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The category of good modules over a quasi–hereditary algebra has almost split sequences

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Abstract. Let A be a quasi–hereditary algebra. The aim of this paper is to show that the category of all A–modules with good filtrations is functorially finite in A–mod, thus it has (relative) almost split sequences. This follows from a general result dealing with arbitrary artin algebras. For quasi–hereditary algebras, we will consider the relation between four rather interesting subcategories, one of them being the category of modules with good filtrations, and we will exhibit one particular module which is both a tilting and a cotilting module. It turns out that the quasi–hereditary algebras always come in pairs.

Part I

Let A be an artin algebra, and A–mod the category of (finitely generated left) A–modules. [Maps will be written on the opposite site of the scalars, thus the composition of two maps $\alpha : M_1 \longrightarrow M_2, \beta : M_2 \longrightarrow M_3$ in A–mod is denoted by $\alpha\beta$.]

1. The main theorem

[We recall some definitions from [AS]. Let $\mathcal{X}$ be a full subcategory of A–mod. Let M be an A–module. A *right $\mathcal{X}$–approximation of M* is a map $\gamma : X \longrightarrow M$ with $X \in \mathcal{X}$ such that for any map $\gamma' : X' \longrightarrow M$ with $X' \in \mathcal{X}$ there exists a map $\xi : X' \longrightarrow X$ satisfying $\gamma' = \xi\gamma$. A *left $\mathcal{X}$–approximation of M* is a map $\beta : M \longrightarrow X$ with $X \in \mathcal{X}$ such that for any map $\beta' : M \longrightarrow X'$ with $X' \in \mathcal{X}$ there exists a map $\xi : X \longrightarrow X'$ satisfying $\beta' = \beta\xi$. The subcategory $\mathcal{X}$ is *closed under direct summands* provided for every module $X \in \mathcal{X}$, any direct summand of X belongs to $\mathcal{X}$, and $\mathcal{X}$ is *closed under extensions* provided for every exact sequence $0 \longrightarrow X_1 \longrightarrow M \longrightarrow X_2 \longrightarrow 0$ with $X_1, X_2 \in \mathcal{X}$, also $M \in \mathcal{X}$.] A full subcategory $\mathcal{X}$ is said to be *functorially finite* in A–mod provided every A–module M has both a right $\mathcal{X}$–approximation and a left $\mathcal{X}$–approximation. In contrast to [AS], we do not assume that $\mathcal{X}$ is closed under direct summands.

Let $\Theta = \{\Theta(1), \dots, \Theta(n)\}$ be a finite set of A–modules with $Ext^1_A(\Theta(j), \Theta(i)) = 0$ for $j \geq i$. We denote by $\mathcal{F}(\Theta)$ the full subcategory of A–mod of direct summands of modules having a filtration with factors in Θ. [Thus, M belongs to $\mathcal{F}(\Theta)$ if and only if M has submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ such that M_s/M_{s-1} is isomorphic to a module in Θ.]

Theorem 1. *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in A–mod.*

The reader should observe that in this way we obtain a large variety of functorially finite subcategories of A–mod which usually will not be closed under submodules or factor modules.

Auslander and Smalø([AS], theorem to 2.4) have shown that a functorially finite subcategory which is closed under extensions and direct summands has (relative) almost split sequences. Let $\mathcal{X}(\Theta)$ be the full subcategory of A–mod of all modules which are direct summands of modules in $\mathcal{F}(\Theta)$. Then $\mathcal{X}(\Theta)$ is closed under extensions and direct summands, and with $\mathcal{F}(\Theta)$ also $\mathcal{X}(\Theta)$ is functorially finite in A–mod. Therefore we obtain the following consequence:

Corollary 1. *The category $\mathcal{X}(\Theta)$ has almost split sequences.*

[We recall the definitions from [AS], see also [R]. Let $\mathcal{X}$ be a full subcategory of A–mod closed under direct summands. Let X be in $\mathcal{X}$. A map $\gamma : Y \longrightarrow X$ is said to be a *sink map* for X (or to be right almost split) provided γ is not a split epimorphism, and given a map $\gamma' : Y' \longrightarrow X$ which is not a split epimorphism, there exists $\eta : Y' \longrightarrow Y$ with $\gamma' = \eta\gamma$. A map $\beta : X \longrightarrow Y$ is said to be a *source map for* X (or to be left almost split) provided β is not a split monomorphism,, and given a map $\beta' : X \longrightarrow Y'$ which is not a split monomorphism, there exists $\eta : Y \longrightarrow Y'$ with $\beta' = \beta\eta$. A (relative) *almost split sequence* in $\mathcal{X}$ is an exact sequence $0 \longrightarrow X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \longrightarrow 0$ in A–mod with X, Y, Z in $\mathcal{X}$ such that β is a source map, and γ a sink map. An object $X \in \mathcal{X}$ is said to be *Ext–injective in* $\mathcal{X}$ provided $Ext^1_A(M, X) = 0$ for all $M \in \mathcal{X}$; an object $Z \in \mathcal{X}$ is said to be *Ext–projective in* $\mathcal{X}$ provided $Ext^1_A(Z, M) = 0$ for all $M \in \mathcal{X}$. We say that $\mathcal{X}$ *has* (relative) *almost split sequences* provided the following three conditions are satisfied: first, every indecomposable object $X \in \mathcal{X}$ has a sink map and a source map; second, if X is indecomposable in $\mathcal{X}$ and not Ext–injective in $\mathcal{X}$, then there exists an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{X}$, and third, if Z is indecomposable in $\mathcal{X}$ and not Ext–projective in $\mathcal{X}$, then there exists an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{X}$.]

[Once the $\mathcal{X}(\Theta)$–approximations of the A–modules are known, it is rather easy to construct the sink maps and the source maps in $\mathcal{X}(\Theta)$. Namely, given an indecomposable module X in $\mathcal{X}(\Theta)$, let $\psi : U \longrightarrow X$ be its sink map in A–mod, and $\varphi : X \longrightarrow V$ its source map in A–mod; let $\gamma : X' \longrightarrow U$ be a right $\mathcal{X}$–approximation of U and $\beta : V \longrightarrow X''$ a left $\mathcal{X}$–approximation of V. Then a right minimal version of $\gamma\psi : X' \longrightarrow X$ is a sink map for X in $\mathcal{X}(\Theta)$, a left minimal version of $\varphi\beta : X \longrightarrow X''$ is a source map for X in $\mathcal{X}(\Theta)$.]

