily lead to a specific value of the fugacity which in this instance should be interpreted as a coupling constant of a pion to its parent fireball. Therefore the fireball system is not fully determined by its volume. The Boltzmann version of the bootstrap, however, depends only on the product $B'=B\times z$, because $l=n$ in Eq. (10), leaving us essentially with one parameter. In the Bose case the cluster contributions lead to a genuine dependence on the two parameters B and z. Therefore it is possible in principle to determine the fireball volume from experimental distributions.

III. SOLUTION OF THE BOOTSTRAP

The bootstrap equation (7) can be solved on the basis of the Boltzmann bootstrap

$$2Y(\beta,z)=t(\beta,m,z)+\exp[Y(\beta,z)]-1 \ . \qquad (11)$$

The solution to Eq. (11) is given by the Yellin expansion in the driving term $t(\beta,m,z)$:

$$Y(\beta,z)=\sum_{l=1}^{\infty}g_l t^l(\beta,m,z)\equiv Z_Y[t(\beta,m,z)] \ , \qquad (12)$$

where the g_l are determined by

$$g_1=1, \quad g_{l+1}=-\frac{1}{l+1}\left(lg_l-2\sum_{k=1}^{l} kg_k g_{l+1-k}\right). \tag{13}$$

The Bose bootstrap equation (7) can be slightly rewritten as

$$2\sum_{k=1}^{\infty}\frac{Z(k\beta,z^k)}{k}=t(\beta,m,z)+2\sum_{k=2}^{\infty}\frac{Z(k\beta,z^k)}{k} + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\sum_{k=1}^{\infty}\frac{Z(k\beta,z^k)}{k}\right]^n . \tag{14}$$

Equation (13) has the same structure as Eq. (11). Therefore, the function $\sum_{k=1}^{\infty}[Z(k\beta,z^k)/k]$ can be written as a Yellin expansion as well, with a modified driving term

$$\sum_{k=1}^{\infty}\frac{Z(k\beta,z^k)}{k}=Z_{\mathrm{Y}}\left[t(\beta,m,z)+2\sum_{k=2}^{\infty}\frac{Z(k\beta,z^k)}{k}\right]. \tag{15}$$

With the abbreviation

$$R(\beta,z)=\sum_{k=2}^{\infty}\frac{Z(k\beta,z^k)}{k} , \tag{16}$$

Eq. (15) becomes

$$Z(\beta,z)=Z_{\mathrm{Y}}[t(\beta,m,z)+2R(\beta,z)]-R(\beta,z) . \tag{17}$$

Combining Eqs. (16), (17), and (12) one can construct an iteration process by defining a sequence $Z^{(N)}$ through

$$Z^{(N)}(\beta,z)=\sum_{l=1}^{\infty}g_l[t(\beta,m,z)+2R^{(N-1)}(\beta,z)]^l - R^{(N-1)}(\beta,z) \tag{18}$$

and

$$R^{(N)}(\beta,z)=\sum_{k=2}^{\infty}\frac{Z^{(N)}(k\beta,z^k)}{k} , \quad R^{(0)}(\beta,z)\equiv 0 . \tag{19}$$

The convergence of the sequence $Z^{(N)}$ to the solution of Eq. (7) has been proved in Ref. 6. In the following we shall show that the solution can be written as an expansion of the form

$$Z(\beta,z)=\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k=0}^{\infty}\bar{h}(n_1,\ldots,n_k) \times\prod_{i=1}^{k}t^{n_i}(i\beta,m,z^i) \tag{20}$$

with coefficients

$$\bar{h}(n_1,\ldots,n_k)=h(n_1,\ldots,n_k)\prod_{i=1}^{k}i^{2n_i} . \tag{21}$$

The expansion Eq. (20) is indeed the cluster expansion of Sec. II, as is seen from the identity

$$t(i\beta,m,z^i)=\frac{z^i}{i^2}\,t(\beta,im,z=1) . \tag{22}$$

As they stand, Eqs. (18) and (19) are not particularly useful for a recursive calculation of the coefficients. However, we make the important observation that the Nth iteration step leaves the coefficients belonging to total pion numbers $n\leq N-1$ unaltered. Furthermore, step N yields the exact values for coefficients up to total pion number $n=N$ from the knowledge of the coefficients with $n<N$.

