Inverse Iteration for Calculating the Spectral Radius of a Non-Negative Irreducible Matrix

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ABSTRACT

Noda established the superlinear convergence of an inverse iteration procedure for calculating the spectral radius and the associated positive eigenvector of a non-negative irreducible matrix. Here a new proof is given, based completely on the underlying order structure. The main tool is Hopf's inequality. It is shown that the convergence is quadratic.

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1. INTRODUCTION

Throughout this paper A will denote a non-negative irreducible $N \times N$ matrix with spectral radius ρ and associated positive eigenvector p.

In [5], Noda established the convergence of an inverse iteration procedure for the determination of ρ and p. He also showed that the convergence is superlinear. Here we shall prove that it is at least quadratic.

This is an easy by-product of our proof of convergence, which uses only the underlying order structure and not (as in [5]) the Jordan form. The main tool is Hopf's inequality. As it has been used for bounding the eigenvalues $\neq \rho$, it is quite natural to use it for convergence proofs, too.

2. DEFINITIONS; TWO LEMMAS

An $N \times N$ matrix $B = (b_{ik})$ is called positive (non-negative) if $b_{ik} > 0$ ($\geqslant 0$), $i,k = 1,\ldots,N$. We write $B > 0$ ($\geqslant 0$). For vectors, $y > 0$, $y \geqslant 0$ are defined in an analogous way.

For a pair of vectors x, y with $y > 0$, we define

$$\max\left(\frac{x}{y}\right) = \max_i \frac{x_i}{y_i}, \qquad \min\left(\frac{x}{y}\right) = \min_i \frac{x_i}{y_i},$$

$$\operatorname{osc}\left(\frac{x}{y}\right) = \max\left(\frac{x}{y}\right) - \min\left(\frac{x}{y}\right).$$

Hopf's inequality [1, 3, 6] states: For $B > 0$ and any pair of vectors x, y, where $y > 0$,

$$\operatorname{osc}\left(\frac{Bx}{By}\right) \leqslant N(B) \operatorname{osc}\left(\frac{x}{y}\right). \tag{1}$$

Here

$$N(B) = \frac{\sqrt{K(B)} - 1}{\sqrt{K(B)} + 1}$$

and

$$K(B) = \sup_{\substack{u>0 \\ v>0}} \left\{ \max\left(\frac{Bu}{Bv}\right) \max\left(\frac{Bv}{Bu}\right) \right\}. \tag{2}$$

It is obvious that

$$N(tB) = N(B), \qquad t > 0, \tag{3}$$

$$N(pq^T) = 0, \qquad p > 0, \quad q > 0, \tag{4}$$

$$N(D_1 B D_2) = N(B), \tag{5}$$

where D_i $(i = 1, 2)$ are diagonal matrices with positive diagonal entries. A bound for $N(B)$ is [3, 6]

$$N(B) \leqslant \frac{m_1 - m_2}{m_1 + m_2}, \qquad m_1 = \max_{i,k} b_{ik}, \quad m_2 = \min_{i,k} b_{ik}. \tag{6}$$

LEMMA 1. *Let $p>0$, $q>0$, $\tilde{p}=\min p_i$, $\tilde{q}=\min q_i$, and $B=(b_{ik})$ be a positive matrix such that*

$$|b_{ik}-p_iq_k|\leqslant\varepsilon. \tag{7}$$

Then

$$N(B)\leqslant\frac{\varepsilon}{\tilde{p}\tilde{q}}. \tag{8}$$

Proof. Define $\tilde{B}=(\tilde{b}_{ik})$, $\tilde{b}_{ik}=b_{ik}/p_iq_k$, $\tilde{\varepsilon}=\varepsilon/\tilde{p}\tilde{q}$. Then by (7)

$$\max\tilde{b}_{ik}\leqslant 1+\tilde{\varepsilon},\qquad \min\tilde{b}_{ik}\geqslant 1-\tilde{\varepsilon},$$

and hence by (5) and (6)

$$N(B)=N(\tilde{B})\leqslant\tilde{\varepsilon}.$$ ■

REMARK. By taking $p_i=(m_1+m_2)/2$, $q_i=1$, $\varepsilon=(m_1-m_2)/2$ in Lemma 1, (8) yields the bound (6).

LEMMA 2. *For a given number $\lambda_0>\rho$ there is an $M>0$ such that*

$$N\big((\lambda I-A)^{-1}\big)\leqslant M(\lambda-\rho),\qquad \rho<\lambda\leqslant\lambda_0. \tag{9}$$

Proof. The adjoint $\mathrm{adj}(B)$ of a square matrix B satisfies the relation [4, p. 13]

$$B\,\mathrm{adj}(B)=\mathrm{adj}(B)B=(\det B)I$$

In particular, for $\lambda>\rho$,

$$(\lambda I-A)^{-1}=\frac{1}{\det(\lambda I-A)}\mathrm{adj}(\lambda I-A).$$

Hence by (3),

$$N\big((\lambda I-A)^{-1}\big)=N(\mathrm{adj}(\lambda I-A)). \tag{10}$$

On the other hand,

$$\operatorname{adj}(\rho I - A) = pq^T,$$

where $q > 0$, $A^T q = \rho q$, q suitably normalized. Equation (9) follows now from Lemma 1. ∎