Remarks concerning the definition of $\mathcal{F}(\Theta)$ and $\mathcal{X}(\Theta)$: The reader should be aware that categories of modules with prescribed filtrations usually will not be closed under direct summands even if Θ consists of indecomposable modules. A typical example is given by $A = \begin{bmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$, where k is a field, $\Theta(2)$ the simple projective A–module, $\Theta(1)$ its injective hull; here the indecomposable modules of length 2 belong to $\mathcal{X}(\Theta)$, but not to $\mathcal{F}(\Theta)$. — On the other hand, an A–module M belongs to $\mathcal{F}(\Theta)$ if and only if M has a filtration $0 = M_{n+1} \subseteq M_n \subseteq \ldots \subseteq M_1 = M$ with M_i/M_{i+1} isomorphic to a direct sum of copies of $\Theta(i)$, for all $1 \leq i \leq n$. This is an immediate consequence of our assumption $Ext^1_A(\Theta(j), \Theta(i)) = 0$ for $j \geq i$.

2. Proof of the main theorem.

We start with an arbitrary full subcategory $\mathcal{X}$ of A–mod, and we denote by $\mathcal{Y}$ the full subcategory of A–mod of all modules Y satisfying $Ext^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$.

Lemma 1. *Let* $0 \longrightarrow Y \longrightarrow X \xrightarrow{\gamma} M \longrightarrow 0$ *be exact, with* $X \in \mathcal{X}$ *and* $Y \in \mathcal{Y}$. *Then* γ *is a right* $\mathcal{X}$*-approximation of* M.

[This is a converse of Wakamatsu's lemma as stated in [AR].]

[**Proof.** Let $\gamma' : X' \longrightarrow M$ be a map with $X' \in \mathcal{X}$. Since $Y \in \mathcal{Y}$, the sequence induced from the given one by γ' splits, thus we obtain $\xi : X' \longrightarrow X$ with $\gamma' = \xi\gamma$.]

Lemma 2. *Assume that* $\mathcal{X}$ *is closed under extensions, and that for every* A*-module* N *there exists an exact sequence* $0 \longrightarrow N \longrightarrow Y^N \longrightarrow X^N \longrightarrow 0$ *with* $X^N \in \mathcal{X}$ *and* $Y^N \in \mathcal{Y}$. *Then every* A*-module* M *has a right* $\mathcal{X}$*-approximation.*

Proof: Let M be an A–module. First, we assume that there is an epimorphism $\pi : X \longrightarrow M$ with $X \in \mathcal{X}$; let $K = Ker\ \pi$. The exact sequence $0 \longrightarrow K \longrightarrow Y^K \longrightarrow X^K \longrightarrow 0$ gives rise to a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & Y^K & \longrightarrow & X^K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & X^K & \longrightarrow & 0 \\
 & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\gamma} & & & & \\
 & & M & = & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since X, X^K belong to $\mathcal{X}$ and $\mathcal{X}$ is closed under extensions, $Z \in \mathcal{X}$. Since $Y^K \in \mathcal{Y}$, we use lemma 1 for the exact sequence which appears as middle column and conclude that $\gamma : Z \longrightarrow M$ is a right $\mathcal{X}$–approximation.

In general, let M' be the submodule of M generated by the images of maps $X' \longrightarrow M$ with $X' \in \mathcal{X}$. There is a finite set of maps $\pi_i : X_i \longrightarrow M$, with $X_i \in \mathcal{X}$ such that the images of π_i generate M'. Since $\mathcal{X}$ is closed under direct sums, $X = \bigoplus_i X_i$ belongs to $\mathcal{X}$, and there is an epimorphism $\pi : X \longrightarrow M'$. The previous considerations yield a right $\mathcal{X}$–approximation $\gamma' : Z \longrightarrow M'$. We denote by $\mu : M' \longrightarrow M$ the inclusion map; clearly, $\gamma'\mu$ is a right $\mathcal{X}$–approximation of M.

Now, let $\mathcal{X} = \mathcal{F}(\Theta)$. Then $\mathcal{Y} = \mathcal{Y}(\Theta)$ may be characterized alternatively as the full subcategory of A–mod of all modules Y satisfying $Ext^1_A(\Theta(i), Y) = 0$ for $1 \leq i \leq n$.

Lemma 3. *Let* $1 \leq t \leq n$. *let* N *be an* A*-module with* $Ext^1_A(\Theta(j), N) = 0$ *for all* $j > t$. *Then there exists an exact sequence* $0 \longrightarrow N \longrightarrow N' \longrightarrow Q \longrightarrow 0$ *with* Q *a direct sum of copies of* $\Theta(t)$ *and* $Ext^1_A(\Theta(j), N') = 0$ *for all* $j \geq t$.

Proof: Let

$$\varepsilon = (0 \to N \to N' \to Q \to 0)$$

be a universal extension of N from above by copies of $\Theta(t)$ [this means the following: take exact sequences $\varepsilon_s = (0 \longrightarrow N \longrightarrow T_s \longrightarrow \Theta(t) \longrightarrow 0)$ so that the corresponding equivalence classes $[\varepsilon_1], \dots, [\varepsilon_m]$ generate $Ext^1_A(\Theta(t), N)$ as left $End_A(\Theta(t))$–module, and let ε be "the" exact sequence so that the s–th inclusion of $\Theta(t)$ into $Q = \Theta(t)^m$ induces the sequence ε_s]. Thus, the connecting homomorphism

$$\delta : Hom_A(\Theta(t), Q) \longrightarrow Ext^1_A(\Theta(t), N)$$

induced by ε is surjective. We show that $Ext^1_A(\Theta(t), N') = 0$ for $j \geq t$. We consider the exact sequence

$$Hom_A(\Theta(j), Q) \longrightarrow Ext^1_A(\Theta(j), N) \longrightarrow Ext^1_A(\Theta(j), N') \longrightarrow Ext^1_A(\Theta(j), Q).$$

Since $j \geq t$, the last term $Ext^1_A(\Theta(j), \Theta(t)^m)$ vanishes. For $j > t$, we know by induction that $Ext^1_A(\Theta(j), N) = 0$. For $j = t$, the first map is just δ, thus surjective. Therefore, for all $j \geq t$, we have $Ext^1_A(\Theta(j), N') = 0$.

Lemma 4. *Let $1 \leq t \leq n$. Let N be an A–module with $Ext^1_A(\Theta(j), N) = 0$ for all $j > t$. Then there exists an exact sequence $0 \longrightarrow N \longrightarrow Y \longrightarrow X \longrightarrow 0$ with $X \in \mathcal{F}(\{\Theta(1), \dots, \Theta(t)\})$ and $Y \in \mathcal{Y}(\Theta)$.*

Proof: By reverse induction, we construct monomorphisms

$$N = N_{t+1} \xrightarrow{\mu_t} N_t \xrightarrow{\mu_{t-1}} \dots \xrightarrow{\mu_1} N_1 = Y$$

with $Q_i = coker\ \mu_i$ a direct sum of copies of $\Theta(i)$, and $Ext^1_A(\Theta(j), N_i) = 0$ for all $j \geq i$. Let $\mu = \mu_t \dots \mu_1 : N \longrightarrow Y$, and $X = Cok\ \mu$. Then Y belongs to $\mathcal{Y}(\Theta)$, and X has a filtration with factors Q_i [without loss of generality, we can assume that all μ_i are inclusion maps; the filtration of $X = Y/N$ is given by the submodules N_i/N, with $1 \leq i \leq t+1$, and $(N_i/N)/(N_{i+1}/N) \cong N_i/N_{i+1} \cong Q_i$ for $1 \leq i \leq t$], thus $X \in \mathcal{F}(\{\Theta(1), \dots, \Theta(t)\})$.

