On Matrices Leaving Invariant a Nontrivial Convex Set

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ABSTRACT

A real square matrix A leaves a nontrivial convex set invariant if there exists a convex set C, which is not a linear subspace, such that $A(C)\subset C$. It is shown that this is equivalent to the statement that A has an eigenvalue λ with $\lambda\geqslant 0$ or $|\lambda|\leqslant 1$.

INTRODUCTION

There are some results in matrix theory relating geometrical and spectral properties of a matrix. We mention only the following:

THEOREM α. *A real square matrix leaves a bounded symmetric convex set with nonvoid interior invariant if and only if all eigenvalues have moduli less or equal to* 1 *and the eigenvalues with modulus* 1 *have index* 1.

THEOREM β. *A real square matrix leaves a solid pointed closed (convex) cone invariant if and only if the spectral radius is an eigenvalue and has maximal index among all eigenvalues with the same modulus.*

Theorem α is only a reformulation of the result in [3, p. 47] using the well-known relation between vector norms and convex bodies in R^n, while Theorem β can be found in [5], [1, Theorems (3.2), (3.5)], and for the case of compact operators also in [2, Theorem 3.1]. Several other results of similar type are mentioned or proved in [2], [5], [7].

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What can be said in general about the spectrum of a matrix A if A leaves some convex set invariant? To exclude trivialities we restrict ourselves to *nontrivial* convex sets. Here we call a convex set *nontrivial* if it is nonvoid and not a linear subspace. The purpose of this note is to prove that A leaves a nontrivial convex set invariant if and only if A has an eigenvalue λ such that either $\lambda \geqslant 0$ or $|\lambda| \leqslant 1$ (Theorem 1).

This will be shown using two results which characterize the existence of a nonnegative real eigenvalue, and of an eigenvalue with modulus less than or equal to 1, of a matrix A by the existence of certain sets invariant under A (Theorems 2 and 3).

Before stating the results explicitly, we collect the necessary notations and concepts, which can be found mostly in [4, §8, pp. 60–65], or in [6].

We call a nonvoid set $K \subset R^n$ a *cone* if for $\alpha, \beta \geqslant 0$, $\alpha K + \beta K \subset K$. The linear subspace $L = K \cap (-K)$ is called its *lineality space*. According to the definition given above, the cone K is nontrivial if and only if $K \neq L$. The *recession cone* K_1 of a convex set C is the set of all y such that $x + \lambda y \in C$ for all $x \in C$ and all $\lambda \geqslant 0$. K_1 is a convex cone which is closed for C closed. If C is closed and $A(C) \subset C$, then also $A(K_1) \subset K_1$. This can be seen easily by using Theorem 8.3 in [4, p. 63].

A set of the form $\{z: y^T z \geqslant 0\}$, where $y \neq 0$, is called a *half space*. We finally remark that if C is a nontrivial convex set, then its closure is also nontrivial. This can be shown by using a separation theorem for convex sets (e.g., Theorem 11.3, p. 97 in [4]).

RESULTS

We now state the results.

THEOREM 1. *For a real square matrix A the following are equivalent:*

(a) *There exists a nontrivial convex set C such that* $A(C) \subset C$.
(b) *A has an eigenvalue* λ *such that* $\lambda \geqslant 0$ *or* $|\lambda| \leqslant 1$.

THEOREM 2. *For a real square matrix A the following are equivalent:*

(a) *A leaves a half space invariant.*
(b) *A leaves a nontrivial cone invariant.*
(c) *A has a nonnegative real eigenvalue.*

THEOREM 3. *For a real square matrix A the following are equivalent:*

(a) *A leaves a nonvoid bounded set S,* $S \neq \{0\}$, *invariant.*

(b) *A leaves a nontrivial bounded convex set C invariant.*

(c) *A leaves a closed nontrivial convex set with trivial recession cone invariant.*

(d) *A has an eigenvalue* λ *such that* $|\lambda|\leq 1$.

PROOFS

We start with the proofs of Theorems 2 and 3.

Proof of Theorem 2. We shall show (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(a).

(a)$\Rightarrow$(b): Trivial, since a half space is a nontrivial cone.

(b)$\Rightarrow$(c): Let K be a nontrivial cone and $A(K)\subset K$. By restricting A to the invariant subspace $L_1=K-K$, if necessary, we may assume that K has interior points. Hence the dual cone $K^*=\{l\in L_1^*,\ l(x)\geq 0 \text{ for all } x\in K\}$ is a closed, pointed, nonvoid cone (see e.g., [1, (2.8)]). A^T, the transpose of A considered as mapping $L_1\to L_1$, leaves K^* invariant. Again, we may assume that K^* has interior points, and by Theorem β above, A^T has a nonnegative eigenvalue μ. This shows that (c) holds.

