Block Scaling with Optimal Euclidean Condition

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ABSTRACT

Let $\mathscr{M}$ denote the set of all complex $n \times n$ matrices whose columns span certain given linear subspaces. The minimal Euclidean condition number of matrices in $\mathscr{M}$ is given in terms of the canonical angles between the linear subspaces, and optimal matrices in $\mathscr{M}$ are described. The result is also stated in terms of norms of certain projections.

1. INTRODUCTION

This note is a contribution to the following problem: For given linear subspaces $\mathscr{X}_i$, $i = 1,\dots,k$, of $\mathbf{C}^n$, $\dim \mathscr{X}_i = n_i$, such that

$$\mathbf{C}^n = \mathscr{X}_1 + \mathscr{X}_2 + \cdots + \mathscr{X}_k, \tag{1}$$

consider the set of matrices

$$\mathscr{M} = \{A \in \mathbf{C}^{n,n} : A = (A_1,\dots,A_k), A_i \in C^{n,n_i}, \operatorname{Im}(A_i) = \mathscr{X}_i\}. \tag{2}$$

Find $X \in \mathscr{M}$ such that

$$\kappa_\nu(X) = \min\{\kappa_\nu(A) : A \in \mathscr{M}\}. \tag{3}$$

Here ν is a given matrix norm and $\kappa_\nu(A) = \nu(A)\nu(A^{-1})$ is the condition of A (with respect to ν). We shall consider in particular the Euclidean norm

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$\|X\|_F = (\Sigma_{i,j=1}^n |x_{ij}|^2)^{1/2}$ and

$$\kappa_F(X) = \|X\|_F \|X^{-1}\|_F. \tag{4}$$

The problem formulated above includes two important special cases:

(a) Let A be a given matrix, and $\mathscr{X}_i$ invariant subspaces of A of dimensions n_i. Then $X \in \mathscr{M}$ iff $X^{-1}AX = \mathrm{diag}(B_i)$, where $B_i \in C^{n_i, n_i}$. Hence the problem is to find a block-diagonalizing X of minimal ν-condition.

(b) Let $A = (A_1, \dots, A_k)$ be invertible, $A_i \in C^{n, n_i}$. Then $\mathscr{X}_i = \mathrm{Im}(A_i)$ satisfy (1). Here $X \in \mathscr{M}$ iff there exists $D = \mathrm{diag}(D_i)$, $D_i \in C^{n_i, n_i}$ such that $X = AD$. So this amounts to the problem of optimal block scaling.

Both problems have been extensively treated in the literature, especially for $k = n$ (see e.g. [2], [1], and the references given there) and special norms. In the case of $n_i > 1$ only few results are known [2].

The formulation used here is that of Demmel in [2]. He gives lower bounds for the minimal condition with respect to the spectral norm. This is done by comparing it with $\kappa_2(\mathrm{S}_{\mathrm{ortho}})$, where $\mathrm{S}_{\mathrm{ortho}} = (S_1, \dots, S_k)$ is any matrix in $\mathscr{M}$ such that the columns of S_i are orthonormal ($i = 1, \dots, k$). He also shows that for $k = 2$ any $\mathrm{S}_{\mathrm{ortho}}$ has minimal condition with respect to the spectral norm. (A result equivalent to this statement is given in [3] by Eisenstat, Lewis, and Schultz.)

We give here a complete solution of the problem for the case of the Euclidean matrix norm. For the special case $n_i = 1$, i.e. $k = n$, this result is due to Smith [6] and Wilkinson [8]. The solution is given in terms of the canonical angles between certain subspaces.