3. THE ITERATIVE PROCEDURE

We define

$$\|x\| = \max\left(\frac{x}{p}\right). \tag{11}$$

Let $\{B_n\}$, $n = 0, 1, \ldots$ be a sequence of positive matrices commuting with A. Assume the existence of γ such that

$$N(B_n) \leqslant \gamma < 1, \qquad n = 0, 1, \ldots . \tag{12}$$

For given $x_0 > 0$, define iteratively

$$\tilde{x}_{n+1} = B_n x_n, \tag{13}$$

$$x_{n+1} = \frac{\tilde{x}_{n+1}}{\|\tilde{x}_{n+1}\|}, \tag{14}$$

$$\bar{\lambda}_{n+1} = \max\left(\frac{Ax_{n+1}}{x_{n+1}}\right), \qquad \underline{\lambda}_{n+1} = \min\left(\frac{Ax_{n+1}}{x_{n+1}}\right). \tag{15}$$

$\bar{\lambda}_0$, $\underline{\lambda}_0$ are defined analogously.

We prove first some useful relations:

LEMMA 3. *For* $n = 1, 2, \ldots,$

$$\bar{\lambda}_n - \rho \leqslant \rho \frac{\operatorname{osc}(x_n/p)}{1 - \operatorname{osc}(x_n/p)} \leqslant C\rho(\bar{\lambda}_n - \rho), \tag{16}$$

$$\rho - \underline{\lambda}_n \leqslant \rho \operatorname{osc}\left(\frac{x_n}{p}\right) \leqslant \tilde{C}(\rho - \underline{\lambda}_n), \tag{17}$$

where C, $\tilde{C}$ *depend on* A, *and* C *also on* $\bar{\lambda}_0$.

Proof. From $\|x_n\|=1$, $n>0$ we get

$$1-\operatorname{osc}\left(\frac{x_n}{p}\right)\leqslant\frac{x_{n,i}}{p_i}\leqslant 1.$$

Hence for suitable s

$$\bar{\lambda}_n-\rho=\sum_k a_{sk}\frac{p_k}{p_s}\left(\frac{x_{n,k}}{p_k}\frac{p_s}{x_{n,s}}-1\right)\leqslant\rho\left(\frac{1}{1-\operatorname{osc}(x_n/p)}-1\right),$$

showing the left inequality of (16). The left inequality of (17) follows in an analogous way. For the other inequalities we use a result in [2, Folgerung 2, p. 72]: Let $x>0$, $z>0$, and $Ax\leqslant\alpha x$, $Az\geqslant\beta z$, and choose i so that x_i/z_i is minimal. For any $k\neq i$ there is an $s\leqslant n-1$ such that $a_{ik}^{(s)}=(A^s)_{ik}>0$ and

$$\frac{x_i}{z_i}\leqslant\frac{x_k}{z_k}\leqslant\left(1+\frac{\alpha^s-\beta^s}{a_{ik}^{(s)}}\frac{z_i}{z_k}\right)\frac{x_i}{z_i}\tag{18}$$

and

$$\left(1-\frac{\alpha^s-\beta^s}{a_{ik}^{(s)}}\frac{x_i}{x_k}\right)\frac{z_i}{x_i}\leqslant\frac{z_k}{x_k}\leqslant\frac{z_i}{x_i}.\tag{19}$$

Taking $x=x_n$, $\alpha=\bar{\lambda}_n$, $z=p$, $\beta=\rho$ in (18), we get

$$\min\left(\frac{x_n}{p}\right)\leqslant\frac{x_{n,k}}{p_k}\leqslant\left[1+C\left(\bar{\lambda}_n-\rho\right)\right]\min\left(\frac{x_n}{p}\right)$$

for suitable C depending on an upper bound for $\bar{\lambda}_n$, $n=1,2,\ldots$. According to Theorem 1 such a bound is provided by $\bar{\lambda}_0$. Thus

$$\operatorname{osc}\left(\frac{x_n}{p}\right)\leqslant C\left(\bar{\lambda}_n-\rho\right)\left[1-\operatorname{osc}\left(\frac{x_n}{p}\right)\right],$$

yielding the right inequality in (16).

