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– The category of good modules over a quasi-hereditary algebra
– On contravariantly finite subcategories

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The category of good modules over a quasi–hereditary algebra

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Let A be an artin algebra. We will consider (finitely generated left) A–modules, maps between A–modules will be written on the right hand of the argument, thus the composition of the maps $f: M_1 \to M_2, \quad g: M_2 \to M_3$ will be denoted by fg. The category of all A–modules will be denoted by A–mod. All subcategories considered will be full and closed under isomorphisms, so usually we will describe subcategories by just specifying their objects (up to isomorphism).

Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the class of all A–modules M which have a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$, such that all factors M_{i-1}/M_i belong to Θ, and we may call these modules the Θ-*good* modules.

Let $E(1), \ldots, E(n)$ be the simple A–modules; note that we fix a particular ordering for labelling the simples A–modules. For any i, let $P(i)$ be the projective cover of $E(i)$, and denote by $\Delta(i)$ the maximal factor module of $P(i)$ in $\mathcal{F}(E(1), \ldots, E(i))$. Let Δ be the subcategory of all $\Delta(i)$, where $1 \leq i \leq n$.

The algebra A, or better the pair (A, E) is called *quasi-hereditary* provided $\mathrm{End}(\Delta(i))$ is a division ring, for any $1 \leq i \leq n$ and the module ${}_AA$ belongs to $\mathcal{F}(\Delta)$.

From now on, we will assume that A is quasi–hereditary. Without loss of generality, we also may assume that A is connected. We are going to investigate the subcategory $\mathcal{F}(\Delta)$ for a quasi–hereditary algebra. By definition, this subcategory is closed under extensions, thus under direct sums, and it is rather easy to see that $\mathcal{F}(\Delta)$ is also closed under direct summands.

We have shown in **[R3]** that $\mathcal{F}(\Delta)$ is functorially finite in A–mod, in particular, $\mathcal{F}(\Delta)$ has (relative) Auslander–Reiten sequences. Also, we have shown that the relative projective objects in $\mathcal{F}(\Delta)$ are just the projective A–modules, and we have constructed the relative injective objects in $\mathcal{F}(\Delta)$. This information is sufficient to establish some fundamental properties of $\mathcal{F}(\Delta)$. Most of these are analogues of properties of the complete module category of an artin algebra. In case there are only finitely many isomorphism classes of indecomposable A–modules which belong to $\mathcal{F}(\Delta)$, we say that A is $\mathcal{F}(\Delta)$-*finite*.

1. Basic results

We are going to review some basic facts of the subcategory $\mathcal{F}(\Delta)$ of A–mod. For the missing proofs, we refer to **[R3]**.

First, we need an additional class of modules. Let $Q(i)$ be the injective envelope of $E(i)$, and $\nabla(i)$ the maximal submodule of $Q(i)$ belonging to $\mathcal{F}(E(1),\ldots,E(i))$, let ∇ be the subcategory of the $\nabla(i)$ where $1 \leq i \leq n$.

1. *We have* $\operatorname{Ext}^t(X,Y) = 0$ *for all* $X \in \mathcal{F}(\Delta), Y \in \mathcal{F}(\nabla)$, *and all* $t \geq 1$. *Conversely, if* $\operatorname{Ext}^1(X, \nabla(j)) = 0$ *for all* j, *then* $X \in \mathcal{F}(\Delta)$, *and if* $\operatorname{Ext}^1(\Delta(i), Y) = 0$ *for all* i, *then* $Y \in \mathcal{F}(\nabla)$.

2. *Let* M *be an* A*-module in* $\mathcal{F}(E(1),...,E(i))$. *There are exact sequences*

$$0 \to {}''M \to {}'M \to M \to 0, \quad \text{and} \quad 0 \to M \to M' \to M'' \to 0,$$

where ${}''M \in \mathcal{F}(\nabla(1),\ldots,\nabla(i-1))$; ${}'M \in \mathcal{F}(\Delta)$, $M' \in \mathcal{F}(\nabla)$, *and* $M'' \in \mathcal{F}(\Delta(1),\ldots,\Delta(i-1))$.

3. *For any* $1 \leq i \leq n$, *there is a unique indecomposable module* $T(i)$ *in* $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \cap \mathcal{F}(E(1),\ldots,E(i))$ *and not in* $\mathcal{F}(E(1),\ldots,E(i-1))$. *There are exact sequences*

$$0 \to \Delta(i) \to T(i) \to X(i) \to 0, \quad \text{and} \quad 0 \to Y(i) \to T(i) \to \nabla(i) \to 0,$$

with $X(i) \in \mathcal{F}(\Delta(1),\ldots,\Delta(i-1))$, *and* $Y(i) \in \mathcal{F}(\nabla(1),\ldots,\nabla(i-1))$.

Let $T = \bigoplus T(i)$, thus $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \operatorname{add} T$.

4. *The* Ext*-projective modules in* $\mathcal{F}(\Delta)$ *are the projective* A*-modules, the* Ext*-injective modules in* $\mathcal{F}(\Delta)$ *are the* A*-modules in* $\operatorname{add} T$.

5. *Let* d *be the maximum of* $\operatorname{proj.dim} \Delta(i)$. *For any* A*-module* $M \in \mathcal{F}(\Delta)$, *there exists a* T*-coresolution of* M *of length* d *(i.e. an exact sequence* $0 \to M \to T_0 \to \cdots \to T_d \to 0$ *with all* $T_i \in \operatorname{add} T$*) and there are* A*-modules in* $\mathcal{F}(\Delta)$ *with no* T*-coresolution of length* $d-1$.

Proof: Assertion 2 yields an exact sequence $0 \to M \to T_0 \to M' \to 0$ with $T_0 \in \mathcal{F}(\nabla)$ and $M' \in \mathcal{F}(\Delta)$. Assertion 1 shows that $\operatorname{Ext}^t(\Delta(i), M') \cong \operatorname{Ext}^{t+1}(\Delta(i), M)$ for all $t \geq 1$. Since $\mathcal{F}(\Delta)$ is closed under extensions, we see that T_0 also belongs to $\mathcal{F}(\Delta)$, thus to $\operatorname{add} T$. Inductively, we obtain an exact sequence $0 \to M \to T_0 \to \cdots \to T_d \to 0$ with $T_i \in \operatorname{add} T$ for $1 \leq i < d$, and $T_d \in \mathcal{F}(\Delta)$. Also, $\operatorname{Ext}^1(\Delta(i), T_d) \cong \operatorname{Ext}^{d+1}(\Delta(i), M) = 0$, thus $T_d \in \mathcal{F}(\nabla)$, according to assertion 1. A similar argument shows that $\operatorname{proj.dim} \Delta(i) \leq m$ for all i, in case any module in $\mathcal{F}(\Delta)$ has a T-coresolution of length m.

6. $\mathcal{F}(\Delta)$ *has (relative) Auslander-Reiten sequences.*

7. *Let* $U(i)$ *be the submodule of* $P(i)$ *with* $P(i)/U(i) = \Delta(i)$. *The sink map for* $P(i)$ *in* $\mathcal{F}(\Delta)$ *is of the form* $g(i) : R(i) \to P(i)$, *where* $R(i)$ *has* $U(i)$ *as a* *submodule, so that all composition factors of $R(i)/U(i)$ are of the form $E(j)$ with $j < i$, and $g(i)|U(i)$ is the identity.*

We consider the algebra $B = \mathrm{End}(T)$, the bimodule ${}_AT_B$, and the functor $\mathrm{Hom}(T, -)$ from A–mod to B–mod. The algebra B has n simple modules, and we order them so that the indecomposable projective B–module $\mathrm{Hom}(T, T(n+1-i))$ has the label $n+1-i$. When dealing with B–modules, we will add an index B, say we write $P_B(i), \Delta_B(i)$, and so on.

8. *The algebra B is quasi-hereditary, and $\mathrm{Hom}_A({}_AT_B, -)$ yields an equivalence from $\mathcal{F}(\nabla)$ onto $\mathcal{F}(\Delta_B)$ and it maps exact sequences in $\mathcal{F}(\nabla)$ to exact sequences in $\mathcal{F}(\Delta_B)$.*

Note that k–duality shows that $\mathcal{F}(\Delta)$ is the opposite of the category of ∇–good modules for A^{op}.

We denote by $\mathcal{P}_{>i}$ the set of modules $P(j)$ with $j > i$. Let J_i be the trace ideal of $\mathcal{P}_{>i}$ in A. (We recall that the *trace* of a set $\mathcal{X}$ of modules in a module M is the largest submodule of M generated by modules from $\mathcal{X}$; the trace of $\mathcal{X}$ in ${}_AA$ is a twosided ideal, the *trace ideal* of $\mathcal{X}$ in A.) For any A–module M, the submodule J_iM is the trace of $\mathcal{P}_{>i}$ in M. Note that an A–module N belongs to $\mathcal{F}(\Delta)$ if and only if $J_{i-1}N/J_iN$ is a projective A/J_i–module, for all $1 \leq i \leq n$, thus if and only if $J_{i-1}N/J_iN$ is a direct sum of copies of $\Delta(i)$. As an immediate consequence of this characterization, we see that $\mathcal{F}(\Delta)$ is closed under direct summands.

2. The Auslander–Reiten quiver of $\mathcal{F}(\Delta)$

Let X, Y be A–modules in $\mathcal{F}(\Delta)$. A map $f : X \to Y$ is said to be *(relative) irreducible in $\mathcal{F}(\Delta)$*, provided f is neither a split monomorphism nor a split epimorphism, and given any factorization $f = f_1 f_2$ in $\mathcal{F}(\Delta)$, then f_1 is a split monomorphism, or f_2 is a split epimorphism. We define $\mathrm{rad}^2_{\mathcal{F}(\Delta)}(X, Y)$ as the set of maps $f : X \to Y$ which are of the form $f = f_1 f_2$, with $f_1 \in \mathrm{rad}(X, M), f_2 \in \mathrm{rad}(M, Y)$, where M is a module in $\mathcal{F}(\Delta)$. We define the *bimodule of (relative) irreducible maps* $\mathrm{Irr}_{\mathcal{F}(\Delta)}(X, Y) = \mathrm{rad}(X, Y)/\mathrm{rad}^2_{\mathcal{F}(\Delta)}(X, Y)$.