With the definition

$$t_i\equiv t(\beta,im,z=1) \tag{23}$$

the iteration starts as follows:

$$Z^{(1)}(\beta,z)=zt_1+O(z^2) , \tag{24}$$

$$Z^{(2)}(\beta,z)=zt_1+z^2(\tfrac{1}{2}t_1^{\,2}+\tfrac{1}{8}t_2)+O(z^3) , \tag{25}$$

where the higher-order terms have no influence on the next step and are therefore discarded. For illustration, we quote the expansion of $Z(\beta,z)$ for up to 4 particles

$$Z(\beta,z)=zt_1+z^2(\tfrac{1}{2}t_1^{\,2}+\tfrac{1}{8}t_2)+z^3(\tfrac{2}{3}t_1^{\,3}+\tfrac{1}{4}t_1t_2+\tfrac{1}{27}t_3) + z^4(\tfrac{13}{12}t_1^{\,4}+\tfrac{3}{64}t_2^{\,2}+\tfrac{2}{27}t_1t_3+\tfrac{1}{2}t_1^{\,2}t_2+\tfrac{1}{32}t_4)+\cdots . \tag{26}$$

We managed to calculate all coefficients up to $N=17$, which amounts to 1211 terms instead of 17 for for the Boltzmann case.

The partition function $Z(\beta,z)$ has a square-root branch point at $\beta=\beta_{\mathrm{H}}$, which leads to the ultimate temperature $T_{\mathrm{H}}=1/\beta_{\mathrm{H}}$. As a consequence of this singularity, the expansion Eq. (8) for Z diverges at $\beta=\beta_{\mathrm{H}}$. The location of the singularity β_{H} is determined by[5]

$$R(\beta_{\mathrm{H}},z)=\sum_{k=2}^{\infty}\frac{Z(k\beta_{\mathrm{H}},z^k)}{k}=\ln 2-\tfrac{1}{2}[1+t(\beta_{\mathrm{H}},m,z)] . \tag{27}$$

To solve for β_{H}, we need the function Z, which is obtained by the iteration process defined by

$$Z^{(N)}(\beta,z)=Z_{\mathrm{Y}}[t(\beta,m,z)+2R^{(N-1)}(\beta,z)]-R^{(N-1)}(\beta,z) \tag{28}$$

and Eq. (19). Basically this is the same iteration process which was used to construct the expansion. However, this time we calculate the full function $Z(\beta,z)$ numerically, starting from

$$Z^{(1)}(\beta,z)=Z_{\mathrm{Y}}[t(\beta,m,z)] . \tag{29}$$

In each step one calculates a value $\beta_{\mathrm{H}}^{(N)}$ by

$$R^{(N)}(\beta_{\mathrm{H}}^{(N)},z)=\ln 2-\tfrac{1}{2}[1+t(\beta_{\mathrm{H}}^{(N)},m,z)] . \tag{30}$$

The sequence $\{\beta_H^{(N)}\}$ converges monotonically to β_H.[6] As a result we obtain $Z(\beta, z)$ in the range $\beta_H < \beta < \infty$. In Fig. 1 we show for $B = 3.145\ \mathrm{GeV}^{-2}$ the partition functions $Z(z=1)$, $Z(z=2)$, Z_Y and for comparison $Z_f(z=1)$ as functions of $t(\beta, m, z)$. The curves for the bootstrap partition functions exhibit square-root behaviors, but for small t follow fairly well the ideal-gas result $Z_f(z=1)$.

From the expansion Eq. (8) it is evident that an increase of z will decrease the convergence radius in $t(\beta, m, z=1)$ and therefore lower the highest temperature, all expansion coefficients being positive. For the B value chosen the new Hagedorn temperatures are $T_H(z=1) = 160$ MeV, $T_H(z=2) = 119.2$ MeV, whereas in the Boltzmann case one has $T_{YH}(z=1) = 169$ MeV and $T_{YH}(z=2) = 130.6$ MeV.