2. RESULTS

Let $\mathscr{M}_1, \mathscr{M}_2$ be two linear subspaces in $\mathbf{C}^n$, $\dim \mathscr{M}_i = m_i$. It is well known that there exist orthonormal bases $x_1, \dots, x_{m_1}$ of $\mathscr{M}_1$ and $y_1, \dots, y_{m_2}$ of $\mathscr{M}_2$ and numbers θ_i, $i = 1, \dots, \min(m_1, m_2)$, $0 \leqslant \theta_i \leqslant \pi/2$, such that

$$x_i^H y_i = \cos\theta_i \cdot \delta_{ij}, \qquad i = 1, \dots, m_1, \quad j = 1, \dots, m_2. \tag{5}$$

Here the θ_i, the so-called canonical angles, depend only on $\mathscr{M}_1$ and $\mathscr{M}_2$. In fact the numbers $\cos\theta_i$ are just the singular values of $M_1^H M_2$ where the M_i are $n \times m_i$ with orthonormal columns spanning $\mathscr{M}_i$ [7].

With $\mathscr{X}_1,\ldots,\mathscr{X}_k$ satisfying (1) we associate the set of subspaces

$$\mathscr{Y}_j = \bigcap_{\substack{i=1 \\ i \neq j}}^{k} \mathscr{X}_i^{\perp}, \qquad \dim \mathscr{Y}_j = n_j, \tag{6}$$

and the canonical angles θ_r^j, $r = 1,\ldots,n_j$, between $\mathscr{X}_j$ and $\mathscr{Y}_j$ $(j = 1,\ldots,k)$. From (1) we get $0 \leqslant \theta_r^j < \pi/2$.

THEOREM. *Let $\mathscr{X}_i$ $(i = 1,\ldots,k)$ be linear subspaces of C^n with dimensions n_i satisfying* (1), *$\mathscr{Y}_i$ defined by* (6) *and $\mathscr{M}$ by* (2). *Then*

$$\min\{\kappa_F(A) : A \in \mathscr{M}\} = \sum_{j=1}^{k} \sum_{r=1}^{n_j} \frac{1}{\cos\theta_r^j}. \tag{7}$$

Proof. It proceeds partly along the lines of Smith's proof in [6]. As outlined above there exist X_j, Y_j such that the columns of X_j, Y_j are orthonormal bases of $\mathscr{X}_j, \mathscr{Y}_j$ and satisfy

$$\begin{aligned} Y_j^H X_j &= \operatorname{diag}\left(\cos\theta_1^j,\ldots,\cos\theta_{n_j}^j\right) \equiv \Sigma_j, \\ Y_j^H X_i &= 0 \qquad \text{if} \quad i \neq j. \end{aligned} \tag{8}$$

Define the $n \times n$ matrices $X = (X_1,\ldots,X_k)$, $Y = (Y_1,\ldots,Y_k)$. By (8) we have

$$Y^H X = \operatorname{diag}(\Sigma_1,\ldots,\Sigma_k) \equiv \Sigma, \tag{9}$$

where Σ is diagonal. Hence

$$X^{-1} = \Sigma^{-1} Y^H. \tag{10}$$

Now $A \in \mathscr{M}$ has the form

$$A = X \operatorname{diag}(D_1,\ldots,D_k) = (X_1 D_1,\ldots,X_k D_k) = XD, \tag{11}$$

where D_j are $n_j \times n_j$ nonsingular. Hence by (10)

$$A^{-1} = D^{-1} X^{-1} = D^{-1} \Sigma^{-1} Y^H. \tag{12}$$

Introducing the singular value decomposition of D_j

$$D_j = U_j \Lambda_j V_j, \qquad \Lambda_j = \operatorname{diag}\left(\lambda_1^j, \dots, \lambda_{n_j}^j\right), \tag{13}$$

we have at once by (11), (12)

$$\|A\|_F^2 = \sum_{j=1}^{k} \|X_j D_j\|_F^2 = \sum_{j=1}^{k} \|D_j\|_F^2 = \sum_{j=1}^{k} \|\Lambda_j\|_F^2, \tag{14}$$

$$\|A^{-1}\|_F^2 = \sum_{j=1}^{k} \|D_j^{-1} \Sigma_j^{-1} Y_j^H\|_F^2 = \sum_{j=1}^{k} \|\Lambda_j^{-1} U_j^H \Sigma_j^{-1}\|_F^2. \tag{15}$$