The *Auslander Reiten quiver* $\Gamma_{\mathcal{F}(\Delta)}$ of $\mathcal{F}(\Delta)$ is a valued translation quiver defined as follows: Its vertices are the isomorphism classes $[X]$ of the indecomposable A–modules in $\mathcal{F}(\Delta)$. There is an arrow $[X] \to [Y]$ provided there exists a (relative) irreducible map $X \to Y$ in $\mathcal{F}(\Delta)$, thus, if and only if $\mathrm{Irr}_{\mathcal{F}(\Delta)}(X, Y) \neq 0$. Given an arrow $[X] \to [Y]$ in $\Gamma_{\mathcal{F}(\Delta)}$, we add the valuation (d_{XY}, d'_{XY}), where d_{XY} is the length of $\mathrm{Irr}_{\mathcal{F}(\Delta)}(X, Y)$ as a right $\mathrm{End}(Y)$–module, and d'_{XY} is the length of $\mathrm{Irr}_{\mathcal{F}(\Delta)}(X, Y)$, as a left $\mathrm{End}(X)$–module. Finally, the translation τ is defined by $\tau[X] = [\tau_\Delta X]$, for X a non-projective indecomposable A–module in $\mathcal{F}(\Delta)$, where $\tau_\Delta X$ is the left hand term in a relative Auslander–Reiten sequence $0 \to \tau_\Delta X \to X' \to X \to 0$ in $\mathcal{F}(\Delta)$.

A path $x_0 \to x_1 \to \cdots \to x_n = x_0$ in a quiver, with $n \geq 1$ is called *cyclic*. A cyclic path $x_0 \to x_1 \to \cdots \to x_n = x_0$ in $\Gamma_{\mathcal{F}(\Delta)}$ is called *sectional* provided $\tau x_{i+1} \neq x_{i-1}$ for all $1 \leq i \leq n$, where $x_{n+1} = x_1$.

Theorem. *The translation quiver $\Gamma_{\mathcal{F}(\Delta)}$ has no loops and no sectional cyclic paths.*

The proof of the Theorem will occupy the rest of this section.

First, let us show that there are no loops. This will be an immediate consequence of the following lemma. The length of an A-module M will be denoted by $l(M)$.

Lemma. *Let X, Y be indecomposable A-modules in $\mathcal{F}(\Delta)$, with $l(X) \leq l(Y)$. If $f : X \to Y$ is an irreducible map in $\mathcal{F}(\Delta)$, then $f|J_{n-1}X$ is injective.*

Proof: If $f|J_{n-1}X$ is not injective, then $\operatorname{Ker} f$ contains an indecomposable summand U of $J_{n-1}X$, and U belongs to a Δ-filtration of X, thus $X/U \in \mathcal{F}(\Delta)$. But f factors through X/U, and this contradicts the fact that f is irreducible in $\mathcal{F}(\Delta)$.

Next, we consider the existence of sectional cyclic paths in $\Gamma_{\mathcal{F}(\Delta)}$. We call $(g_1, \ldots, g_n)$ a *sectional path in* $\mathcal{F}(\Delta)$, provided $g_i : X_{i-1} \to X_i$ is an irreducible map between indecomposable modules, for $1 \leq i \leq n$, and $\tau_\Delta X_{i+1} \not\cong X_{i-1}$ for $0 < i < n$.

Warning: *The composition $g_1 \cdots g_n$ of a sectional path $(g_1, \ldots, g_n)$ may be zero.* Let A have three simple modules $E(1), E(2), E(3)$, with $P(1) = E(1) = \operatorname{rad} P(2)$, and $\operatorname{rad} P(3) = E(2)$. Then $\mathcal{F}(\Delta)$ is the class of all projective A-modules, and there is a sectional path $(P(1) \to P(2), P(2) \to P(3))$ in $\mathcal{F}(\Delta)$ with zero composition.

However, there is the following result:

Proposition. *Let $g_i : X_{i-1} \to X_i$ be maps such that $(g_1, \ldots, g_n)$ is a sectional path in $\mathcal{F}(\Delta)$ of length $n \geq 1$, and assume the following condition is satisfied: in case at least one of the modules X_i is projective, say $X_i = P(t_i)$, also X_0 is projective, say $X_0 = P(t_0)$, and $t_0 \geq t_i$. Let $g'_n : X'_{n-1} \to X_n$ be a map such that $\begin{bmatrix} g_n \\ g'_n \end{bmatrix} : X_{n-1} \oplus X'_{n-1} \to X_n$ is a sink map for X_n in $\mathcal{F}(\Delta)$. Then $g_1 \cdots g_n$ does not factor through g'_n. In particular, $g_1 \cdots g_n \neq 0$.*

Proof: We use induction on n. Assume there exists h_n such that $g_1 \cdots g_n = h_n g'_n$. Let $f_{n-1} : Y \to X_{n-1}$, and $f'_{n-1} : Y \to X'_{n-1}$, be maps so that $[f_{n-1}, f'_{n-1}] : Y \to X_{n-1} \oplus X'_{n-1}$ is the kernel of $\begin{bmatrix} g_n \\ g'_n \end{bmatrix}$. Since $g_1 \cdots g_n = h_n g'_n$, there is a map $h_{n-1} : X_0 \to Y$, such that $h'_{n-1} f_{n-1} = g_1 \cdots g_{n-1}$ (and $h'_{n-1} f'_{n-1} = -h_n$).

First, assume $X_n = P(t_n)$ is projective. In this case, also $X_0 = P(t_0)$ is projective and $t_0 \geq t_n$. On the other hand, the kernel Y of the sink map for $P(t_n)$

belongs to $\mathcal{F}(E(1),\ldots,E(t_n-1))$, according to Assertion 7 in Section 1. Since $t_0 > t_n - 1$, it follows that $\mathrm{Hom}(P(t_0),Y) = 0$, thus $g_1\cdots g_{n-1} = 0$. For $n = 1$, this would mean that $1_{X_0} = 0$, impossible, for $n \geq 2$, we see that $g_1\cdots g_{n-1}$ can be factorized through the corresponding map g'_{n-1}.

Next, assume X_n is not projective, thus $Y = \tau_\Delta X_n$, and f_{n-1} is an irreducible map. If $n = 1$, there is the factorization $1_{X_0} = g_1\cdots g_{n-1} = h'_0 f_0$, but then f_0 is a split epimorphism, impossssible. Thus $n \geq 2$. Since $(g_1,\ldots,g_n)$ is a sectional path in $\mathcal{F}(\Delta)$, we know that $X_{n-2} \not\cong \tau_\Delta X_n = Y$, and therefore the sink map for X_{n-1} in $\mathcal{F}(\Delta)$ is of the form $\begin{bmatrix} g_{n-1} \\ f_{n-1} \\ g''_{n-1} \end{bmatrix}$ for some map $g''_{n-1} : X''_{n-2} \to X_{n-1}$. Let $X'_{n-2} = Y \oplus X''_{n-2}$, and $g'_{n-2} = \begin{bmatrix} f_{n-1} \\ g''_{n-1} \end{bmatrix} : X'_{n-2} \to X_{n-1}$, and $h_{n-1} = [h'_{n-1}, 0] : X_0 \to X'_{n-2}$. Then $\begin{bmatrix} g_{n-1} \\ g'_{n-1} \end{bmatrix} : X_{n-2} \oplus X'_{n-2} \to X_{n-1}$ is a sink map for X_{n-1} in $\mathcal{F}(\Delta)$, and $g_1\cdots g_{n-1} = h_{n-1}g'_{n-1}$, but by induction, this is impossible, too.

Proof of Theorem: Let $X_0,\ldots,X_{n-1},X_n = X_0$ be indecomposable modules in $\mathcal{F}(\Delta)$, and assume $[X_0] \to [X_1] \to \cdots \to [X_n]$ is a sectional cyclic path in $\Gamma_{\mathcal{F}(\Delta)}$. Note that with $(g_1,\ldots,g_n)$ also $(g_1,\ldots,g_n,g_1,\ldots,g_n)$ is a sectional cyclic path; thus we may suppose that $n \geq 2^b$, where b is an upper bound for the length of the modules X_i. In case one of the modules X_i is projective, we may rotate the indices so that $X_0 = P(t_0)$ is projective, and that $t_0 \geq t_j$ for any j with $X_j = P(t_j)$ projective. Choose an irreducible map $g_i : X_{i-1} \to X_i$ in $\mathcal{F}(\Delta)$, for any $1 \leq i \leq m$. The Proposition now asserts that the composition $g_1\cdots g_n$ is non–zero, in contrast to the Harada–Sai lemma. This completes the proof.

3. Brauer–Thrall I

Let Γ be a component of $\Gamma_{\mathcal{F}(\Delta)}$. Let $\mathcal{C}_\Gamma$ be the subcategory closed under direct sums and direct summands whose indecomposable objects are the A–modules M with $[M]$ in Γ. The subcategories of the form $\mathcal{C}_\Gamma$ will be called the *Auslander–Reiten components of* A–mod. Investigations of Auslander yield the following result:

Theorem. *Let $\mathcal{C}$ be an Auslander–Reiten component of $\mathcal{F}(\Delta)$, and assume the indecomposable modules in $\mathcal{C}$ are of bounded length. Then $\mathcal{C} = \mathcal{F}(\Delta)$, and $\mathcal{F}(\Delta)$ is finite.*

In particular, there is the following analogue to the assertion of the first Brauer–Thrall conjecture:

Corollary. *Assume the indecomposable modules in $\mathcal{F}(\Delta)$ are of bounded length. Then A is $\mathcal{F}(\Delta)$–finite and the Auslander–Reiten quiver $\Gamma_{\mathcal{F}(\Delta)}$ is connected.*

Proof of Theorem: There is the following general assertion due to Auslander [A]:

Theorem. *Let B be a connected artin algebra. Let $\mathcal{X}$ be a subcategory of B-mod which is functorially finite, closed under extensions and direct summands, and suppose $\mathcal{X}$ contains all projective B-modules. Let $\mathcal{X}'$ be an Auslander–Reiten component of $\mathcal{X}$, and assume the indecomposable modules in $\mathcal{X}'$ are of bounded length. Then $\mathcal{X}' = \mathcal{X}$ and $\mathcal{X}$ is finite.*

Let us outline a proof of the general assertion following Yamagata (see [R1]): Assume the indecomposable modules in $\mathcal{X}'$ have length at most b. Let M be an indecomposable module in $\mathcal{X}'$ and assume $\mathrm{Hom}(P(i), M) \neq 0$. The Harada–Sai lemma implies that there is a path in $\Gamma_{\mathcal{X}}$ of length at most $2^b - 2$ from $P(i)$ to M, here we work inductively with factorizations which are given by using the minimal right almost split maps in $\mathcal{X}$. In particular, $P(i)$ belongs to $\mathcal{X}'$. On the other hand, let X be an indecomposable module in $\mathcal{X}$ and assume $\mathrm{Hom}(P(i), X) \neq 0$. Again, we use the Harada–Sai lemma in order to obtain a path in $\Gamma_{\mathcal{X}}$ of length at most $2^b - 2$ from $P(i)$ to X, but now we work inductively with factorizations which are given by using the minimal left almost split maps in $\mathcal{X}$. Since we assume that B is connected, there are sufficiently many non–zero maps between the indecomposable projective B–modules, thus all $P(j)$ belong to $\mathcal{X}'$. And any indecomposable module in $\mathcal{X}$ is joined by a path of length at most $2^b - 2$ to some $P(i)$, thus all belong to $\mathcal{X}'$ and there are only finitely many.