IV. SIZE OF BOSE-EINSTEIN EFFECTS

While one-particle spectra are little affected by quantum statistics within the bootstrap picture,[8] one expects, of course, sizeable effects in two-particle distributions from fireball decay. In the following we shall present some exploratory calculations for the decay of a fireball into identical pions.

The inclusive distributions from fireball decay can be obtained from $\tau(Q^2, z)$, Eq. (10), by omitting an appropriate number of integrations in the phase-space integrals Ω^l (the omitted integrations are indicated by square brackets):

$$\frac{2p_0}{\sigma_{\rm tot}}\frac{d^3\sigma}{d^3p} = \tau^{-1}(Q^2, z)\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k=0}^{\infty} h(n_1,\ldots,n_k)B^{l-1}z^n\sum_{j=1}^{k} j^3 n_j \Omega^{l-1}((Q-jp)^2;\ldots,[jm],\ldots) \tag{31}$$

and

$$\begin{aligned} F_2(\vec{p}, \vec{p}\,') \equiv \frac{4p_0p_0'}{\sigma_{\rm tot}}\frac{d^6\sigma}{d^3p\,d^3p'} &= \tau^{-1}(Q^2, z)\sum_{k=1}^{\infty}\sum_{n_1,\ldots,n_k=0}^{\infty} h(n_1,\ldots,n_k)B^{l-1}z^n \\ &\times\Bigg[\sum_{j,j'=1}^{k} j^3 j'^3 n_j(n_{j'} - \delta_{jj'})\Omega^{l-2}((Q-jp-j'p')^2;\ldots,[jm],\ldots,[j'm],\ldots) \\ &+ 2p_0\delta^{(3)}(\vec{p}-\vec{p}\,')\sum_{j=1}^{k} j^3(j-1)n_j\Omega^{l-1}((Q-jp)^2;\ldots,[jm],\ldots)\Bigg]. \end{aligned} \tag{32}$$

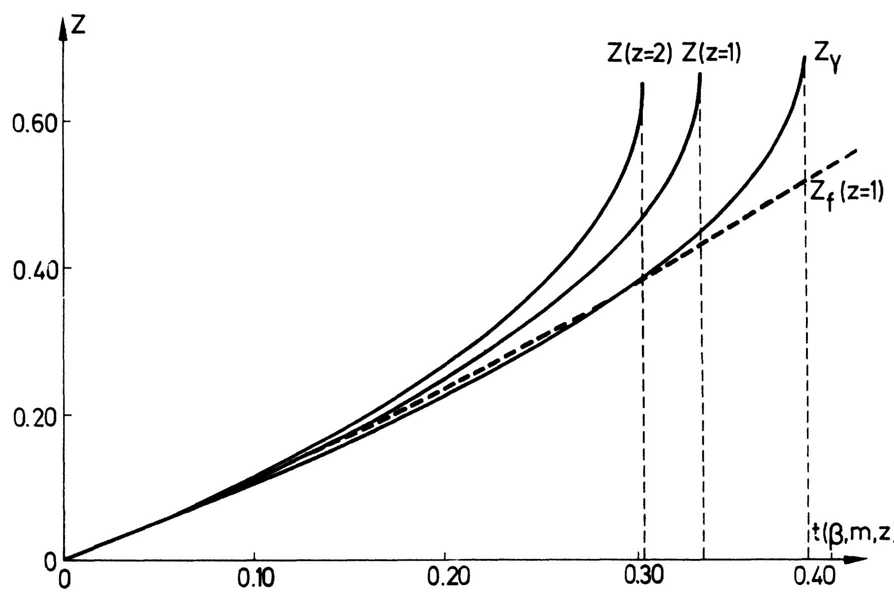

FIG. 1. Partition functions for the ideal relativistic gas Z_f for $z=1$ (dashed curve) of the Boltzmann bootstrap Z_Y, and of the Bose bootstrap Z for $z=1$ and $z=2$ (solid curves) plotted vs $t(\beta, m, z)$. All curves are calculated with $B = 3.145\ \mathrm{GeV}^{-2}$.

The j and j' summations run over the observed clusters, and the powers in j and j' originate from the conversion of cluster to pion phase-space measure. The $\delta^{(3)}$-function term in F_2 is due to the pions coming out of one cluster.