Of particular interest is the case $t = n$ which may be formulated as follows:

Lemma 4'. *For every A–module N, there exists an exact sequence $0 \longrightarrow N \longrightarrow Y \longrightarrow X \longrightarrow 0$ with $X \in \mathcal{F}(\Theta)$ and $Y \in \mathcal{Y}(\Theta)$.*

The proof of the main theorem is now straight–forward. Lemma 3 asserts that the assumptions of lemma 4' are satisfied for $\mathcal{X} = \mathcal{F}(\Theta)$ and $\mathcal{Y} = \mathcal{Y}(\Theta)$, thus every A–module has a right $\mathcal{F}(\Theta)$–approximation. Since the construction of $\mathcal{F}(\Theta)$ is self–dual, we may use duality in order to obtain also left $\mathcal{F}(\Theta)$–approximations. This finishes the proof.

We should remark that our proof, in particular lemma 2, is inspired by a recent paper of Auslander and Reiten [AR].

We may reformulate lemma 4' as follows. [Recall that a full subcategory $\mathcal{Z}$ of A–mod closed under direct summands is said to be *contravariantly finite in* A–mod provided every A–module has a right $\mathcal{Z}$–approximation, and to be *covariantly finite in* A–mod provided every A–module has a left $\mathcal{Z}$–aproximation.]

Proposition 1. *The subcategory $\mathcal{Y}(\Theta)$ is covariantly finite in A–mod.*

Proof: Let $0 \longrightarrow N \xrightarrow{\beta} Y \longrightarrow X \longrightarrow 0$ be exact, with $X \in \mathcal{F}(\Theta)$ and $Y \in \mathcal{Y}(\Theta)$. Since $Ext^1_A(X, Y') = 0$ for all $Y' \in \mathcal{Y}(\Theta)$, we can use the dual of lemma 1 in order to conclude that β is a left $\mathcal{Y}(\Theta)$–approximation.

Proposition 1*. *The full subcategory $\mathcal{W}(\Theta)$ of all A–modules W with $Ext^1_A(W, \Theta(i)) = 0$ for $1 \leq i \leq n$, is contravariantly finite in A–mod.*

[We remark that the proposition and its dual can also be obtained as direct consequences of the assertion of our main theorem, see section 1 of [**AR**]].

Note that, by definition, the modules in $\mathcal{F}(\Theta) \cap \mathcal{Y}(\Theta)$ are the Ext–projective objects of $\mathcal{F}(\Theta)$, the modules in $\mathcal{F}(\Theta) \cap \mathcal{W}(\Theta)$ are the Ext–injective objects of $\mathcal{F}(\Theta)$.

Remark. If we weaken the assumptions on Θ, then the subcategory $\mathcal{F}(\Theta)$ no longer has to be functorially finite in A–mod. For example, let $A = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$ be the Kronecker algebra, k a field. Let S be an indecomposable A–module of length 2. For $\Theta = \{S\}$, the category $\mathcal{F}(\Theta)$ of modules having filtrations with factors S, is neither covariantly, nor contravariantly finite in A–mod. For $\Theta = \{P, S\}$, with P the simple projective A–module, $\mathcal{F}(\Theta)$ is a covariantly finite (and resolving) subcategory, but not contravariantly finite in A–mod.

3. Application

Let $E(1), \ldots, E(n)$ be the simple A–modules; note that we fix a particular ordering. For $1 \leq i \leq n$, let $P(i)$ be the projective cover of $E(i)$, and $Q(i)$ the injective envelop of $E(i)$. We denote by $U(i)$ the sum of all images of maps $P(j) \longrightarrow P(i)$ with $j > i$, and $\Delta(i) = P(i)/U(i)$. Also, let $\nabla(i)$ be the intersection of all kernels of maps $Q(i) \longrightarrow Q(j)$ with $j > i$. Then, we have

$$Ext^1_A(\Delta(j), \Delta(i)) = 0 \quad \text{for} \quad j \geq i$$

and

$$Ext^1_A(\nabla(j), \nabla(i)) = 0 \quad \text{for} \quad j \leq i\ ,$$

thus, we can apply theorem 1 both to $\Delta = \{\Delta(1), \ldots, \Delta(n)\}$ and to $\nabla = \{\nabla(1), \ldots, \nabla(n)\}$.

An alternative description of $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ is as follows. Let J_i be the image of all maps $P(j) \longrightarrow {}_AA$ with $j \leq i$, thus

$$A = J_1 \supset J_2 \supset \ldots \supset J_n \supset J_{n+1} = 0.$$

A module M belongs to $\mathcal{F}(\Delta)$ if and only if $J_iM/J_{i+1}M$ is projective as an A/J_{i+1}-module, for $1 \le i \le n$. And similarly, M belongs to $\mathcal{F}(\nabla)$ if and only if $J_iM/J_{i+1}M$ is injective as an A/J_{i+1}-module, for $1 \le i \le n$. It follows that $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ both are closed under direct summands, thus $\mathcal{F}(\Delta) = \mathcal{X}(\Delta)$, and $\mathcal{F}(\nabla) = \mathcal{X}(\nabla)$.

Theorem 2. *The subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite subcategories which are closed under extensions and direct summands. In particular, both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ have (relative) almost split sequences.*

We should remark that for $j < i$, we have

$$Hom_A(P(i), \nabla(j)) = 0, \quad \text{and} \quad Hom_A(\Delta(j), Q(i)),$$

since $E(i)$ does not occur as composition factor of $\nabla(j)$ or $\Delta(j)$. Consequently, we have

$$Hom_A(\Delta(i), \nabla(j)) \neq 0 \quad \text{if and only if} \quad i = j.$$

For later reference, we also note the following:

Lemma 5. *Let M be an A-module. If $Hom_A(\Delta(i), M) = 0$ for all $1 \le i \le n$, then $M = 0$. If $Hom_A(M, \nabla(i)) = 0$ for all $1 \le i \le n$, then $M = 0$.*

Proof: Assume $Hom_A(\Delta(i), M) = 0$ for all $1 \le i \le n$. Since $Hom_A(\Delta(n), M) = 0$, it follows that M is annihilated by J_n, thus an A/J_n-module. By induction, $M = 0$. The second assertion follows by duality.