For any $\Lambda = \operatorname{diag}(\lambda_i)$, $M = \operatorname{diag}(\mu_i)$, and unitary U, Birkhoff's theorem (e.g. [4, p. 97]) implies

$$\|\Lambda U M\|_F^2 = \sum_{i,k} |u_{ik}|^2 |\lambda_i \mu_k|^2 \geqslant \sum_i |\lambda_i|^2 |\mu_{\pi(i)}|^2 \tag{16}$$

for a suitable permutation $\pi(i)$. Applying (16) to $\Lambda_j^{-1} U_j^H \Sigma_j^{-1}$ in (15) and using Cauchy's inequality, we get

$$[\kappa_F(A)]^2 = \|A\|_F^2 \|A^{-1}\|_F^2 \geqslant \sum_{j=1}^{k} \left(\sum_{r=1}^{n_j} |\lambda_r^j|^2 \right) \sum_{j=1}^{k} \left(\sum_{r=1}^{n_j} |\lambda_r^j|^{-2} |\cos \theta_{\pi_j(r)}^j|^{-2} \right)$$

$$\geqslant \left(\sum_{j=1}^{k} \sum_{r=1}^{n_j} |\cos \theta_r^j|^{-1} \right)^2 \tag{17}$$

and equality is attained for $\lambda_r^j = (\cos \theta_r^j)^{-1/2}$, $U_j = I_{n_j}$, and any V_j. Hence (7) is proved. ∎

Remark 1. We have also proved that any A of the form $A = X\Sigma^{-1/2} \operatorname{diag}(V_j)$, where V_j is an arbitrary $n_j \times n_j$ unitary matrix, has minimal condition with respect to the Euclidean matrix norm.

Remark 2. We can formulate the result of the theorem also in terms of norms of projections. Consider the decomposition $x = x_1 + x_2 = P_i x + x_2$

where $x_1 \in \mathscr{X}_i$, $x_2 \in \text{span}(\cup \mathscr{X}_j,\ j \neq i) = \mathscr{Y}_i^{\perp}$. Then with X_i, Y_i as defined above, we get $P_i = X_i(Y_i^H X_i)^{-1} Y_i^H$. Hence the numbers $(\cos\theta_r^i)^{-1}$ are just the singular values of P_i. It is well known that for a matrix $B \in C_{m,m}$ with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m$, $\sum_{i=1}^{m}\sigma_i$ is a unitarily invariant norm, which is just the dual norm $\| \ \|_2^D$ of the spectral norm $\|B\|_2 = \sigma_1$ [5]. Therefore (7) can be written as

$$\min\{\kappa_F(A) : A \in \mathscr{M}\} = \sum_{i=1}^{k} \|P_i\|_2^D.$$

Note added in proof: The author has learnt that the Theorem has been proved independently by Dr. Paul Van Dooren (personal communication). Dr. Van Dooren has also given a description of all minimizing matrices, see Remark 1.

REFERENCES

1 F. L. Bauer, Optimally scaled matrices, *Numer. Math.* 5:73–87 (1963).
2 J. Demmel, The condition number of equivalence transformations that block diagonalize matrix pencils, in *Matrix Pencils* (B. Kagström and A. Ruhe, Eds.), Lecture Notes in Mathematics 973, Springer, 1982, pp. 2–16.
3 S. C. Eisenstat, J. W. Lewis, and M. H. Schultz, Optimal block diagonal scaling of block 2-cyclic matrices, *Linear Algebra Appl.* 44:181–186 (1982).
4 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, 1964.
5 R. Schatten, *A Theory of Cross-Spaces*, Princeton U.P., 1950.
6 R. A. Smith, The condition numbers of the matrix eigenvalue problem, *Numer. Math.* 10:232–240 (1967).
7 G. W. Stewart, Error and perturbation bounds for subspaces associated with certain eigenvalue problems, *SIAM Rev.* 15:727–764 (1973).
8 J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford U.P., 1965.

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