In order to demonstrate the effects of the interaction induced by the bootstrap, we have evaluated the ideal-gas expressions as well. They are obtained from Eqs. (31) and (32) by the replacement

$$h(n_1,\ldots,n_k)\to\prod_{i=1}^{k}\frac{k^{-3n_i}}{n_i\,!}\,,\quad B\to B_f\;. \tag{33}$$

One constraint for the determination of the parameters B and z in the bootstrap scheme is given by the asymptotic temperature T_H, which is believed to be about 160 MeV. In applications of the ideal gas to particle physics, it is customary to choose $z=1$ ("free-particle creation").[7] For lack of better knowledge about the size of this parameter, we have decided to use this value for comparison in the bootstrap scheme as well. This amounts to a value of $B=3.145\ \text{GeV}^{-2}$ for the Bose and $B=3.655\ \text{GeV}^{-2}$ for the Boltzmann bootstrap (of $T_H=160$ MeV). The comparison to the ideal-gas predictions was done at $Q^2=9\ \text{GeV}^2$, therefore the parameter B_f was fitted to reproduce the Bose bootstrap value of the mean decay multiplicity $\bar{n}$, leading to a value $B_f=65\ \text{GeV}^{-2}$.

In Figs. 2 to 5 we show a representative sample

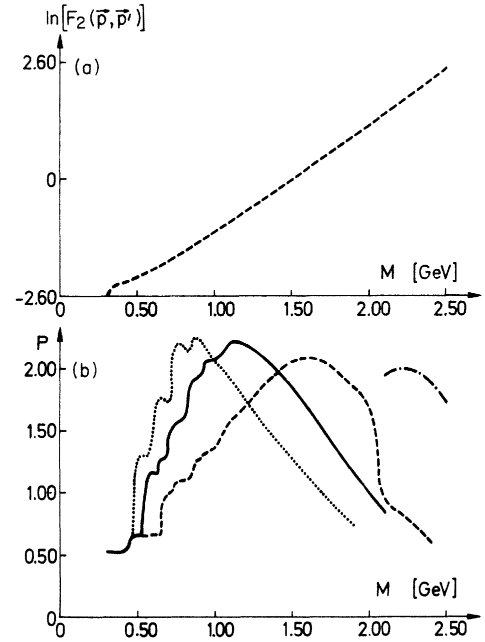

FIG. 3. (a) $\ln F_2$, where F_2 is given in units of GeV^{-4}, for the Boltzmann bootstrap plotted as function of the missing mass $M=[(Q-p-p')^2]^{1/2}$ for fixed $(Q^2)^{1/2}=3$ GeV, $\phi=180°$. This curve is universal in $m_{2\pi}$. (b) Ratio $P=F_2^{BSB}/F_2^{BZB}$ plotted vs M for various values of $m_{2\pi}$. $m_{2\pi}=500$ MeV (dot-dashed curve), $m_{2\pi}=700$ MeV (dashed curve) $m_{2\pi}=900$ MeV (solid curve), $m_{2\pi}=1100$ MeV (dotted curve) and fixed $(Q^2)^{1/2}=3$ GeV, $\phi=180°$.

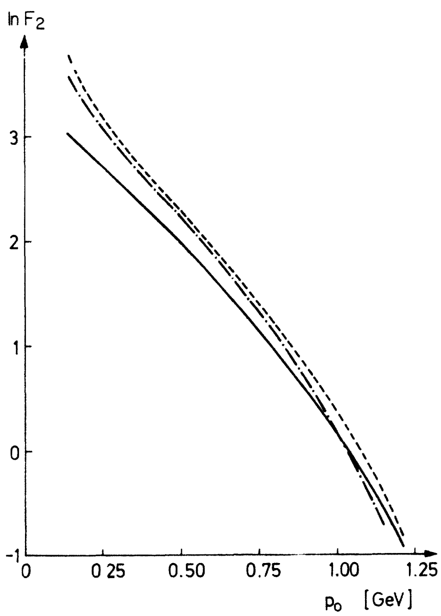

FIG. 2. Two-particle distribution $\ln F_2$, where F_2 is given in units of GeV^{-4} [see Eq. (32)], predicted by Bose bootstrap (dashed curve), Boltzmann bootstrap (solid curve), and the ideal gas (dot-dashed curve) plotted as a function of p_0 for fixed $p_0'=m$, $(Q^2)^{1/2}=3$ GeV, $\phi=180°$.