4. The stable Auslander–Reiten quiver

The *stable Auslander-Reiten quiver* $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is the full translation subquiver of $\Gamma_{\mathcal{F}(\Delta)}$ obtained by deleting all vertices of the form $\tau_\Delta^{-t}p$ where p is a projective vertex, and $t \in \mathbb{N}_0$, or of the form $\tau_\Delta^n q$ where q is an injective vertex, and $t \in \mathbb{N}_0$.

Recall that a vertex x of a translation quiver is said to be *periodic* provided there is some $t \geq 1$ such that $\tau^t x = x$. Let Γ be a component of the stable translation quiver $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$. Since $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is locally finite, the existence of a periodic vertex in Γ implies that all vertices of Γ are periodic, and, in this case, Γ is said to be periodic.

Given a valued quiver Q, we may form the stable translation quiver $\mathbb{Z}Q$, as introduced by Riedtmann (see [HPR]). The same reference may be used for looking up the well–known list of Dynkin diagrams, Euclidean diagrams and the graph A_∞. A valued quiver with underlying graph a Dynkin diagram, or a Euclidean diagram, or A_∞, will be called a Dynkin quiver, a Euclidean quiver, or to be of the form A_∞.

Theorem. *A periodic component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is of the form $\mathbb{Z}Q/G$, where Q is either a Dynkin quiver or a quiver of the form A_∞, and G is a non–trivial group of automorphisms of $\mathbb{Z}Q$.*

In particular, we have the analogue of Riedtmann's theorem: a finite component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is of the form $\mathbb{Z}Q/G$ with Q a Dynkin quiver, and G a non-trivial group of automorphisms of $\mathbb{Z}Q$.

The proof uses the existence of the length function on the component Γ, it is a subadditive function on Γ_0 with values in $\mathbb{N}_1$. In case this function is bounded, it cannot be additive (since otherwise Γ would be a component of $\Gamma_{\mathcal{F}(\Delta)}$ itself, in contrast to Auslander's theorem). The combinatorial considerations of **[HPR]** yield the result.

The structure of non-periodic components has been studied by Zhang **[Z]**. Her investigations yield the following result:

Theorem. *A non-periodic component of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$ is of the form $\mathbb{Z}Q$ where Q is a connected valued quiver without cyclic paths.*

For the proof, we use again the existence of the length function on the component, Auslander's theorem, and the non-existence of loops and sectional cyclic paths.

Recall that a component of $\Gamma_{\mathcal{F}(\Delta)}$ which does not contain projective or injective vertices, is called a *stable component of* $\Gamma_{\mathcal{F}(\Delta)}$. Clearly, stable components of $\Gamma_{\mathcal{F}(\Delta)}$ are components of $\Gamma^{(s)}_{\mathcal{F}(\Delta)}$, but for stable components, the length function is additive, and not only subadditive.

Theorem. *A stable component of $\Gamma_{\mathcal{F}(\Delta)}$ is either periodic and then of the form $\mathbb{Z}A_\infty/G$ for some non-trivial automorphism group G, or else non-periodic, and then of the form $\mathbb{Z}Q$ for some connected valued quiver Q without cyclic paths, and Q cannot be a Dynkin or a Euclidean quiver.*

This is an immediate consequence of the previous results: Assume the component Γ is of the form $\mathbb{Z}Q/G$ with Q a quiver and G a group of automorphisms. If Q is a Dynkin quiver, then there is no additive function on Γ with values in $\mathbb{N}_1$. If Q is a Euclidean quiver, we consider the so called "defect" δ of the restriction of the length function l to some copy of Q. If $\delta \neq 0$, then the additivity of l enforces that l takes negative values, impossible. If $\delta = 0$, then l is bounded, but then Auslander's theorem implies that Γ contains projective vertices, again a contradiction.

5. The multiplicities of $\Delta(i)$

Given $M \in \mathcal{F}(\Delta)$, say with a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$ with factors $M_{s-1}/M_s \in \Delta$, for all $1 \leq s \leq t$, we denote by $[M : \Delta(i)]$ the number of factors M_{s-1}/M_s isomorphic to $\Delta(i)$; note that this number is independent of the particular filtration which we have used. There are different ways for calculating $[M : \Delta(i)]$, as we want to show.

Let $d_i = \dim_k \operatorname{End}(E(i)) = \dim_k \operatorname{End}(\Delta(i))$.

Proposition. *Let $M \in \mathcal{F}(\Delta)$. Then $d_i[M : \Delta(i)] = \dim_k \operatorname{Hom}(M, \nabla(i))$.*

Proof: Consider a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$ with factors $M_{s-1}/M_s \cong \Delta(i_s)$. We use induction on t, the case $t = 0$ being trivial. We apply $\operatorname{Hom}(-, \nabla(i))$ to the exact sequence $0 \to M_1 \to M \to \Delta(i_1) \to 0$. On the one hand, we have $\dim_k \operatorname{Hom}(\Delta(i_1), \Delta(i)) = d_i$ for $i = i_1$, and zero otherwise, on the other hand, $\operatorname{Ext}^1(\Delta(i_1), \nabla(i)) = 0$. This completes the proof.

Corollary. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\mathcal{F}(\Delta)$. Then $[M : \Delta(i)] = [M' : \Delta(i)] + [M'' : \Delta(i)]$ for all i.*

Proof: We use the previous formula and the fact that $\operatorname{Ext}^1(M'', \nabla(i)) = 0$.

Corollary. *The function $[M] \mapsto [M : \Delta(i)]$ is an additive function on $\Gamma_{\mathcal{F}(\Delta)}$.*

Recall that an A-module N belongs to $\mathcal{F}(\Delta)$ if and only if $J_{i-1}N/J_iN$ is a projective A/J_i-module, for all $1 \le i \le n$, and, in this case, $J_{i-1}N/J_iN \cong [N : \Delta(i)] \cdot \Delta(i)$.

Given A-modules X, Y, and a class $\mathcal{M}$ of A-modules, let $\operatorname{Hom}(X, \mathcal{M}, Y)$ be the set of maps $X \to Y$ which factor through $\operatorname{add}\mathcal{M}$. (For example, $\operatorname{Hom}(X, {}_AA, Y)$ is the set of maps $X \to Y$ which factor through a projective A-module.) Note that, for any A-module M, a map $f : P(i) \to M$ belongs to $\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M)$ if and only if $P(i)f \subseteq J_iM$. (For, J_iM is generated by $\mathcal{P}_{>i}$, thus the projective cover of J_iM belongs to $\operatorname{add}\mathcal{P}_{>i}$.)

Proposition. *Let $M \in \mathcal{F}(\Delta)$. Then*

$$d_i[M : \Delta(i)] = \dim_k \operatorname{Hom}(P(i), M)/\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M).$$

Proof: For any i, choose a primitive idempotent e_i such that $P(i) \cong Ae_i$. The evaluation map $\operatorname{Hom}(P(i), M) \to M$ sending f to e_if has as image e_iM, and it sends $\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M)$ onto e_iJ_iM. In this way, we see that

$$\frac{1}{d_i} \dim_k \operatorname{Hom}(P(i), M)/\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M)$$

counts the Jordan-Hölder multiplicity of $E(i)$ in M/J_iM, thus the multiplicity of $\Delta(i)$ in a direct decomposition of $J_{i-1}M/J_iM$.

We also introduce $\mathcal{T}_{<i}$ as the set of modules $T(j)$ with $j < i$.

Proposition. *For any $M \in \mathcal{F}(\Delta)$, the composition of maps yields a non-degenerate bilinear map on*

$$(\operatorname{Hom}(P(i), M)/\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M)) \times (\operatorname{Hom}(M, T(i))/\operatorname{Hom}(M, \mathcal{T}_{<i}, T(i)))$$

with values in $\operatorname{Hom}(P(i), T(i))$.

Proof: First, we have to show that $\operatorname{Hom}(P(i), \mathcal{P}_{>i}, T(i)) = 0$. But, for $j > i$, we know that $[T(i) : E(j)] = 0$, therefore $\operatorname{Hom}(P(j), T(i)) = 0$. Similarly, $\operatorname{Hom}(P(i), \mathcal{T}_{<i}, T(i)) = 0$, since for $j < i$, we have $\operatorname{Hom}(P(i), T(j)) = 0$. This shows that the composition of maps yields a bilinear form as stated.

It remains to be seen that this bilinear form is non-degenerate. Let $f : P(i) \to M$ be a map which does not belong to $\operatorname{Hom}(P(i), \mathcal{P}_{>i}, M)$. Let $g : M \to M/J_iM$ be the canonical projection. The image of the map $fg : P(i) \to M/J_iM$ is isomorphic to $\Delta(i)$, and the cokernel Q of fg belongs to $\mathcal{F}(\Delta)$. Let $fg = f_1f_2$ be a factorization of f with $f_1 : P(i) \to \Delta(i)$, and $f_2 : \Delta(i) \to M/J_iM$. Let $u : \Delta(i) \to T(i)$ be the canonical embedding. Since $\operatorname{Ext}^1(Q, T(i)) = 0$, it follows that there is $h : M/J_iM \to T(i)$ such that $f_2h = u$. Altogether we see that $fgh = f_1f_2h = f_1u \neq 0$.