Part II. Quasi–hereditary algebras

[We recall some definitions from [S], see also **[PS]**, **[DR1]**. As before, let A be an artin algebra, and we fix some ordering of the simple A-modules $E(1), \dots, E(n)$, thus the modules $\Delta(i)$ and $\nabla(i)$ are defined. The algebra A is said to be *quasi-hereditary* provided first, ${}_AA$ belongs to $\mathcal{F}(\Delta)$, and second, $E(i)$ occurs with multiplicity one in $\Delta(i)$, (or, equivalently, $End_A(\Delta(i))$ is a division ring) for every $1 \le i \le n$,. If we want to stress that we have fixed the ordering $E(1), \dots, E(n)$, we say that we deal with the quasi–hereditary algebra (A, E).]

We assume from now on that A, or better (A, E), is quasi–hereditary. The modules in Δ are said to be the *standard* (or Verma, or Weyl) modules, those in ∇ will be called the *costandard* (or induced) modules. [We prefer the notation $\Delta(i), \nabla(i)$ introduced in **[DR3]** in contrast to the notation $V(i), A(i)$ of Cline–Parshall–Scott **[CPS1, CPS2]**, since $\Delta(i)$ always has simple top, $\nabla(i)$ simple socle: so the shape of the letters visualizes the shape of the modules.]

The modules in $\mathcal{F}(\Delta)$ will be said to be *good* (or to have a good filtration, or a Weyl filtration), those in $\mathcal{F}(\nabla)$ will be said to be *cogood.*

[The usual definition of a quasi–hereditary algebra uses induction. First, one introduces the notion of a heredity ideal, this is an idempotent ideal J satisfying $JNJ = 0$, where N is the radical of A, which is projective when considered as a left A–module, and one calls A quasi–hereditary provided there exists a heredity ideal J in A such that A/J is quasi–hereditary. With the notation introduced above, we always will work with the heredity ideal $J = J_n = AeA$, where e is an idempotent of A such that $\Delta(n)$ is isomorphic to Ae; note that A/J is quasi–hereditary with respect to $E(1), \ldots, E(n-1)$, so that we can use induction.]

4. The good modules and the cogood modules

As we have seen, the subcategory $\mathcal{F}(\Delta)$ of good A–modules and the subcategory $\mathcal{F}(\nabla)$ of cogood modules are functorially finite in A–mod.

In their recent paper **[AR]**, Auslander and Reiten have drawn attention to subcategories which are both contravariantly finite and resolving. Now, the category $\mathcal{F}(\Delta)$ of good modules is functorially finite, thus contravariantly finite. It is also resolving as we want to add. [Recall that a full subcategory $\mathcal{X}$ of A–mod is said to be *resolving* if $\mathcal{X}$ is closed under extensions, closed under kernel of surjective maps, and contains the projective A–modules.]

Theorem 3. *Let A be quasi–hereditary. The category $\mathcal{F}(\Delta)$ of good A–modules is a resolving subcategory of A–mod.*

Proof: Clearly $\mathcal{F}(\Delta)$ is closed under extensions and contains the projective A–modules. It remains to show that $\mathcal{F}(\Delta)$ is closed under kernels of surjective maps. Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be exact, with M_2, M_3 in $\mathcal{F}(\Delta)$. Let $J = AeA$ be the heredity ideal of A where e is an idempotent of A with Ae isomorphic to $\Delta(n)$. We denote by $\mu_i : Ae \underset{eAe}{\otimes} eM_i \longrightarrow M_i$ the multiplication map. Its image is JM_i. Since M_2, M_3 are good, the maps μ_2, μ_3 are monomorphism (**[DR2]**, lemma 2). We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ae \otimes eM_1 & \longrightarrow & Ae \otimes eM_2 & \longrightarrow & Ae \otimes eM_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0\,, \end{array}$$

where the upper sequence is obtained from the lower by multiplying with e and tensoring with Ae over eAe. Since eAe is a semisimple ring, the upper sequence is exact. Since μ_2 is a monomorphism, the same is true for μ_1, thus $JM_1 \cong Ae \otimes eM_1$ is a projective A-module. Since μ_3 is a monomorphism, the cokernel sequence

$$0 \longrightarrow M_1/JM_1 \longrightarrow M_2/JM_2 \longrightarrow M_3/JM_3 \longrightarrow 0$$

is exact. Now, M_2/JM_2 and M_3/JM_3 are good A/J–modules, by induction M_1/JM_1 is a good A/J–module. Altogether we see that M_1 is a good A–module.

We want to discuss some consequences.

Corollary 2. *The Ext–projective object in $\mathcal{F}(\Delta)$ are just the projective modules.*

Corollary 3. *Let $X \in \mathcal{F}(\Delta), Y \in \mathcal{Y}(\Delta)$. Then $Ext^i_A(X, Y) = 0$ for all $i \geq 1$.*

[**Proof.** Let $0 \longrightarrow X' \longrightarrow P \longrightarrow X \longrightarrow 0$ be exact, with P projective. Then for $i \geq 2$ $Ext^i_A(X,Y) \cong Ext^{i-1}_A(X',Y)$, and $Ext^{i-1}_A(X',Y) = 0$ by induction, since with X and P also $X \in \mathcal{F}(\Delta)$.]

Let us formulate the corresponding result for $\mathcal{F}(\nabla)$.

Theorem 3*. *Let A be quasi-hereditary. The category $\mathcal{F}(\nabla)$ is a coresolving subategory of A-mod.*

[We recall that a full subcategory $\mathcal{X}$ of A-mod is said to be *coresolving* if it is closed under extensions, cokernels of injective maps, and contains all injective A-modules.]

We now consider the relationship between $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$.

Theorem 4. *Let A be quasi-hereditary. Then $\mathcal{Y}(\Delta) = \mathcal{F}(\nabla)$.*

Proof: First, we recall that $Ext^1_A(\Delta(i), \nabla(j)) = 0$ for all i, j (see [**CPS1**]): Since $\Delta(n)$ is a projective, and $\nabla(n)$ is an injective A-module, we may assume $1 \leq i, j < n$. But for $1 \leq i, j < n$, we have $Ext^1_A(\Delta(i), \nabla(j)) = Ext^1_{A/J}(\Delta(i), \nabla(j))$, the latter being zero by induction. This implies that $\mathcal{F}(\nabla) \subseteq \mathcal{Y}(\Delta)$.

For the converse, let $Y \in \mathcal{Y}(\Delta)$. Let Y' be the maximal A/J-submodule of Y, thus $Y'' = Y/Y'$ can be embedded into a direct sum of copies of $\nabla(n)$. It follows that there is an exact sequence $0 \longrightarrow Y'' \longrightarrow Z \longrightarrow Z' \longrightarrow 0$ where Z is a direct sum of copies of $\nabla(n)$ and Z' is an A/J-module. The inclusion $Y'' \longrightarrow Z$ yields a monomorphism

$$Hom_A(\Delta(i), Y'') \longrightarrow Hom_A(\Delta(i), Z)\ ,$$

and $Hom_A(\Delta(i), Z) = 0$ for $1 \leq i < n$, since Z is a direct sum of $\nabla(n)$, therefore $Hom_A(\Delta(i), Y'') = 0$ for $1 \leq i < n$. The exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ yields an exact sequence

$$Hom_A(\Delta(i), Y'') \longrightarrow Ext^1_A(\Delta(i), Y) \longrightarrow Ext^1_A(\Delta(i), Y)\ ;$$

the last term is zero for all i, since $Y \in \mathcal{Y}(\Delta)$, the first term is zero for $1 \leq i < n$, thus $Ext^1_A(\Delta(i), Y') = 0$ for $1 \leq i < n$. By induction, it follows that the A/J-module Y' has a filtration with factors in $\{\nabla(1), \ldots, \nabla(n-1)\}$. In particular, $Y' \in \mathcal{F}(\Delta)$, thus $Y' \in \mathcal{Y}(\Delta)$ by the first part of the proof.