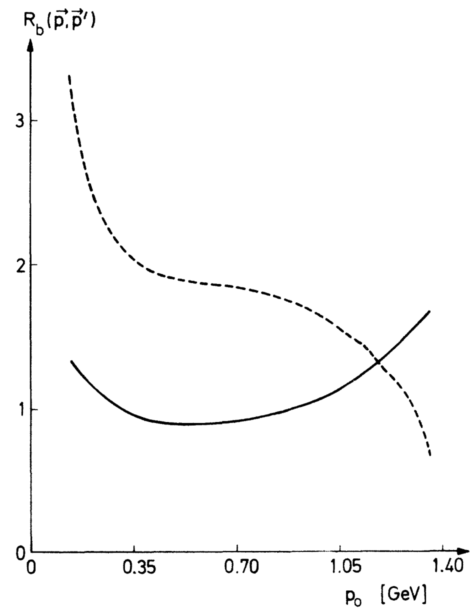

FIG. 4. $R_b=(F_2^{BSB}/F_2^{BZB})\,[1-\exp(-p_0/0.16\ \text{GeV})]$ plotted vs p_0 for $p_0'=m$ (dashed curve) and $p_0'=355$ MeV (solid curve), and fixed $(Q^2)^{1/2}=3$ GeV, $\phi=180°$.

of results obtained at 3-GeV fireball mass for the inclusive two-particle distribution function that depends essentially on three variables, e.g., on p_0, p_0' and the angle ϕ between the momenta p and p'. Figure 2 contains a comparison between the Bose (BSB) and Boltzmann (BZB) bootstrap predictions and the ideal Bose gas at 3-GeV fireball mass. Here we have plotted the invariant cross section F_2, Eq. (32), with $p_0'=m$ fixed, against the energy p_0 of the other secondary. We observe that for both particles at rest there is a factor 6 difference between BSB and BZB. In addition there is the $\delta^{(3)}$-function contribution that is due to both particles emanating from one k cluster. In other regions of $(\vec{p},\vec{p}')$ space, the differences between BSB and BZB are less dramatic, yet they still amount to 100%.

The most comprehensive comparison between BSB and BZB can be made, when we plot F_2 as a function of the missing mass $M=[(Q-p-p')^2]^{1/2}$ and, say $m_{2\pi}=(p+p')^2$ and ϕ. Since the BZB predictions depend only on M and not on $m_{2\pi}$ and ϕ, this choice of variables shows Bose effects most clearly in the form of $m_{2\pi}$ and ϕ dependence of BSB predictions and deviations of BSB distributions from the universal BZB curve. This situation is displayed in Fig. 3, which shows in its top part the BZB distribution as function of M and the ratio between BSB and BZB for $\phi=180°$ and various values of $m_{2\pi}$ in its bottom part. Again, we realize 100% effects due to Bose statistics. We find strong deviations from a factorizing form of the two-particle distribution. The kinematical part of this correlation is divided out by calculating the ratio BSB to BZB. This ratio, normalized for convenience by the respective ratio of thermodynamic expressions for Bose and Boltzmann spectra $[\exp(p_0/kT)-1]^{-1}$ and $\exp(-p_0/kT)$ (with $T=160$ MeV) is shown in Fig. 4. The remaining correlations which are due to statistics are seen in form of a strong p_0 dependence.

In practical applications, it is frequently assumed[10] that F_2 can be written as

$$F_2^{\text{naive}}(\vec{p},\vec{p}')=[\exp(p_0/kT)-1]^{-1}[\exp(p_0'/kT)-1]^{-1}f\ , \qquad (34)$$
$$f=1+\exp[-(\vec{p}-\vec{p}')^2/\sigma^2]\,,$$

with $\sigma=200$ MeV. We tested this assumption for the ideal Bose gas by plotting in Fig. 5 $R_i = F_2^{\text{ideal gas}}/F_2^{\text{naive}}$ for various values of ϕ and $p_0'=355$ MeV. For $\phi=90°$, we achieved a fairly flat R_i distribution by choosing $T=140$ MeV. The remaining strong ϕ dependence demonstrates clearly that the factor f in Eq. (34) does not reproduce our correlations.