Taking $x=p$, $\alpha=\rho$, $z=x_n$, $\beta=\underline{\lambda}_n$ in (19), we get for a suitable $\tilde{C}$

$$\left[1-\tilde{C}\left(\rho-\underline{\lambda}_n\right)\right]\max\left(\frac{x_n}{p}\right)\leqslant\frac{x_{n,k}}{p_k}\leqslant\max\left(\frac{x_n}{p}\right)$$

or

$$\operatorname{osc}\left(\frac{x_n}{p}\right) \leqslant \tilde{C}\,(\rho - \underline{\lambda}_n).$$

This is the second inequality in (17). ■

THEOREM 1. *Consider the procedure* (13)–(15). *For* $n = 0, 1, 2, \ldots,$

$$\underline{\lambda}_n \leqslant \underline{\lambda}_{n+1} \leqslant \rho \leqslant \bar{\lambda}_{n+1} \leqslant \bar{\lambda}_n, \tag{20}$$

$$\lim \underline{\lambda}_n = \lim \bar{\lambda}_n = \rho, \tag{21}$$

$$\lim x_n = p. \tag{22}$$

If $x_n \neq p$ *for all* n, *then the inequalities of* (20) *are strict.*

Proof. If we multiply the relation

$$\underline{\lambda}_n x_n \leqslant A x_n \leqslant \bar{\lambda}_n x_n$$

by B_n and use $B_n A = A B_n$, we get

$$\underline{\lambda}_n \tilde{x}_{n+1} \leqslant A\tilde{x}_{n+1} \leqslant \bar{\lambda}_n \tilde{x}_{n+1}$$

and hence $\underline{\lambda}_n \leqslant \underline{\lambda}_{n+1}$, $\bar{\lambda}_{n+1} \leqslant \bar{\lambda}_n$. If $x_n \neq p$, then $\bar{\lambda}_n x_n - A x_n \neq 0$; hence $\bar{\lambda}_n \tilde{x}_{n+1} - A\tilde{x}_{n+1} > 0$ and $\bar{\lambda}_{n+1} < \bar{\lambda}_n$. Similarly $\underline{\lambda}_n < \underline{\lambda}_{n+1}$. The remaining inequalities $\underline{\lambda}_n \leqslant \rho \leqslant \bar{\lambda}_n$ follow from the quotient theorem (e.g., [4], II, 5.5.2). From (16), (17) we infer the strict inequalities for $x_n \neq p$. Now

$$\bar{\lambda}_{n+1} - \underline{\lambda}_{n+1} = \operatorname{osc}\left(\frac{A x_{n+1}}{x_{n+1}}\right) = \operatorname{osc}\left(\frac{B_n A x_n}{B_n x_n}\right) \leqslant N(B_n)\operatorname{osc}\left(\frac{A x_n}{x_n}\right)$$

$$= N(B_n)\left(\bar{\lambda}_n - \underline{\lambda}_n\right). \tag{23}$$

From (12) and (20), we infer (21). From (17), we get

$$\lim_{n\to\infty} \operatorname{osc}\left(\frac{x_n}{p}\right)=0$$

or

$$\lim_{n\to\infty} \frac{x_{n,i}}{p_i}=1, \qquad i=1,\dots,N.$$

Hence, we get (22). ∎

In the case of $A>0$, taking $B_n=A$, Theorem 1 gives the convergence of the usual power method. If only $A^m>0$ for a suitable integer m, i.e., if A is primitive, the proof given above can be easily adapted to yield the same result. In fact,

$$\bar{\lambda}_{n+m}-\underline{\lambda}_{n+m}=\operatorname{osc}\left(\frac{A^mAx_n}{A^mx_n}\right)\leqslant N(A^m)(\bar{\lambda}_n-\underline{\lambda}_n).$$

More interesting is the case

$$B_n=(\bar{\lambda}_nI-A)^{-1}, \qquad n=0,1,\dots . \tag{24}$$

If we start with an x_0 such that $Ax_0\neq\rho x_0$, then $x_n\neq p$, $\bar{\lambda}_n>\rho$, $B_n>0$ for all n, as can be proved by induction.

Hence, Theorem 1 can be applied and gives the convergence of the inverse iteration procedure considered by Noda [5]. Additionally, we have the following statement about the rate of convergence:

THEOREM 2. *In the iteration procedure* (13)–(15) *with*

$$B_n=(\bar{\lambda}_nI-A)^{-1},$$

the sequences $\{\underline{\lambda}_n\}$, $\{\bar{\lambda}_n\}$ *converge quadratically to* ρ *and the* $\{x_n\}$ *quadratically to the eigenvector* p.

Proof. From (23) and (9) we get

$$\bar{\lambda}_{n+1}-\underline{\lambda}_{n+1}\leqslant M(\bar{\lambda}_n-\rho)(\bar{\lambda}_n-\underline{\lambda}_n)\leqslant M(\bar{\lambda}_n-\underline{\lambda}_n)^2,$$

i.e., $\{\bar{\lambda}_n - \underline{\lambda}_n\}$ converges quadratically to zero. It is now obvious from (16) and (17) that the sequences

$$\{\bar{\lambda}_n - \rho\}, \qquad \{\rho - \underline{\lambda}_n\}, \qquad \operatorname{osc}\left(\frac{x_n}{p}\right)$$

also converge quadratically. ■

Note added in proof.

The author learned that Theorem 2 has also been proved in Stephen M. Robinson-Karl Nickel: Computation of the Perron root and vector of a nonnegative matrix, MRC Technical Summary Report #1100, September 1970, Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706.

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