We want to apply the first part of lemma 5 in oder to show that $Z' = 0$. Since Z' is an A/J-module, $Hom_A(\Delta(n), Z') = 0$. For $i < n$, we have $Ext^2_A(\Delta(i), Y') = 0$ according to corollary 3. The exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ yields an exact sequence

$$Ext^1_A(\Delta(i), Y) \longrightarrow Ext^1_A(\Delta(i), Y'') \longrightarrow Ext^2_A(\Delta(i), Y')\ ;$$

the first term is zero, since $Y \in \mathcal{Y}(\Delta)$, and we have just observed that the last term is zero for $i < n$. Thus $Ext^1_A(\Delta(i), Y'') = 0$ for $i < n$. From the exact sequence $0 \longrightarrow Y'' \longrightarrow Z \longrightarrow Z' \longrightarrow 0$, we obtain an exact sequence

$$Hom_A(\Delta(i), Z) \longrightarrow Hom_A(\Delta(i), Z') \longrightarrow Ext^1_A(\Delta(i), Y'') .$$

Since both end terms vanish for $i < n$, the same is true for the middle term. We see that $Hom_A(\Delta(i), Z') = 0$ for all $1 \leq i \leq n$, and therefore $Z' = 0$. This shows that $Y/Y' \cong Z$ is a direct sum of copies of $\nabla(n)$, thus $Y \in \mathcal{F}(\nabla)$. This finishes the proof.

Theorem 4*. *Let A be a quasi-hereditary. Then $\mathcal{F}(\Delta) = \mathcal{W}(\nabla)$.*

This is the dual assertion. Taking into account theorem 4, this just asserts that a module X belongs to $\mathcal{F}(\Delta)$ if and only if $Ext^1_A(X, Y) = 0$ for all $Y \in \mathcal{Y}(\Delta)$. [We observe that this also follows directly from theorem 3 using proposition 3.3 of [**AR**]].

Let us summarize: If A is quasi-hereditary, the two sets Δ and ∇ of modules yield four interesting full subcategories, namely

$$\mathcal{W}(\Delta), \quad \mathcal{F}(\Delta) = \mathcal{W}(\nabla), \quad \mathcal{Y}(\Delta) = \mathcal{F}(\nabla), \quad \mathcal{Y}(\Delta) .$$

The first three are contravariantly finite in A–mod, the last three are covariantly finite in A–mod, $\mathcal{F}(\Delta)$ is a resolving subcategory, $\mathcal{F}(\nabla)$ is a coresolving subcategory. The pair $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ seems to be a rather pretty example for the bijection between the resolving contravariantly finite subcategories and the coresolving covariantly finite subcategories, as studied by Auslander and Reiten [**AR**]. The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ have almost split sequences, the *Ext*–projective objects in $\mathcal{F}(\Delta)$ are the projective A–modules, the *Ext*–injective objects in $\mathcal{F}(\nabla)$ are the injective A–modules, finally, the modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ are just the *Ext*–injective objects in $\mathcal{F}(\Delta)$ and also precisely the *Ext*–projective objects in $\mathcal{F}(\nabla)$. We will deal with $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ below.

Here are the recipes for obtaining the various approximations. Let M be an A–module. In order to obtain a right $\mathcal{F}(\Delta)$–approximation of M, we use the dual of lemma 3 for $\mathcal{F}(\nabla)$: it gives an exact sequence $0 \longrightarrow M'' \longrightarrow M' \longrightarrow M \longrightarrow 0$ where $M'' \in \mathcal{F}(\Delta)$ and $M' \in \mathcal{W}(\nabla) = \mathcal{F}(\Delta)$, the map $M' \longrightarrow M$ is the right $\mathcal{F}(\Delta)$–approximation of M; note that M' is obtained from M by a universal extension from below, using the modules in ∇. Similarly, we obtain a left $\mathcal{F}(\nabla)$–approximation of M by using lemma 3 for $\mathcal{F}(\Delta)$, here we extend M from above by the modules in Δ. In order to obtain a left $\mathcal{F}(\Delta)$–approximation of M, we have to use the construction described in the proof of lemma 2, for a right $\mathcal{F}(\nabla)$–approximation of M, we use the dual construction. Observe that lemma 3 applied to $\mathcal{F}(\nabla)$ yields the right $\mathcal{Y}(\nabla)$–approximation of M, and its dual applied to $\mathcal{F}(\Delta)$ yields the left $\mathcal{W}(\Delta)$–approximation of M.

5. The characteristic module

Let $\omega = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, thus ω is the full subcategory of all A–modules which have both a filtration with factors in Δ and a filtration with factors in ∇. Note that ω depends on the ordering of the simple A–modules, thus we should write $\omega(E)$ instead of ω. Auslander and Reiten show (on the basis of previous investigations of Auslander and Buchweitz) that theorem 4 has the following consequence:

Theorem 5. *There is a (uniquely defined) basic module T with $\omega = add\ T$ and T is both a tilting and a cotilting module.*

[We recall the definitions. Given a module M, we denote by *add* M the category of all direct sums of direct summands of M. The module M is said to be *basic* provided M has no direct summand of the form $N \oplus N$, with non-zero N. A (generalized) *tilting module* T is a module with finite projective dimension, $Ext^i_A(T,T) = 0$, for all $i \geq 1$, and such that for any projective module P, there exists an exact sequence $0 \longrightarrow P \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \ldots \longrightarrow T_m \longrightarrow 0$ with all $T_i \in add\ T$. Similarly, a (generalized) *cotilting module* T is a module with finite injective dimension, $Ext^i_A(T,T) = 0$, for all $i \geq 1$, and such that for any injective module I, there exists an exact sequence $0 \longrightarrow T_m \longrightarrow \ldots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow I \longrightarrow 0$ with all $T_i \in add\ T$.]

Proof: We use the notations of [**AR**] with $\mathcal{X} = \mathcal{F}(\Delta)$. According to proposition 1.9. of [**AR**] we have $\mathcal{X} \subseteq \mathcal{X}_\omega$, in particular all projective modules are in $\mathcal{X}_\omega$. Since A has finite global dimension, $\hat{\mathcal{X}}_\omega = A$–mod. It follows from theorem 5.3 of [**AR**] that $\omega = add\ T$, with T a cotilting module. By duality, T is also a tilting module.