Conversely, assume that $f' : M \to T(i)$ does not belong to $\operatorname{Hom}(M, \mathcal{T}_{<i}, T(i))$. There is a surjective map $g' : T(i) \to \nabla(i)$ with kernel $V(i) \in \mathcal{F}(\nabla(1), \ldots, \nabla(i-1))$. We claim that f' does not map into $V(i)$. So assume f' maps into $V(i)$. There is an exact sequence $0 \to'' V(i) \to' V(i) \to V(i) \to 0$, with $''V(i) \in \mathcal{F}(\nabla(1), \ldots, \nabla(i-1))$, and $'V(i) \in \mathcal{F}(\Delta)$. Since $'V(i)$ belongs both to $\mathcal{F}(\Delta)$ and to $\mathcal{F}(\nabla)$, it is in $\operatorname{add} T$, and, in fact in $\mathcal{T}_{<i}$. On the other hand, we know that $\operatorname{Ext}^1(M,'' V(i)) = 0$, since $M \in \mathcal{F}(\Delta)$, and $''V(i) \in \mathcal{F}(\nabla)$. This implies that the map $f' : M \to V(i)$ can be lifted to $'V(i)$, thus f' factors through $\mathcal{T}_{<i}$, in contrast to our assumption. It follows that $f'g' \neq 0$, thus we see that the image of $f' : M \to T(i)$ has $E(i)$ as a composition factor. Therefore, there is a map $P(i) \to M$ whose composition with f' is non-zero. This completes the proof.

In order to understand the behaviour of the function $[M] \mapsto [M : \Delta(i)]$ on $\Gamma_{\mathcal{F}(\Delta)}$, it remains to consider the sink maps in $\mathcal{F}(\Delta)$ for the projective modules $P(i)$, and the source maps in $\mathcal{F}(\Delta)$ for the relative injective modules $T(i)$.

Proposition. *Let $R(j) \to P(j)$ be the sink map in $\mathcal{F}(\Delta)$ for $P(j)$. Then*

$$[P(j) : \Delta(i)] = 0 \quad \textit{for} \quad i < j$$
$$[P(j) : \Delta(j)] = 1, \quad [R(j) : \Delta(j)] = 0,$$
$$[P(j) : \Delta(i)] = [R(j) : \Delta(i)] \quad \textit{for} \quad i > j.$$

Similarly, for the source map $T(j) \to S(j)$ for $T(j)$ in $\mathcal{F}(\Delta)$, we have

$$[T(j) : \Delta(i)] = [S(j) : \Delta(i)] \quad \textit{for} \quad i < j$$
$$[T(j) : \Delta(j)] = 1, \quad [S(j) : \Delta(j)] = 0,$$
$$[T(j) : \Delta(i)] = 0 \quad \textit{for} \quad i > j.$$

Proof: Let $U(j)$ be the submodule of $P(j)$ with $P(j)/U(j) = \Delta(j)$. Then $U(j) \in \mathcal{F}(\Delta(j+1), \ldots, \Delta(n))$, thus $[P(j) : \Delta(i)] = 0$ for $i < j$, and $[P(j) : \Delta(j)] = 1$. Also, $[P(j) : \Delta(i)] = [U(j) : \Delta(i)]$ for $i > j$. As we know, the sink map for $P(j)$ in $\mathcal{F}(\Delta)$ is of the form $g(j) : R(j) \to P(j)$, where $R(j)$ has $U(j)$ as a submodule, and all composition factors of $R(j)/U(j)$ are of the form $E(t)$ with $t < j$. Thus $[R(j) : \Delta(i)] = [U(j) : \Delta(i)] + [R(j)/U(j) : \Delta(i)]$ and $[R(j)/U(j) : \Delta(i)] = 0$ for $i \geq j$.

The second assertion will be derived from the first one using duality and the equivalence $F = \mathrm{Hom}(T, -) : \mathcal{F}(\nabla) \to \mathcal{F}(\Delta_B)$. Given a module M in $\mathcal{F}(\nabla)$, let $[M : \nabla(i)]$ be the multiplicity of $\nabla(i)$ in a ∇–filtration of M. Consider the source map $S'(j) \to T(j)$ for $T(j)$ in $\mathcal{F}(\nabla)$. We have $FT(j) = P_B(n+1-j); FS'(j) = R_B(n+1-j)$ and $F\nabla(i) = \Delta(n+1-i)$. For an arbitrary module $M \in \mathcal{F}(\nabla)$; we have $[M : \nabla(i)] = [FM : \Delta(n+1-i)]$, so we can transform the assertions concerning $P_B(n+1-j)$ and $R_B(n+1-j)$ to corresponding assertions for $T(j)$ and $R'(j)$. Using k–duality we know that $\mathcal{F}(\Delta)$ is the opposite of the category of ∇–good modules for A^{op}, and under this duality, the relative injective objects of $\mathcal{F}(\Delta)$ correspond to the relative projective objects in the category of ∇–good modules for A^{op}.

6. Multiple arrows in the Auslander–Reiten quiver

Let A be a finite–dimensional k–algebra where k is an algebraically closed field.

Warning: *Even if A is $\mathcal{F}(\Delta)$–finite, there may exist indecomposable A–modules $X_0, X_1 \in \mathcal{F}(\Delta)$ such that* $\dim_k \mathrm{Irr}_{\mathcal{F}(\Delta)}(X_0, X_1) \geq 2$. We exhibit the following examples:

First, assume that $\mathrm{rad}(P(i), P(j)) = 0$ for $i \geq j$. In this case, we have $\Delta(i) = P(i)$ for all i, thus the modules in $\mathcal{F}(\Delta)$ are the projective A–modules, and clearly $\dim_k \mathrm{Irr}_{\mathcal{F}(\Delta)}(X_0, X_1)$ may be arbitrarily large. Of course, in this case all objects of $\mathcal{F}(\Delta)$ are both relative projective and relative injective in $\mathcal{F}(\Delta)$.

A less trivial example is given as follows: Let A be a quasi-hereditary algebra with two simple modules $E(1), E(2)$ such that

$$\dim_k \mathrm{Hom}(E(1), E(2)) = 1 \quad \text{and} \quad \dim_k \mathrm{Hom}(E(2), E(1)) = d.$$

Then the indecomposable modules in $\mathcal{F}(\Delta)$ are $E(1), P(1), P(2)$ and the vector space $\mathrm{rad}(E(1), P(2)) = \mathrm{Hom}(E(1), P(2))$, is d-dimensional, whereas we observe that $\mathrm{rad}^2_{\mathcal{F}(\Delta)}(E(1), P(2)) = 0$, thus $\dim_k \mathrm{Irr}_{\mathcal{F}(\Delta)}(E(1), P(2)) = d$.

Theorem. *Let k be an algebraically closed field. Let A be $\mathcal{F}(\Delta)$–finite. Let X_0, X_1 be indecomposable A–modules in $\mathcal{F}(\Delta)$, with* $\dim_k \mathrm{Irr}_{\mathcal{F}(\Delta)}(X_0, X_1) \geq 2$. *Then $X_0 = T(j)$, and $X_1 = P(i)$ for some $j < i$.*

Proof: We are going to define modules X_i for certain $i \geq 0$ inductively as follows: Assume X_i and X_{i+1} are already defined, and X_i is not relative injective in

$\mathcal{F}(\Delta)$, then $X_{i+2} = \tau^{-}_{\mathcal{F}(\Delta)} X_i$. Note that the modules obtained in this way are indecomposable, belong to $\mathcal{F}(\Delta)$ and $\dim_k \operatorname{Irr}_{\mathcal{F}(\Delta)}(X_i, X_{i+1}) = \dim_k \operatorname{Irr}_{\mathcal{F}(\Delta)}(X_0, X_1)$.

We claim that there is some $t \geq 0$ such that the modules $X_0, \dots, X_{t+1}$ are defined and X_t is relative injective in $\mathcal{F}(\Delta)$. Otherwise, we have an infinite sequence $X_i, i \geq 0$. Let $l(X_i) \leq l(X_{i+1})$, for some i. There is a relative almost split sequence $0 \to X_i \to Y_i \to X_{i+2} \to 0$ and X^2_{i+1} is a direct summand of Y_i, thus $l(X_{i+1}) \leq l(X_{i+2})$. Deleting, if necessary, finitely many of these modules, we can assume that $l(X_i) \leq l(X_{i+1})$ for all $i \geq 0$. Let $f_i : X_i \to X_{i+1}$ be an irreducible map. The Lemma asserts that $f_i | JX_i$ is injective. We can assume that $JX_i \neq 0$ for some i, otherwise we replace A by A/J. Since with $JX_i \neq 0$, also $JX_{i+1} \neq 0$, we can assume that $JX_i \neq 0$ for all $i \geq 0$, deleting, if necessary, finitely many of the modules. It follows that the composition $f_0 f_1 ... f_i$ is non-zero for all $i \geq 0$, a contradiction to the Harada–Sai lemma.

Duality shows that we can assume, in addition, that X_1 is projective. (For, the equivalence of the category $\mathcal{F}(\nabla)$ of ∇–good modules with a category of Δ–good modules for some other quasi–hereditary algebra shows that the same assertion is true for $\mathcal{F}(\nabla)$, and k–duality shows that $\mathcal{F}(\Delta)$ is the opposite of the category of ∇–good modules for A^{op}. It just remains to renumber the modules X_i.)

Let $X_1 = P(i)$. Note that X_0^2 is a direct summand of $R(i)$, say $R(i) = X_0 \oplus X_0 \oplus R'$, and we can write $g(i) = [g_1, g_2, g']$ with $g_1, g_2 : X_0 \to P(i)$ and $g' : R' \to P(i)$. For $\alpha \in k$, consider the maps $g_\alpha = g_1 + \alpha g_2 : X_0 \to P(i)$. We will use the following fact: if $\alpha, \beta \in k$ are given such that $g_\alpha h - g_\beta$ is not irreducible, for some automorphism h of $P(i)$, then $\alpha = \beta$. For, since $g = g_\alpha h - g_\beta : X_0 \to P(i)$ is not irreducible in $\mathcal{F}(\Delta)$, the residue class of $g = g_1(h-1) + g_2(\alpha h - \beta 1)$ in $\operatorname{Irr}_{\mathcal{F}(\Delta)}(X_0, P(i))$ is zero. But this implies that $h - 1 \in \operatorname{rad} \operatorname{End} P(i)$ and $\alpha = \beta$.