(c)$\Rightarrow$(a) : If $\mu\geq 0$ is an eigenvalue of A, then there exists a left-hand eigenvector $y^T\neq 0: y^TA=\mu y^T$. As $y^Tx\geq 0$ implies $y^TAx=\mu y^Tx\geq 0$, A leaves the half space $\{z: y^Tz\geq 0\}$ invariant. ∎

Proof of Theorem 3. We shall show (a)$\Leftrightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(b).

(a)$\Rightarrow$(b): Denote by C the convex hull of S. Then C is a nontrivial bounded convex set, and from $A(S)\subset S$ we have $A(C)\subset C$. (b)$\Rightarrow$(a) is trivial.

(b)$\Rightarrow$(c): We may assume that C is closed. As C is bounded, its recession cone K is $\{0\}$ (see [4, Theorem 8.4, p. 64]), hence trivial. This proves (c).

(c)$\Rightarrow$(d): Let C be closed and convex, $A(C)\subset C$, and its recession cone K trivial. If $K=\{0\}$, then C is bounded and so is the convex hull H of $C\cup(-C)$. As $A(H)\subset H$, Theorem α gives the existence of an eigenvalue λ with $|\lambda|\leq 1$. So suppose $K\neq\{0\}$. As $K\neq R^n$ (otherwise $C=R^n$), the decomposition $R^n=K\oplus K^\perp$ is nontrivial. Let $C_1=K^\perp\cap C$. Then $C_1\neq\{0\}$ and C_1 is bounded. Let P denote the orthogonal projection of R^n onto $K^\perp$. Then from the well-known relation (see [4, p. 65]) $C=L\oplus(L^\perp\cap C)$, where L is the lineality space of K, and from the fact that $L=K$, we have $C=K\oplus C_1$; hence $P(C)=C_1$ and $PA(C_1)\subset PA(C)\subset P(C)=C_1$.

PA maps the bounded (closed) convex set C_1 into itself, and hence [see the first part of (c)$\Rightarrow$(d)] PA restricted to $K^\perp$ has an eigenvalue λ, with $|\lambda|\leq 1$. If we choose a basis $b_1,\ldots,b_n$ of R^n, where $b_1,\ldots,b_r\in K$ and $b_{r+1},\ldots,b_n\in K^\perp$, then in the block decomposition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \}_r \\ \}_{n-r} \end{matrix}$$

$A_{21} = 0$ because $A(K) \subset K$ and $A_{22} = PA/K^{\perp}$. This shows that the eigenvalues of $PA/K^{\perp}$ are also eigenvalues of A.

(d)⇒(b): Let $x \neq 0$ be an eigenvector corresponding to λ, $Ax = \lambda x$. If λ is real, then x can be chosen real and A leaves the bounded convex set $C = \{\alpha x: -1 \leqslant \alpha \leqslant 1\}$ invariant. If λ is complex, $\lambda \neq \bar{\lambda}$, then $x = y + iz$ with y, z real and linearly independent. A leaves the subspace L spanned by y, z invariant and has there two different eigenvalues λ, $\bar{\lambda}$, $|\lambda| = |\bar{\lambda}| \leqslant 1$. Hence by Theorem α, applied to the restriction of A to L, we see that A leaves a convex bounded set invariant. ■

Remark. If (d) holds, it is also possible, by using the "real" Jordan canonical form, to construct a convex set C with trivial recession cone and interior points, such that $A(C) \subset C$.

Proof of Theorem 1. (a)⇒(b): We may assume that C is closed. Then A leaves the recession cone K_1 of C invariant. If K_1 is nontrivial, then we see from Theorem 2 that A has a nonnegative real eigenvalue. If K_1 is trivial, then we see from Theorem 3 that A has an eigenvalue λ, with $|\lambda| \leqslant 1$.

(b)⇒(a): This is an immediate consequence of Theorems 2 and 3. ■

I thank Hans Schneider for his valuable comments. The proof of Theorem 3, (c)⇒(d), presented here is essentially his and simplifies my original argument considerably.

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This note is based partly on results obtained in 1968–69 during a visit of the National Research Institute for Mathematical Sciences (NRIMS) of the Council for Scientific and Industrial Research (CSIR) in Pretoria (South Africa). Discussing [5] *revived the interest in these results.*

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