Corollary 4. *The tilting–cotilting module T with $\omega = add\ T$ determines $\mathcal{F}(\Delta), \mathcal{F}(\nabla)$ as follows:*

$$\mathcal{F}(\Delta) = \{X \mid Ext^i_A(X,T) = 0 \quad \textit{for all} \quad i \geq 1\}\ ,$$
$$\mathcal{F}(\nabla) = \{Y \mid Ext^i_A(T,Y) = 0 \quad \textit{for all} \quad i \geq 1\}\ .$$

As a consequence, T determines Δ and ∇.

Proof: Theorem 5.2 of [**AR**], and its dual assert the stated description of $\mathcal{F}(\Delta)$ and $\mathcal{Y}(\Delta) = \mathcal{F}(\nabla)$.

We obtain the set Δ from $\mathcal{F}(\Delta)$ as follows. Recall that $\Delta(i) = P(i)/U(i)$. We can describe $U(i)$ as the sum of the kernels of non–zero surjective maps $\psi : P(i) \longrightarrow X$ with $X \in \mathcal{F}(\Delta)$. [For, let $\psi : P(i) \longrightarrow X$ be a non–zero surjective map with $X \in \mathcal{F}(\Delta)$. Since $0 \neq X \in \mathcal{F}(\Delta)$, there is a submodule $X' \subset X$ with $X/X' \in \Delta$. Since $P(i)$ maps onto X/X', it follows that $X/X' \cong \Delta(i)$. But then $Hom_A(P(j), X/X') = 0$ for all $j > i$, therefore $U(i) \subseteq Ker\ \varphi\pi$, where $\pi : X \longrightarrow X/X'$ is the canonical projection. But this is possible only for $U(i) = Ker\ \varphi\pi$, since $P(i)/U(i)$ and X/X' have the same length. Thus $U(i) = Ker\ \varphi\pi \supseteq Ker\ \varphi$.] Similarly, we obtain ∇ from $\mathcal{F}(\nabla)$.

Corollary 5. *There are precisely n isomorphism classes of indecomposable modules in ω.*

Proof: A basic tilting module is the direct sum of n indecomposable modules [**H**].

It remains to describe the indecomposable modules in ω.

Proposition 2. *The basic module T with add $T = \omega$ can be decomposed $T = \bigoplus_{i=1}^{n} T(i)$ into indecomposable modules $T(i)$ such that there are exact sequences*

$$0 \longrightarrow \Delta(i) \xrightarrow{\beta(i)} T(i) \longrightarrow X(i) \longrightarrow 0,$$

$$0 \longrightarrow Y(i) \longrightarrow T(i) \xrightarrow{\gamma(i)} \nabla(i) \longrightarrow 0,$$

where $\beta(i)$ is a left $\mathcal{F}(\nabla)$-approximation, and $X(i)$ belongs to $\mathcal{F}(\{\Delta(j)|j < i\})$, and where $\gamma(i)$ is a right $\mathcal{F}(\Delta)$-approximation and $Y(i)$ belongs to $\mathcal{F}(\{\nabla(j)|j < i\})$.

Proof: We consider some fixed $\Delta(i)$. Lemma 3' gives an exact sequence $0 \longrightarrow \Delta(i) \xrightarrow{\beta} Y \longrightarrow X \longrightarrow 0$ with $X \in \mathcal{F}(\{\Delta(j)|j < i\})$, and $Y \in \mathcal{Y}(\Delta) = \mathcal{F}(\nabla)$, since $Ext^1_A(\Delta(j), \Delta(i)) = 0$ for $j \geq i$. In particular, the simple module $E(i)$ appears with multiplicity 1 in Y. Let $Y = \bigoplus_{s=1}^{t} Y_s$ with Y_s indecomposable. We can assume that $E(i)$ appears as a composition factor of Y_1, thus $Hom_A(\Delta(i), Y_s) = 0$ for $2 \leq s \leq t$, and therefore we may assume that $Y = Y_1$ is indecomposable. Since $\mathcal{F}(\Delta)$ is closed under extensions, $Y \in \mathcal{F}(\Delta)$, thus $Y \in \omega$. It follows that $T(i) := Y$ is a direct summand of T. The various $T(i)$, $1 \leq i \leq n$, are pairwise non–isomorphic, since $T(i)$ has a composition factor $E(i)$, all other composition factors being of the form $E(j)$, with $j < i$. Thus T is isomorphic to $\bigoplus_{i=1}^{n} T(i)$. Using duality, and the characterization of $T(i)$ by its composition factors, we also obtain the second assertion.

[Let us add direct proofs of theorem 5 and its first corollary.

Lemma. *Let gl.dim $A = d$. Then, for $X \in \mathcal{F}(\Delta)$, there exists an exact sequence*

$$0 \longrightarrow X \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \dots \longrightarrow T_d \longrightarrow 0$$

with $T_i \in \omega$ for all $0 \leq i \leq d$.

Proof: Let $X_{-1} = X$. Using inductively lemma 2, we obtain exact sequences $\varepsilon_i = (0 \longrightarrow X_{i-1} \longrightarrow T_i \longrightarrow X_i \longrightarrow 0)$ with $X_i \in \mathcal{F}(\Delta)$, $T_i \in \mathcal{Y}(\Delta)$. Since $\mathcal{F}(\Delta)$ is closed under extensions, and also $X_{-1} \in \mathcal{F}(\Delta)$, we see that $T_i \in \mathcal{F}(\Delta) \cap \mathcal{Y}(\Delta) = \omega$. Applying $Hom_A(X_d, -)$ to ε_i, we obtain an exact sequence

$$Ext^j_A(X_d, T_i) \longrightarrow Ext^j_A(X_d, X_i) \longrightarrow Ext^{j+1}_A(X_d, X_{i-1}) \longrightarrow Ext^{j+1}_A(X_d, T_i).$$

Here, the end terms vanish, since $X_d \in \mathcal{F}(\Delta)$, $T_i \in \mathcal{Y}(\Delta)$. It follows that $Ext^1_A(X_d, X_{d-1}) \cong Ext^{d+1}_A(X_d, X_{-1}) = 0$, therefore ε_d splits. Thus $X_{d-1} \in \omega$. Fitting together the sequences ε_i with $0 \leq i \leq d-1$, we obtain the desired exact sequence, where $T_d = X_{d-1}$.

For the proof of theorem 5, we apply this lemma for $X = {}_AA$, and form $T' = \bigoplus_{i=0}^{d} T_i$. Then T' is a tilting module. Deleting multiple summands from T', we obtain a basic tilting module $T \in \omega$. If M is a module in ω, also $T \oplus M$ satisfies the axioms of a tilting module, thus $M \in$ *add* T, therefore $\omega =$ *add* T. Using duality, we see that T is also a cotilting module.