We claim that the composition factors of X_0 are of the form $E(j)$ with $j < i$. For the proof, assume that X_0 has a composition factor $E(j)$ with $j \geq i$. Then, a Δ–good filtration of X_0 has some factor of the form $\Delta(j)$ with $j > i$, since X_0 is a direct summand of $R(i)$. Thus X_0 has a submodule X' isomorphic to some $\Delta(j), j > i$, such that $X_0/X' \in \mathcal{F}(\Delta)$. Let $u : X' \to X_0$ be the embedding. Note that $X' \oplus X'$ is contained in $U(i) \subseteq R(i)$, and $U(i)/(X' \oplus X') \in \mathcal{F}(\Delta)$. For $\alpha \in k$, let $u_\alpha = u g_\alpha$, clearly, this is an injective map. We denote by Q_α the cokernel of u_α, so that Q_α is indecomposable and in $\mathcal{F}(\Delta)$. We claim that for $\alpha \neq \beta \in k$, the modules Q_α and Q_β are not isomorphic. Assume they are. Since $P(i)$ is projective, we find an automorphism h of $P(i)$ such that $u_\alpha h = h' u_\beta$ for some automorphism h' of X'. However, $\operatorname{End}(X') = \operatorname{End}(\Delta(j)) = k$, thus h' is scalar multiplication by some non-zero element of k, and we can assume $h' = 1$. It follows that $g = g_\alpha h - g_\beta$ is not irreducible, since $ug = 0$, so that g factors over the cokernel X_0/X' of u. As a consequence, $\alpha = \beta$. The existence of this one–parameter family of indecomposable modules Q_α in $\mathcal{F}(\Delta)$ contradicts the assumption that $\mathcal{F}(\Delta)$ is of finite type.

Next, we claim that $\operatorname{Ext}^1(\Delta(j), X_0) = 0$ for $j < i$. Assume not, let $v : X_0 \to Y$

be a non-split embedding with cokernel Y/X_0 isomorphic to $\Delta(j)$, with $j < i$. For any $\alpha \in k$, we may consider the induced exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_0 & \xrightarrow{v} & Y & \longrightarrow & \Delta(j) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & P(i) & \xrightarrow{v_\alpha} & Y_\alpha & \xrightarrow{p_\alpha} & \Delta(j) & \longrightarrow & 0 \end{array}$$

Note that v_α cannot split, since otherwise $g_\alpha = vw$, for some $w : Y \to P(i)$, but by assumption, v is not a split monomorphism, and w is not surjective, since $E(i)$ is not a composition factor of Y. Since $\mathrm{Hom}(P(i), \Delta(j)) = 0$, it follows that Y_α is indecomposable. The module Y_α is an extension of $P(i)$ by $\Delta(j)$, thus it belongs to $\mathcal{F}(\Delta)$. We claim that for $\alpha \neq \beta \in k$, the modules Y_α and Y_β are not isomorphic. Assume there is an isomorphism $f : Y_\alpha \to Y_\beta$. Since $\mathrm{Hom}(P(i), \Delta(j)) = 0$, f induces an automorphism f' of $P(i)$ such that $v_\alpha f = f' v_\beta$, and an automorphism f'' of $\Delta(j)$, such that $p_\alpha f'' = f p_\beta$. Since $\mathrm{End}(\Delta(j)) = k$, we can assume that $f'' = 1$. The exact sequence $0 \to X_0 \to Y \to \Delta(j) \to 0$ gives rise to the exact sequence

$$\mathrm{Hom}(Y, P(i)) \xrightarrow{v^*} \mathrm{Hom}(X_0, P(i)) \xrightarrow{\delta} \mathrm{Ext}^1(\Delta(j), P(i))$$

with δ the connecting homomorphism, and, as we have seen, $\delta(g_\alpha f') = \delta(g_\beta)$, thus $g_\alpha f' - g_\beta$ is in the image of v^*, so there is a map $y : Y \to P(i)$, such that $g_\alpha f' - g_\beta = vy$. But, by assumption, v is not a split monomorphism, and y is not surjective, since $E(i)$ is not a composition factor of Y. As a consequence, $g_\alpha f' - g_\beta$ is not irreducible. Again, we conclude that $\alpha = \beta$, so that we obtain a one-parameter family of indecomposable modules Y_α in $\mathcal{F}(\Delta)$ contradicting the fact that $\mathcal{F}(\Delta)$ is of finite type.

Since the composition factors of X_0 are of the form $E(j)$, with $j < i$, it follows that $\mathrm{Ext}^1(\Delta(j), X_0) = 0$ for all $j \geq i$, thus $\mathrm{Ext}^1(\Delta(j), X_0) = 0$ for all j, and therefore $X_0 = T(j)$ for some j. Clearly, $X_0 = T(j)$ for some $j < i$, since the composition factors of X_0 are of the form $E(j)$, with $j < i$. Also, since $X_0 = T(j)$ is relative injective in $\mathcal{F}(\Delta)$, we see that $t = 0$. This completes the proof.

Since we assume that k is algebraically closed, the valuation of the arrows of $\Gamma_{\mathcal{F}(\Delta)}$ is symmetric (i.e we have $d_{XY} = d'_{XY}$ for all X, Y) As usual, we may replace an arrow $[X] \to [Y]$ with $d_{XY} = m$ by m arrows $[X] \to [Y]$ (and delete the valuation). We are happy to know that for an $\mathcal{F}(\Delta)$–finite algebra A, multiple arrows in $\Gamma_{\mathcal{F}(\Delta)}$, say from x to y, exist only in case x is an injective vertex and y is a projective vertex. For translation quivers (with possibly multiple arrows) with this property we may define the corresponding mesh category as usual (without having to choose a "polarization" [**R2**]).

7. Hammocks

Let $\pi : \tilde{\Gamma}_{\mathcal{F}(\Delta)} \to \Gamma_{\mathcal{F}(\Delta)}$ be the universal cover of $\Gamma_{\mathcal{F}(\Delta)}$ as defined in [**BG**], but with the valuation of $\Gamma_{\mathcal{F}(\Delta)}$ lifted to $\tilde{\Gamma}_{\mathcal{F}(\Delta)}$ (i.e., if $x \to y$ is an arrow of $\tilde{\Gamma}_{\mathcal{F}(\Delta)}$, let $d_{xy} = d_{\pi x, \pi y}, d'_{xy} = d'_{\pi x, \pi y}$). We will consider only the case of a translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau, d, d')$ with symmetric valuation, thus $d = d'$. In this case, we are tempted to replace any arrow $x \to y$ by d_{xy} arrows, but note that the universal cover $\tilde{\Gamma}$ of Γ will be formed *before* we insert multiple arrows. The valuation of the translation quiver $\tilde{\Gamma}$ again will be symmetric, and we may do the corresponding replacements for $\tilde{\Gamma}$. Considering Γ and $\tilde{\Gamma}$ as translation quivers with multiple arrows, the map $\pi : \tilde{\Gamma} \to \Gamma$ still is a covering map (but no longer "universal").

Assume now that k is an algebraically closed field and that A is $\mathcal{F}(\Delta)$-finite.

Let $\Gamma = \Gamma_{\mathcal{F}(\Delta)}$, and $\tilde{\Gamma} = \tilde{\Gamma}_{\mathcal{F}(\Delta)}$. Fix some $1 \le i \le n$, and let $\tilde{\mathcal{P}}_{>i}$ be the set of all vertices $p \in \tilde{\Gamma}$ such that $\pi p = [P(j)]$ with $j > i$. In view of the second characterization of $[M : \Delta(i)]$ for $M \in \mathcal{F}(\Delta)$, it seems to be reasonable to consider besides $\tilde{\Gamma}$ also the full translation subquivers $\tilde{\Gamma}^{(i)}$ obtained from $\tilde{\Gamma}$ by deleting all vertices $\tilde{\mathcal{P}}_{>i}$. We consider the mesh categories $k(\tilde{\Gamma})$ and $k(\tilde{\Gamma}^{(i)})$ (taking into account the possible multiple arrows).

Theorem. *Let $p \in \pi^{-1}([P(i)])$, for some i. Define $h_p : \Gamma_0 \to \mathbb{N}_0$ by*

$$h_p(x) = \dim_k \operatorname{Hom}_{\tilde{\Gamma}^{(i)}}(p, x).$$

Then the support of h_p is a hammock, and h_p is the corresponding hammock function.

We extend h_p to $\tilde{\Gamma}$ by $h_p(x) = 0$ for $x \notin \tilde{\Gamma}_0^{(i)}$. Then, for $M \in \mathcal{F}(\Delta)$, we have $[M : \Delta(i)] = \sum_{x \in \pi^{-1}([M])} h_p(x)$.

The proof will occupy the rest of this section.

Given any path w in $\tilde{\Gamma}$, we denote the corresponding residue class in the mesh category $k(\tilde{\Gamma})$ by $\bar{w}$; in particular, $\bar{\alpha}$ denotes the residue class of the arrow α of $\tilde{\Gamma}$. A functor $F : k(\tilde{\Gamma}) \to \mathcal{F}(\Delta)$ will be called *well-behaved* provided the following two properties are satisfied: First, for any object x of $k(\tilde{\Gamma})$, the module $F(x)$ belongs to the isomorphism class πx, and second, if $\alpha_1, \ldots, \alpha_r$ are the arrows $x \to y$ in $\tilde{\Gamma}$, then the residue classes of $F(\bar{\alpha}_1), \ldots, F(\bar{\alpha}_r)$ modulo $\operatorname{rad}^2_{\mathcal{F}(\Delta)}$ yield a k-basis of $\operatorname{Irr}_{\mathcal{F}(\Delta)}(F(x), F(y))$.

According to [**BG**], there exists a well-behaved functor $F : k(\tilde{\Gamma}) \to \mathcal{F}(\Delta)$ (the existence of multiple arrows does not add any difficulty), and we may assume that for any projective vertex p of $\tilde{\Gamma}$, with $\pi p = [P(i)]$, we have $F(p) = P(i)$.

For p in $\tilde{\Gamma}$, with $\pi p = [P(i)]$, and z an arbitrary vertex of $\tilde{\Gamma}$, we set

$$\mathcal{H}_i(p, z) = \operatorname{Hom}(p, z) / \operatorname{Hom}(p, \tilde{\mathcal{P}}_{>i}, z).$$

We should remark, that we may identify $\mathcal{H}_i(p,z)$ with $\mathrm{Hom}_{\tilde{\Gamma}(i)}(p,z)$, in particular, we have $\dim_k \mathcal{H}_i(p,z) = h_p(z)$. Similarly, for $Z \in \mathcal{F}(\Delta)$, let

$$\mathcal{H}_i(P(i), Z) = \mathrm{Hom}(P(i), Z)/\,\mathrm{Hom}(P(i), \mathcal{P}_{>j}, Z),$$

thus $\dim_k \mathcal{H}_i(P(i), Z) = [Z : \Delta(i)]$.