Since k clusters are important and phase space cuts them out fairly fast at low fireball mass, we

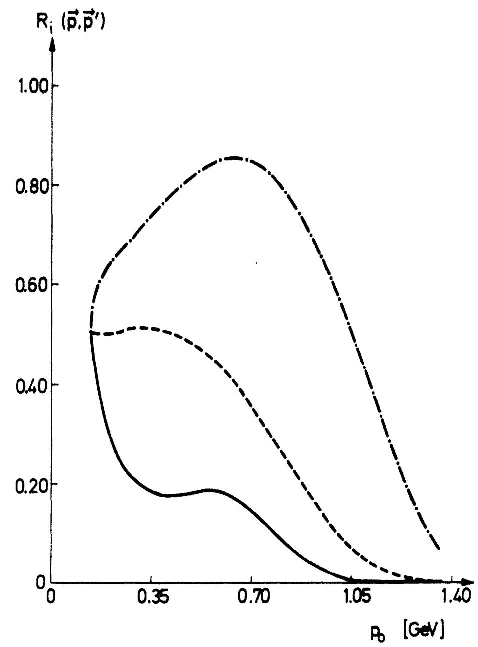

FIG. 5. $R_i = F^{\text{ideal gas}}/F_2^{\text{naive}}$ in arbitrary scale [see Eq. (34)] plotted vs p_0 for $\phi=0°$ (solid curve), $\phi=90°$ (dashed curve), $\phi=180°$ (dot-dashed curve) for $(Q^2)^{1/2}=3$ GeV and $p_0'=355$ MeV. A δ contribution [see Eq. (32)] is to be added to the $\phi=0°$ curve at $p_0=355$ MeV.

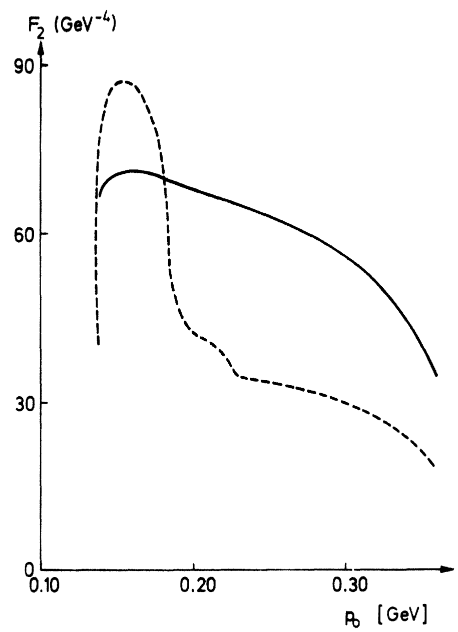

FIG. 6. The structure induced by individual phase-space integrals in low-mass fireball decay. $F_2(\vec{p},\vec{p}')$ plotted vs p_0 at fixed $(Q^2)^{1/2}=1$ GeV, $\phi=180°$, $p_0'=334$ MeV for Bose bootstrap (dashed curve) and Boltzmann bootstrap (solid curve).

expect to see structures in lower-mass fireball decay induced by individual k clusters. We performed calculations at 1-GeV fireball mass and present some of the resulting p_0 distributions for BSB and BZB in Fig. 6. There is a very marked kink in the p_0 distribution predicted by BSB near $p_0 = 191$ MeV that is related to the kinematic limit of the particular phase-space integral $\Omega^2((Q-2p-p')^2, m, m)$ contributing to $F_2(\vec{p}, \vec{p}')$. This structure is of course absent in the corresponding BZB curve.

At large fireball mass such structures due to individual k cluster contributions to F_2 occur very close to $p_0 = (Q^2)^{1/2}/2$ and are therefore much less striking. This is the region of low missing mass M. The variable M is therefore suitable to present the effect in magnified form, as is seen in the bottom part of Fig. 3.

We conclude that correct counting of quantum states indeed leads to sizable effects in two-particle spectra from statistical models.

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[9]We thank Dr. Hagedorn for an enlightening discussion which inspired the preceding presentation.

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