In order to show that $\mathcal{F}(\nabla) = \{Y \mid Ext_A^i(T,Y) = 0 \text{ for all } i \geq 1\}$, let M be a module with $Ext_A^i(T,M) = 0$ for all $i \geq 1$. Lemma 2 yields an exact sequence $0 \longrightarrow M \longrightarrow Y \longrightarrow X \longrightarrow 0$ with $Y \in \mathcal{Y}(\Delta)$ and $X \in \mathcal{F}(\Delta)$. The lemma above gives an exact sequence $0 \longrightarrow X \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \dots \longrightarrow T_m \longrightarrow 0$ with all $T_i \in \omega = add\ T$. Since $Ext_A^i(T,M) = 0$ for all $i \geq 1$, we conclude that $Ext_A^i(X,M) = 0$ for all $i \geq 1$. (We use induction. The case $m = 0$ is trivial. If $m \geq 1$, there is an exact sequence $0 \longrightarrow X \longrightarrow T_0 \longrightarrow X' \longrightarrow 0$ with $Ext_A^i(X',M) = 0$ for all $i \geq 1$, and it yields an exact sequence $0 = Ext_A^i(T_0,M) \longrightarrow Ext_A^i(X,M) \longrightarrow Ext_A^{i+1}(X',M) = 0$, for all $i \geq 1$.) It follows that the sequence $0 \longrightarrow M \longrightarrow Y \longrightarrow X \longrightarrow 0$ splits, thus $M \in \mathcal{Y}(\Delta) = \mathcal{F}(\nabla)$. By duality, we also have $\mathcal{F}(\Delta) = \{X \mid Ext_A^i(X,T) = 0 \text{ for all } i \geq 1\}$.]

6. The endomorphism ring of the characteristic module.

Let ${}_AT = \bigoplus_{i=1}^{n} T(i)$ be the characteristic module, and $A' = End({}_AT)$. We denote by F the functor $F = Hom_A(T,-) : A\text{–mod} \longrightarrow A'\text{–mod}$.

Theorem 6. *The ring A' is quasi-hereditary with $\Delta' = \{F\nabla(i); 1 \leq i \leq n\}$ the set of standard modules. The functor F yields an equivalence between $\mathcal{F}(\nabla)$ and the category $\mathcal{F}(\Delta')$ of good A'–modules.*

Remark. Since the standard A'–modules $F\nabla(i)$ satisfy the relations

$$Ext^1(F\nabla(j), F\nabla(i)) = 0 \quad \text{for} \quad j \leq i,$$

it seems appropriate to use the following numbering: let $\Delta'(i) = F\nabla(n-i+1)$, for $1 \leq i \leq n$, and $E'(i) = top\Delta'(i)$. In this way, the indices of the simple A'–modules are in accordance with the rule specified for A at the beginning of part II.

Proof of theorem 6. Since ${}_AT$ is a tilting module and $A' = End({}_AT)$, we know (see [M] or [H]) that F is a full exact embedding of $\mathcal{F}(\nabla) = \{Y \mid Ext^i(T,Y) = 0 \text{ for all } i \geq 1\}$ onto an extension closed subcategory of A'–mod containing the projective A'–modules. [Here, *exact* means that any sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ which is exact in A–mod, with $Y', Y, Y'' \in \mathcal{F}(\nabla)$ goes under F to an exact sequence in A'–mod.] For $1 \leq i \leq n$, let $i' = n-i+1$. Let $\Delta'(i) = F\nabla(i')$, and $\Delta' = \{\Delta'(1), \dots, \Delta'(n)\}$. Clearly, the image of $\mathcal{F}(\nabla)$ under F is the set of A'–modules having filtrations with factors in Δ', and it is closed under direct summands, thus it is just $\mathcal{F}(\Delta')$. Let us determine the structure of the A'–modules $\Delta'(i)$. We denote by $P'(i) = FT(i')$ the indecomposable projective A'–modules, and $E'(i) = top\ P'(i)$ denotes the corresponding simple A'–module. We claim that $Hom_{A'}(P(j), \Delta(i)) = 0$ for $j > i$; for, we have $Hom_A(T(j'), \nabla(i')) = 0$ for $j' < i'$, since $E(i')$ does not occur as composition factor of $T(j')$, whereas $soc\ \nabla(i') = E(i')$. On the other hand, proposition 2 yields an exact sequence

$$0 \longrightarrow Y(i') \longrightarrow T(i') \longrightarrow \nabla(i') \longrightarrow 0$$

with $Y(i') \in \mathcal{F}(\{\nabla(j') \mid j' < i'\})$. All three terms $Y(i'), T(i'), \nabla(i')$ belong to $\mathcal{F}(\nabla)$, thus under F we obtain an exact sequence

$$0 \longrightarrow FY(i') \longrightarrow P'(i) \longrightarrow \Delta'(i) \longrightarrow 0,$$

where $FY(i')$ belongs to $\mathcal{F}(\{\Delta'(j) \mid j > i\})$. As a first consequence, *top* $\Delta'(i) = E'(i)$. Since $FY(i')$ has a filtration with factors $\Delta'(j), j > i$, and $top\Delta'(j) = E'(j)$, it follows that the top composition factors of $FY(i')$ are of the form $E'(j)$, with $j > i$. As a consequence, $\Delta'(i)$ is the largest factor module of $P'(i)$ with composition factors of the form $E'(j)$, where $j \leq i$, thus it is the indecomposable projective A'/I_{n-i}–module with *top* $E'(i)$, where I_t denotes the ideal of A' of all endomorphisms of ${}_AT$ which factor through a module in $add(T_1 \oplus \ldots \oplus T_t)$. Since $End_{A'}(\Delta'(i)) \cong End_A(\nabla(i'))$ is a division ring, it follows that $E'(i)$ occurs only once as a composition factor of $\Delta'(i)$. Finally, we use that any projective A'–module belongs to the image of $\mathcal{F}(\nabla)$ under F, thus it has a filtration with factors in Δ'. Altogether, we see that A' is quasi–hereditary with standard modules $\Delta'(1), \ldots, \Delta'(n)$.

Starting with the quasi–hereditary algebra (A, E), we have constructed a (uniquely defined) quasi–hereditary algebra (A', E'), where A' is the endomorphism ring of the characteristic module T of (A, E), and we may iterate this procedure: we may consider the characteristic module T' of (A', E'), and its endomorphism ring A'', or better, (A'', E'').

Lemma 7. *We have* $FQ(i) = T'(i')$ *for all* $1 \leq i \leq n$.

Proof: The module $FQ(i)$ is *Ext*–injective in $\mathcal{F}(\Delta')$, thus $FQ(i) \cong T'(r)$, for some r. But we know that $Hom_A(T(i), Q(i)) \neq 0$, and $Hom_A(T(j), Q(i)) = 0$ for $j < i$. Thus $Hom_{A'}(P'(i'), T'(r)) \neq 0$, and $Hom_{A'}(P'(j'), T'(r)) = 0$ for $j < i$, and therefore $r = i'$.