Lemma. *For any vertex u of $\tilde{\Gamma}$, the functor F induces an isomorphism*

$$\bigoplus_p \mathcal{H}_i(p,u) \to \mathcal{H}_i(P(i), F(u)),$$

where p ranges over all p with $\pi p = [P(i)]$.

Proof: Clearly, the functor F is dense, and it maps $\bigoplus_p \mathrm{Hom}(p, \tilde{\mathcal{P}}_{>i}, u)$ onto $\mathrm{Hom}(P(i), \mathcal{P}_{>i}, F(u))$. Consequently, we only have to show that given maps $\phi_p \in \mathrm{Hom}(p,u)$ with $\sum_p F(\phi_p) = 0$, then all ϕ_p factor through $\tilde{\mathcal{P}}_{>i}$.

Let $\phi_p \in \mathrm{Hom}(p,u)$ be maps with $\sum_p F(\phi_p) = 0$. For $t \geq 0$, let $\mathcal{W}_t$ be the set of paths of length t in $\tilde{\Gamma}$ ending in u. For any $w \in \mathcal{W}_t$, let $s(w)$ be its starting vertex. We claim that, for any $t \geq 0$, we can write ϕ_p in the form $\phi_p = \sum_{w\in\mathcal{W}_t} \phi_{p,w}\bar{w} + \phi_p^{(t)}$, where $\phi_{p,w} : p \to s(w)$ and $\phi_p^{(t)}$ are maps in $k(\tilde{\Gamma})$, such that $\phi_p^{(t)}$ factors through $\tilde{\mathcal{P}}_{>i}$, and $F(\sum_p \phi_{p,w}) = 0$ for all $w \in \mathcal{W}_t$. The proof is by induction on t. The case $t = 0$ is trivial.

So assume for some $t \geq 0$, we know that $\phi_p = \sum_{w\in\mathcal{W}_t} \phi_{p,w}\bar{w} + \phi_p^{(t)}$, where $\phi_p^{(t)}$ factors through $\tilde{\mathcal{P}}_{>i}$, and $F(\sum_p \phi_{p,w}) = 0$ for all $w \in \mathcal{W}_t$. Consider a $w \in \mathcal{W}_t$, and let $z = s(w)$. We can assume that $z \notin \tilde{\mathcal{P}}_{>i}$, changing, if necessary $\phi_p^{(t)}$. Let $\alpha_i : y_i \to z$ be the arrows ending in z, where $1 \leq i \leq r$. We claim that

$$\phi_{p,w} = \sum_i \phi_{p,w,i}\bar{\alpha}_i,$$

for suitable morphisms $\phi_{p,w,i} : p \to y_i$. This is trivially true in case $\phi_{p,w}$ is a linear combination of (residue classes of) paths of length at least one, therefore it is true in case $p \neq z$. Thus, we may assume $\pi z = [P(i)]$. Since $0 = F(\sum_p \phi_{p,w}) = F(\phi_{z,w}) + \sum_{p\neq z} F(\phi_{p,w})$, and $\sum_{p\neq z} F(\phi_{p,w})$ belongs to $\mathrm{rad}_{\mathcal{F}(\Delta)}$, we see that $F(\phi_{z,w})$ belongs to $\mathrm{rad}_{\mathcal{F}(\Delta)}$. On the other hand, $\phi_{z,w}$ is a scalar multiple of a path of length zero, thus $F(\phi_{z,w})$ is the corresponding multiple of an identity map. It follows that $\phi_{z,w} = 0$.

Since F is well-behaved, we see that $[F(\bar{\alpha}_i)]_i : Y = \bigoplus_i F(y_i) \to F(z)$ is the sink map for $F(z)$ in $\mathcal{F}(\Delta)$. Let $f : X \to Y$ be the kernel of this map. Since $0 = F(\sum_p \phi_{p,w}) = F(\sum_{p,i} \phi_{p,w,i}\bar{\alpha}_i)$, we see there is a map $h : P(i) \to X$ such that $hf = [F(\sum_p \phi_{p,w,1}), \ldots, F(\sum_p \phi_{p,w,r})]$.

First, consider the case of z being a projective vertex. By assumption, $z \notin \tilde{\mathcal{P}}_{>i}$, thus $\pi z = [P(j)]$ for some $j \leq i$. But X belongs to $\mathcal{F}(E(1), \ldots, E(j-1))$, and therefore $\mathrm{Hom}(P(i), X) = 0$. It follows that $F(\sum_p \phi_{p,w,i}) = 0$ for all $1 \leq i \leq r$. In this case, let $\phi_{p,\alpha_i w} = \phi_{p,w,i}$, where $\alpha_i w$ denotes the path in $\mathcal{W}_{t+1}$ obtained by composing the arrow α_i with the path w. It follows that $F(\sum_p \phi_{p,\alpha_i w}) = 0$, and that

$$\sum_i \phi_{p,\alpha_i w} \bar{\alpha}_i \bar{w} = \sum_i \phi_{p,w,i} \bar{\alpha}_i \bar{w} = \phi_{p,w} \bar{w}.$$

Next, we assume that z is not projective, thus $X \cong \tau_\Delta F(z)$. Note that in this case there is a unique arrow $\beta_i : \tau z \to y_i$, since we know that multiple arrows can occur only from an injective vertex to a projective vertex. We can assume that $f = [F(\beta_1), \ldots, F(\beta_r)] : X = F(\tau z) \to \bigoplus_i F(y_i)$. Also, h may be written in the form $h = F(\sum_p \psi_p)$, thus we have $F(\sum_p \psi_p \beta_i) = F(\sum_p \phi_{p,w,i})$, for all i. In this case, let $\phi_{p,\alpha_i w} = \phi_{p,w,i} - \psi_p \beta_i$. It follows that $F(\sum_p \phi_{p,\alpha_i w}) = 0$. On the other hand, observe that

$$\sum_i \phi_{p,\alpha_i w} \bar{\alpha}_i \bar{w} = \sum_i (\phi_{p,w,i} - \psi_p \bar{\beta}_i) \bar{\alpha}_i \bar{w} = \sum_i \phi_{p,w,i} \bar{\alpha}_i \bar{w} = \phi_{p,w} \bar{w},$$

where we use that $\sum_i \bar{\beta}_i \bar{\alpha}_i = 0$.

For any path $\alpha_i w$, we have defined $\phi_{p,\alpha_i w}$, such that $F(\sum_p \phi_{p,\alpha_i w}) = 0$, and such that $\sum_i \phi_{p,\alpha_i w} \bar{\alpha}_i \bar{w} = \phi_{p,w} \bar{w}$. The latter implies that $\sum_{w \in \mathcal{W}_t} \sum_i \phi_{p,\alpha_i w} \bar{\alpha}_i \bar{w} + \phi_p^{(t)} = \phi_p$. This completes the induction.

However, for large t, we have $\mathrm{Hom}(p, s(w)) = 0$, for any $w \in \mathcal{W}_t$, so in this case $\phi_p = \phi_p^{(t)}$. This shows that ϕ_p factors through $\tilde{\mathcal{P}}_{>i}$, and completes the proof of the Lemma.

Corollary 1. *Let p_0, u_0 be vertices of $\tilde{\Gamma}$, with p_0 projective. Let $F(p_0) = P(i)$, and $F(u_0) = M$. Then*

$$[M : \Delta(i)] = \sum_{p \in \pi^{-1}([P(i)])} h_p(u_0) = \sum_{u \in \pi^{-1}([M])} h_{p_0}(u).$$

Proof: The first equality follows from the Lemma considering k-dimensions. The fundamental group G of Γ operates on $\tilde{\Gamma}$, and the fibers $\pi^{-1}(x)$ with $x \in \Gamma_0$ are just the G-orbits of $\tilde{\Gamma}_0$. Shifting by the various elements of G, the second term is transformed in the third one.

Corollary 2. *The support of h_p is finite, for any projective vertex p of $\tilde{\Gamma}$.*

Proof: For any indecomposable module M in $\mathcal{F}(\Delta)$, there can be only finitely many elements $u \in \pi^{-1}([M])$ with $h_p(u) \neq 0$, since these numbers add up to $[M : \Delta(i)]$.

Corollary 3. *Let p, z be vertices in $\tilde{\Gamma}$, with p projective. Then $h_p(p) = 1$, and, for $z \neq p$,*

$$h_p(z) = \sum_{\alpha : y \to z} h_p(y) - h_p(\tau z),$$

where, by definition, $h_p(\tau z) = 0$ in case z is projective.

Proof: Clearly, $h_p(p) = 1$, thus we may assume $z \neq p$. Let $\alpha_s : y_s \to z$, with $1 \leq s \leq t$ be the arrows ending in z. In case z is projective, the α_s induce an isomorphism

$$\bigoplus_{s=1}^{t} \mathrm{Hom}(p, y_i) \to \mathrm{Hom}(p, z),$$

thus $h_p(z) = \sum_{\alpha : y \to z} h_p(y)$ in this case. It remains to consider the case when z is non–projective. The α_s induce an exact sequence

$$\mathrm{Hom}(p, \tau z) \to \bigoplus_{s=1}^{t} \mathrm{Hom}(p, y_i) \to \mathrm{Hom}(p, z) \to 0,$$

(see [**BG**] and the remarks in [**RV**]), thus we see that

$$h_p(z) \geq \sum_{s=1}^{t} h_p(y) - h_p(\tau z).$$

Now, let $F(z) = Z, F(y_s) = Y_s, F(\tau z) = X$, and add up all these inequalities for $p' \in \pi^{-1}([P(i)])$. Since we obtain as sum the equality

$$[Z : \Delta(i)] = \sum_{s=1}^{t} [Y_s : \Delta(i)] - [X : \Delta(i)],$$

it follows that all the inequalities had been, in fact, equalities. This completes the proof.

It remains to consider the behaviour of h_p at injective vertices of $\tilde{\Gamma}$.

Lemma. *Let $j < i$, and let $[P(j) : \Delta(i)] = t$. There are maps $f : P(i) \to P(j)$, $g_s \in \mathrm{rad}\,\mathrm{End}(P(j)$, with $1 \leq s < t$, and $h : P(j) \to T(i)$, such that $fg_1 \cdot g_{t-1}h$ is non–zero.*

Proof: We want to show that the right $\mathrm{End}(P((j))$–module $\mathcal{H}_i(P(i), P(j))$ is serial.

First, consider the case $i = n$. Note that $\mathcal{H}_i(P(n), P(i)) = \mathrm{Hom}(P(n), P(i))$. Assume there are elements f_1, f_2 in $\mathrm{Hom}(P(n), P(i))$ such that the subspaces

$f_1 \cdot \mathrm{Hom}(P(n), P(i))$ and $f_2 \cdot \mathrm{Hom}(P(n), P(i))$ are incomparable. For $\alpha \in k$, let Q_α be the cokernel of $f_1 + \alpha f_2 : P(n) \to P(i)$. Clearly, Q_α is indecomposable, and belongs to $\mathcal{F}(\Delta)$. And, it is easy to see that for $\alpha \neq \beta$, the modules Q_α and Q_β are non–isomorphic. Thus we obtain a one–parameter family of indecomposable modules in $\mathcal{F}(\Delta)$, in contrast to our assumption on A to be $\mathcal{F}(\Delta)$–finite.

Let $f : P(i) \to P(j)$ be a map whose residue class modulo $\mathrm{Hom}(P(i), \mathcal{P}_{>i}, P(j))$ does not belong to $\mathcal{H}_i(P(i), P(j)) \cdot \mathrm{rad}\,\mathrm{End}(P(j))$. Since $\mathcal{H}_i(P(i).P(j))$ is serial as a right $\mathrm{End}((P(j))$–module, we obtain elements $g_s \in \mathrm{rad}\,\mathrm{End}(P(j))$, so that $fg_1 \cdots g_{s-1}$ does not belong to $\mathrm{Hom}(P(i), \mathcal{P}_{>i}, P(j))$. The bilinear pairing exhibited above yields a map $h : P(j) \to T(i)$ such that the composition $fg_1 \cdots g_{t-1}h$ does not belong to $\mathrm{Hom}(P(i), \mathcal{P}_{>i}, T(i))$.

We will need the dual assertion which may be stated as follows:

Lemma. *Let $j > i$, and let $[T(j) : \Delta(i)] = t$. There are maps $f : P(i) \to T(j)$, $g_s \in \mathrm{rad}\,\mathrm{End}(T(j))$, with $1 \leq s < t$, and $h : T(j) \to T(i)$, such that $fg_1 \cdots g_{t-1}h$ is non–zero.*

Lemma. *Let p be a projective vertex, q an injective vertex of $\tilde{\Gamma}$, say $\pi p = [P(i)]$, and $\pi q = [T(j)]$. If $j < i$, then $h_p(q) = 0$. If $j = i$, then $h_p(y) = 0$ for any vertex $y \in q^+$. If $j > i$ and $h_p(q) \neq 0$, then $\sum_{q \to y} h_p(y) \leq 1$.*

Proof: For $j < i$, we have $[T(j) : \Delta(i)] = 0$, thus $h_p(q) = 0$. For $j = i$, we have $[S(j) : \Delta(i)] = 0$, thus $\sum_{q \to y} h_p(y) = 0$. So let us assume $j > i$. In this case, and let $[T(j) : \Delta(i)] = t$. We know that also $[S(j) : \Delta(i)] = t$. According to the previous lemma, there are elements $f \in \mathrm{Hom}(P(i), T(j))$, $g_s \in \mathrm{rad}\,\mathrm{End}(T(j))$, $h \in \mathrm{Hom}(T(j), T(i))$, with $1 \leq s < t$, such that $fg_1 \cdots g_{t-1}h \neq 0$. This implies that in $\tilde{\Gamma}^{(i)}$, there is a path ϕ from p to q_1, and non–constant paths γ_s from q_s to q_{s+1}, for $1 \leq s < t$, and η from q_t to q', where $\tau q_s = [T(j)]$, for all $1 \leq s \leq t$, and $\tau q' = [T(i)]$, such that $\bar{\phi}\bar{\gamma}_1 \cdots \bar{\gamma}_{t-1}\bar{\eta} \neq 0$ in $k(\tilde{\Gamma}^{(i)})$. Since the paths γ_s and η are of length at least one, let $\alpha_s : q_s \to y_s$, with $1 \leq s < t$ be the first arrow of γ_s, and α_t the first arrow of η. Then $\bar{\phi}\bar{\gamma}_1 \cdots \bar{\gamma}_{s-1}\bar{\alpha}_s \neq 0$ shows that $h_p(y_s) \geq 1$, for all $1 \leq s \leq t$. Thus

$$t = [S(j) : \Delta(i)] = \sum_{z \in \pi^{-1}([T(j)])} \sum_{z \to y} h_p(y) \geq \sum_{s=1}^{t} h_p(y_s) \geq t$$

implies that the values $h_p(y_s)$ are the only non–zero summands in the double sum, and all these values are equal to 1. As a consequence, we have $\sum_{z \to y} h_p(y) = 1$ for $z = q_s$, and $\sum_{z \to y} h_p(y) = 0$ otherwise. On the other hand, the vertices q_s, are the only vertices in $\pi^{-1}([T(j)])$ which belong to the support of h_p.

This finishes the proof.

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References

[A] Auslander, M.: Applications of morphisms determined by objects. In: Representation Theory of Algebras. LN Pure Appl. Math 37. M.Dekker (1978), 245-327.

[BG] Bongartz, K. and Gabriel, P.: Covering spaces in representation theory. Invent. Math.65 (1982), 331–378.

[HPR] Happel,D., Preiser.U., and Ringel,C.M.: Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules. In: Representation Theory II. Springer LNM 832 (1980), 280–294.

[R1] Ringel, C.M.: Report on the Brauer–Thrall conjectures. In: Springer LNM 832 (1980), 104–136.

[R2] Ringel, C.M.: Tame algebras and integral quadratic forms. Springer LNM 1099 (1984).

[R3] Ringel, C.M.: The category of good modules over a quasi-hereditary algebra has almost split sequences. Math.Z. (to appear)

[R3] Ringel, C.M.: On contravariantly finite subcategories. (to appear)

[RV] Ringel,C.M., and Vossieck,D.: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216–246

[Z] Zhang,Y.: The structure of regular components. Can. J. (to appear).

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On contravariantly finite subcategories

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Let A be an artin algebra. We will consider (finitely generated left) A–modules, maps between A–modules will be written on the right hand of the argument, thus the composition of the maps $f : M_1 \to M_2$, $g : M_2 \to M_3$ will be denoted by fg. The category of all A–modules will be denoted by A–mod. All subcategories considered will be full and closed under isomorphisms, so usually we will describe subcategories by just specifying their objects (up to isomorphism).

Let $\mathcal{X}$ be a subcategory of A–mod. Recall that $\mathcal{X}$ is said to be *extension closed* provided for any exact sequence $0 \to X_2 \to E \to X_1 \to 0$ with $X_1, X_2 \in \mathcal{X}$, also $E \in \mathcal{X}$. Given an A–module M, a *right $\mathcal{X}$–approximation* of M is a map $g : X \to M$ with $X \in \mathcal{X}$ such that for any map $h : X' \to M$ with $X' \in \mathcal{X}$, there is a map $h' : X' \to X$ such that $h = h'g$. In case every A–module has a right $\mathcal{X}$–approximation, $\mathcal{X}$ is said to be *contravariantly finite in* A–mod. We write $\operatorname{Ext}^1_A(\mathcal{X}, Y) = 0$ as an abbreviation for $\operatorname{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$, and we use corresponding notation in similar cases.

There is the following criterion:

Proposition. *Let $\mathcal{X}$ be an extension closed subcategory of A–mod. Then $\mathcal{X}$ is contravariantly finite in* A–mod *if and only if any A–module M can be embedded into an A–module $\overline{M}$ such that $\overline{M}/M \in \mathcal{X}$ and* $\operatorname{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$.

Proof: One direction is due to Auslander–Reiten [**AR**], the other one has been shown in [**R**], Lemma 2. For the convenience of the reader, we indicate the arguments of [**AR**], but we delete the functorial and homological interpretations of the individual steps.

So assume that $\mathcal{X}$ is contravariantly finite in A–mod, and let M be an arbitrary A–module. According to Auslander–Smalø[**AS**], there is an embeddding $v : M \hookrightarrow M'$ with $M'/M \in \mathcal{X}$ such that for any embedding $w : M \hookrightarrow Y$ with $Y/M \in \mathcal{X}$, there is a map $f : Y \to M'$ with $wf = v$. Indeed, we just construct a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{v} & M' & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & I & \longrightarrow & \Sigma M & \longrightarrow & 0
\end{array}$$

with exact rows, I an injective A–module, and g a right $\mathcal{X}$–approximation, starting with the lower row. Of course, we can assume that v is an embedding, and one easily checks that v has the desired property.

Recall that a map $y: M \to Y$ is called *left minimal* provided any endomorphism e of Y with $ye = y$ is an automorphism. We can decompose $M' = \overline{M} \oplus M''$ so that the image of v is contained in $\overline{M}$, say $v = [u\,0]$ with an embedding $u : M \hookrightarrow \overline{M}$ which is left minimal. In this way, we obtain a left minimal embedding u of M into $\overline{M}$ with $\overline{M}/M \in \mathcal{X}$ and such that for any embedding $w: M \hookrightarrow Y$ with $Y/M \in \mathcal{X}$, there is a map $f: Y \to \overline{M}$ with $wf = u$.

In order to see that $\mathrm{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$, consider an embedding $h: \overline{M} \hookrightarrow H$ with $H/\overline{M} \in \mathcal{X}$. We claim that h splits. The cokernel H/M of $uh: M \hookrightarrow H$ belongs to $\mathcal{X}$, since both $H/\overline{M}$ and $\overline{M}/M$ belong to $\mathcal{X}$, and $\mathcal{X}$ is extension closed. Thus, there is a map $f: H \to \overline{M}$ with $uhf = u$. But u is minimal, thus hf is an automorphism, and therefore h is a split monomorphism. This completes the proof.