Theorem 7. *Assume that A is basic. Then we may identify the quasi–hereditary ring* (A'', E'') *with* (A, E).

Proof: Let $Q = \oplus Q(i)$, then $A \cong End_A(Q) \cong End_{A'}(FQ) = End_{A'}(T') \cong A''$, and, under this isomorphism, E'' corresponds to E.

Corollary 6. *The categories* $\mathcal{F}(\Delta)$ *and* $\mathcal{F}(\nabla')$ *are equivalent.*

Proof: We apply theorem 6 in order to see that $\mathcal{F}(\nabla')$ is equivalent to $\mathcal{F}(\Delta'')$, thus to $\mathcal{F}(\Delta)$.

References

[AR] Auslander, M. and Reiten, I.: Applications of contravariantly finite subcategories. (To appear)

[AS] Auslander, M. and Smalø, S.: Almost split sequences in subcategories. J. Algebra **69** (1981), 426–454.

[CPS1] Cline, E., Parshall, B. and Scott, L.: Finite dimensional algebras and highest weight categories. J. reine angew. Math. 391 (1988), 277-291

[CPS2] Cline, E., Parshall, B. and Scott, L.: Duality in highest weight categories. (To appear)

[DR1] Dlab, V. and Ringel, C.M.: Quasi–hereditary algebras. Illinois J.Math **33** (1989), 280–291.

[DR2] Dlab, V. and Ringel, C.M.: A construction for quasi–hereditary algebras. Compositio Math. **70** (1989), 155–175.

[DR3] Dlab, V. and Ringel, C.M.: Filtrations of right ideals related to projectivity of left ideals. In: Séminaire d'Algèbre. Springer LNM. (To appear)

[H] Happel, D.: Triangulated categories in the representation theory of finite dimensional algebras. London Math. Soc. Lecture Note Series **119** (1988).

[M] Miyashita, Y.: Tilting modules of finite projective dimension. Math. Z. 193 (1986) 113-146.

[PS] Parshall, B. and Scott, L.: Derived categories, quasi–hereditary algebras and algebraic groups. Proceedings of the Ottawa–Moosonee Workshop in Algebra. Carleton Univ. Notes **3** (1988).

[R] Ringel, C.M.: Tame algebras and integral quadratic forms.Springer LNM **1099** (1984).

[S] Scott, L.L.: Simulating algebraic geometry with algebra I: Derived categories and Morita theory. Proc. Symp. Pure Math. **47.1** (1987), 271–282.

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Appendix

Quasi–hereditary algebras have been introduced by Parshall and Scott ([**S**], [**PS**]) in order to deal with the structure of suitable derived categories using recollements and tilting functors. On the basis of the results above, we are going to provide an explicit description of the quasi–hereditary algebras in terms of tilting modules.

Let A be an artin algebra with simple modules $E(1), \dots, E(n)$. Given an A–module M, we denote by $\mathbf{dim}\, M \in \mathbf{Z}^n$ its dimension vector, thus $(\mathbf{dim}\, M)_i$ is the Jordan–Hölder multiplicity of $E(i)$ in M. Let $P(i)$ be the projective cover of $E(i)$, and J_i the sum of all images of maps $P(j) \longrightarrow {}_AA$, with $j \leq i$. Note that the A/J_i–modules are precisely the A–modules M with $(\mathbf{dim}\, M)_i = 0$ for all $j \geq i$.

Theorem. *The artin algebra A is quasi–hereditary with respect to the ordering $E(1), \dots, E(n)$ if and only if there exist indecomposable A–modules $T(i)$, $1 \leq i \leq n$, such that $(\mathbf{dim}\, T(i))_i = 1$ and $\bigoplus_{j=1}^{i} T(j)$ is a tilting A/J_{i+1}–module, for $1 \leq i \leq n$.*

Proof: If (A, E) is quasi–hereditary, let T be its characteristic module. It is a tilting A–module and Proposition 2 asserts that $T = \bigoplus_{i=1}^{n} T(i)$, where $(\mathbf{dim}\, T(i))_i = 1$ for all i. Note that A/J_{i+1} is quasi–hereditary with respect to $E(1), \dots, E(i)$ and $\bigoplus_{j=1}^{i} T(j)$ is its characteristic module.

Conversely, assume there is a tilting module $T = \bigoplus_{i=1}^{n} T(i)$ with indecomposable modules $T(i)$ such that $(\mathbf{dim}\, T(i))_i = 1$ and $(\mathbf{dim}\, T(i))_n = 0$ for $i < n$. We claim that $J = J_n$ is a heredity ideal. Since T is a tilting module, ${}_AA$ is cogenerated by T, thus $P(n)$ and ${}_AJ$ both are cogenerated by T. Since $Hom_A(P(n), T(i)) = 0$ for $i < n$, it follows that $P(n)$ is cogenerated by $T(n)$. Similarly, ${}_AJ$ is cogenerated by $T(n)$, since ${}_AJ$ is generated by $P(n)$. We are going to show that any non–zero map $P(n) \longrightarrow T(n)$ is a monomorphism. Let $D_i = End_A(E(i))$, and $D_i' = End_A(T(i))/rad\ End_A(T(i))$. Now, $(\mathbf{dim}\, T(i))_i = 1$ means that $Hom_A(P(i), T(i))$ is a one–dimensional D_i–vectorspace. Since it is a D_i–D_i'–bimodule, we see that $dim_k D_i' \leq dim_k D_i$. However, the k–algebras $\prod_{i=1}^{n} D_i$ and $\prod_{i=1}^{n} D_i'$ are isomorphic, since these algebras can be recovered from the derived categories $D^b(A) \cong D^b(End({}_AT))$, see [**H**]. Thus $dim_k D_i = dim_k D_i'$, for all i. It follows that $Hom_A(P(i), T(i))$ is a one–dimensional D_i'–vectorspace. Let $\varphi : P(n) \longrightarrow T(n)$ be a non–zero map and U its image. Since $P(n)$ is cogenerated by $T(n)$, and φ generates $Hom_A(P(n), T(n))$ as an $End_A(T(n))$–module, we see that φ is a monomorphism, thus $U \cong P(n)$. As a consequence, the Jordan–Hölder multiplicity of $E(n)$ in $P(n)$ is 1. We also know that ${}_AJ$ is cogenerated by $T(n)$, let $\psi : {}_AJ \longrightarrow T(n)^m$ be an embedding. Since ${}_AJ$ is generated by $P(n)$, we see that the image of ψ lies in U^m, thus ${}_AJ$ is both generated and cogenerated by $P(n)$. It follows easily [**DR2**] that therefore ${}_AJ$ is a projective left A–module. Thus J is a heredity ideal of A.

Assume now, in addition that $\bigoplus_{j=1}^{i} T(j)$ is a tilting A/J_{i+1}–module for all i. Then we use induction and conclude that A/J is quasi–hereditary, with respect to $E(1), \dots, E(n-1)$. This finishes the proof.

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