Consider now the following situation: Given subcategories $\mathcal{X}_1, \mathcal{X}_2$ of A–mod, let $\mathcal{X}_1 \int \mathcal{X}_2$ be the full subcategory of all A-modules M which have a submodule U belonging to $\mathcal{X}_2$ such that M/U belongs to $\mathcal{X}_1$. One may wonder whether with $\mathcal{X}_1, \mathcal{X}_2$ also $\mathcal{X}_1 \int \mathcal{X}_2$ is contravariantly finite in A–mod. Using the criterion above, we are able to show:

Theorem. *Let $\mathcal{X}_1, \mathcal{X}_2$ be subcategories with $\mathrm{Ext}^1_A(\mathcal{X}_2, \mathcal{X}_1) = 0$. If both $\mathcal{X}_1, \mathcal{X}_2$ are extension closed and contravariantly finite in A–mod, then also $\mathcal{X}_1 \int \mathcal{X}_2$ is extension closed and contravariantly finite in A–mod.*

Proof: Let $\mathcal{X} = \mathcal{X}_1 \int \mathcal{X}_2$. In order to show that $\mathcal{X}$ is extension closed, let M be an A-module with a submodule U such that both U and M/U belong to $\mathcal{X}$. By definition, there are submodules $U' \subseteq U \subseteq U'' \subseteq M$ such that both $U', U''/U \in \mathcal{X}_2$ and both $U/U', M/U'' \in \mathcal{X}_1$. Since $\mathrm{Ext}^1_A(U''/U, U/U') = 0$, there is a submodule M' with $U' \subseteq M' \subseteq U''$ such that $U''/M' \cong U/U'$ and $M'/U' \cong U''/U$. Since $\mathcal{X}_1$ is closed under extensions, and $M/U'', U''/M' \in \mathcal{X}_1$, also $M/M' \in \mathcal{X}_1$. Similarly, since $\mathcal{X}_2$ is closed under extensions, $M' \in \mathcal{X}_2$. Thus, M belongs to $\mathcal{X}$.

In order to show that $\mathcal{X}$ is contravariantly finite in A–mod, we apply the Proposition. Let M be any A-module. We want to show that M can be embedded into an A-module $\overline{M}$ such that $\overline{M}/M \in \mathcal{X}$ and $\mathrm{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$. Since $\mathcal{X}_2$ is extension closed and contravariantly finite in A–mod, there is an embedding $M \hookrightarrow Y$ such that $Y/M \in \mathcal{X}_2$ and $\mathrm{Ext}^1_A(\mathcal{X}_2, Y) = 0$. Since $\mathcal{X}_1$ is extension closed and contravariantly finite in A–mod, there is an embedding $Y \hookrightarrow \overline{M}$ such that $\overline{M}/Y \in \mathcal{X}_1$ and $\mathrm{Ext}^1_A(\mathcal{X}_1, \overline{M}) = 0$. Clearly, $\overline{M}/M \in \mathcal{X}$, since there is the submodule $Y/M \in \mathcal{X}_2$ and $\overline{M}/Y \in \mathcal{X}_1$. It remains to be seen that $\mathrm{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$. We know already $\mathrm{Ext}^1_A(\mathcal{X}_1, \overline{M}) = 0$, thus we have to show that $\mathrm{Ext}^1_A(\mathcal{X}_2, \overline{M}) = 0$. Consider the exact sequence $0 \to Y \to \overline{M} \to \overline{M}/Y \to 0$. Since $\mathrm{Ext}^1_A(\mathcal{X}_2, Y) = 0$ and $\overline{M}/Y \in \mathcal{X}_1$, it follows that $\mathrm{Ext}^1_A(\mathcal{X}_2, \overline{M}) = 0$.

Let us stress that the operation $\int$ on subcategories is obviously associative, so given subcategories $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$, the subcategory $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ consists of the modules M which have a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$ such that

$M_{i-1}/M_1 \in \mathcal{X}_i$ for all $1 \leq i \leq t$. Using induction, we immediately obtain the following result:

Corollary 1. *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$, be subcategories which are extension closed and contravariantly finite in A-mod. Assume that $\mathrm{Ext}^1_A(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $j > i$. Then also $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ is extension closed and contravariantly finite in A-mod.*

There is the dual notion of covariantly finite subcategories: Let $\mathcal{X}$ be a subcategory of A-mod. Given an A-module M, a *left $\mathcal{X}$-approximation* of M is a map $f : M \to X$ with $X \in \mathcal{X}$ such that for any map $h : X \to X'$ with $X' \in \mathcal{X}$, there is a map $h' : X \to X'$ such that $h = fh'$. In case every A-module has a left $\mathcal{X}$-approximation, $\mathcal{X}$ is said to be *covariantly finite in* A-mod. And $\mathcal{X}$ is said to be *functorially finite in* A-mod provided $\mathcal{X}$ is both contravariantly and covariantly finite in A-mod. The dual assertion of Corollary 1 is the following:

Corollary 2. *Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$, be subcategories which are extension closed and covariantly finite in A-mod. Assume that $\mathrm{Ext}^1_A(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $j > i$. Then also $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ is extension closed and covariantly finite in A-mod.*

Applications

As first application, we will obtain Theorem 1 of [R]. Let $\Theta = \{\Theta(1), \ldots, \Theta(n)\}$ be a finite set of A-modules with $\mathrm{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ for $j \geq i$. We denote by $\mathcal{F}(\Theta)$ the full subcategory of A-mod of direct summands of modules having a filtration with factors in Θ, thus, M belongs to $\mathcal{F}(\Theta)$ if and only if M has submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = M$ such that M_{s-1}/M_s is isomorphic to a module in Θ.

Corollary. *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in A-mod.*

Proof: For any $1 \leq i \leq n$, let $\mathcal{X}_i$ be the subcategory of all modules which are direct sums of copies of $\Theta(i)$. Since $\mathrm{Ext}^1_A(\Theta(i), \Theta(i)) = 0$, we see that $\mathcal{X}_i$ is closed under extensions. Also, it is well-known and easy to see that $\mathcal{X}_i$ is functorially finite in A-mod (in order to obtain a right $\mathcal{X}_i$-approximation for a module M, take $[g_1, \ldots, g_t] : M \to \Theta(i)^t$, where $g_1, \ldots, g_t$ is a k-basis of $\mathrm{Hom}_A(M, \Theta(i))$, and similarly, one obtains a left $\mathcal{X}_i$-approximation). The assumption $\mathrm{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ for $j > i$ yields $\mathrm{Ext}^1_A(\mathcal{X}_j, \mathcal{X}_i) = 0$ for $j > i$, thus we can apply Corollary 1 and Corollary 2 in order to conclude that $\mathcal{X} = \mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ is functorially finite in A-mod. But, of course, $\mathcal{X} = \mathcal{F}(\Theta)$.

As a second application, we obtain a recent result of Smalø[S]. Let $e \in A$ be an idempotent such that $eA(1-e) = 0$. Let $R = eAe$, and $S = (1-e)A(1-e)$.

Note that we may write A as a lower triangular matrix ring $A = \begin{bmatrix} R & 0 \\ T & S \end{bmatrix}$ with $T = (1-e)Ae$. We may (and will) consider both R–mod and S–mod as subcategories of A–mod, namely, we identify R–mod with the subcategory of all A–modules M with $eM = M$, and S–mod with the subcategory of all A–modules M with $eM = 0$. In this way, R–mod and S–mod are subcategories which are closed under submodules, factor modules and extensions, and thus they are functorially finite in A–mod. Given an A–module M, then $(1-e)M$ is always an A–submodule which belongs to S–mod, and $M/(1-e)M$ belongs to R–mod. In particular, $\mathrm{Ext}^1_A(S\text{–mod}, R\text{–mod}) = 0$. For, given an A–module M with a submodule U such that U belongs to R–mod and M/U belongs to S–mod, then $(1-e)M$ is a direct complement to U.

Let $\mathcal{R}$ be a subcategory of R–mod, and $\mathcal{S}$ a subcategory of S–mod. Following Smalø[S], we denote $\mathcal{R} \int \mathcal{S}$ by $A\text{–mod}^{\mathcal{R}}_{\mathcal{S}}$.

Corollary. *Let $\mathcal{R}$ be an extension closed subcategory of R–mod, and let $\mathcal{S}$ be an extension closed subcategory of S–mod. If $\mathcal{R}$ is contravariantly finite in R–mod, and $\mathcal{S}$ is contravariantly finite in S–mod, then $A\text{–mod}^{\mathcal{R}}_{\mathcal{S}}$ is contravariantly finite in A–mod. If $\mathcal{R}$ is covariantly finite in R–mod, and $\mathcal{S}$ is covariantly finite in S–mod, then $A\text{–mod}^{\mathcal{R}}_{\mathcal{S}}$ is covariantly finite in A–mod.*

Proof: Clearly, a subcategory $\mathcal{R}$ of R–mod which is extension closed, or contravariantly finite, or covariantly finite in R–mod, has the same property even in A–mod. And similarly, a subcategory $\mathcal{S}$ of S–mod which is extension closed, or contravariantly finite, or covariantly finite in S–mod, has the same property even in A–mod. Also, as we have noted above, we have $\mathrm{Ext}^1_A(S\text{–mod}, R\text{–mod}) = 0$, thus $\mathrm{Ext}^1_A(\mathcal{S}, \mathcal{R}) = 0$.

Both results generalize a previous observation of Grecht [G], de la Pena and Simson [PS], and Vossieck [V] on prinjective modules. Recall that an A–module M is called *prinjective*, provided it belongs to $A\text{–mod}^{\mathcal{P}(R)}_{\mathcal{I}(S)}$, where $\mathcal{P}(R)$ is the subcategory of projective R–modules, $\mathcal{I}(S)$ the subcategory of injective S–modules. Thus, M is prinjective if and only if $(1-e)M$ is an injective S–module, and $M/(1-e)M$ is a projective R–module. Note that we have

$$A\text{–mod}^{\mathcal{P}(R)}_{\mathcal{I}(S)} = \mathcal{F}(\Theta),$$

with $\Theta(1), \ldots, \Theta(m)$ the indecomposable projective R–modules, and $\Theta(m+1), \ldots, \Theta(n)$ the indecomposable injective S–modules.

References

[AR] Auslander, M. and Reiten, I.: Applications of contravariantly finite subcategories. Adv.Math.(To appear)

[AS] Auslander, M. and Smalø, S.: Almost split sequences in subcategories. J. Algebra **69** (1981), 426–454.

[PS] de la Pena, J. and Simson, D.: Prinjective modules, reflection functors, quadratic forms and almost split sequences. Trans.Amer.Math.Soc. (to appear)

[G] Grecht, R.: Kategorien von Moduln mit Unterräumen. Diplomarbeit. Zürich (1986)

[R] Ringel, C.M.: The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. Math.Z. (to appear)

[S] Smalø, S.: (to appear)

[V] Vossieck, D.: Representations de bifuncteurs et interpretation en termes de modules. C.R.Acad.Sci.Paris. 307, Ser. I (1988). 713–716

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