An Optimal Bound for the Spectral Variation of Two Matrices

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Dedicated to Helmut W. Wielandt on the occasion of his 75th birthday.

ABSTRACT

We establish a bound for the spectral variation of two complex $n \times n$ matrices A, B in terms of $\|A\|$, $\|B\|$, and $\|A - B\|$. Here $\| \ \|$ denotes the spectral norm. It is always better than a bound previously given by Bhatia and Friedland, and it is optimal. We describe the set of pairs A, B for which the bound is attained.

1. INTRODUCTION AND NOTATION

Let $C^{n,n}$ denote the set of all complex $n \times n$ matrices. For the description of the distance between the spectra $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$ of A and B, respectively, where $A, B \in C^{n,n}$, we use the spectral variation

$$s_A(B) = \max_j \min_i |\lambda_i - \mu_j| \tag{1}$$

and the eigenvalue variation

$$v(A, B) = \min_\pi \max_i |\lambda_i - \mu_{\pi(i)}| \tag{2}$$

where π runs through all permutations of $\{1, \ldots, n\}$. We denote by $\|x\|$ for $x \in C^n$ the Euclidean vector norm, and by $\|A\| = \sup \|Ax\|/\|x\|$ the spectral norm. I is the unit matrix, e_j the jth unit vector.

Bhatia and Friedland [1] have shown that

$$s_A(B) \leqslant n^{1/n}(2M)^{1-1/n}\|A - B\|^{1/n}, \qquad M = \text{Max}(\|A\|, \|B\|). \qquad (3)$$

See [2] and [3] for different proofs. In Theorem 1 we show that the factor $n^{1/n}$ can be dropped and $2M$ can be replaced by $\|A\| + \|B\|$. Hence apart from the trivial case $A = B$, the new bound is always better. In Theorem 2 we describe the exact set of matrix pairs A, B for which equality is attained. This shows the optimality of the bound of Theorem 1.

We conclude this note with some remarks concerning other norms and bounds of the eigenvalue variation.

2. RESULTS

THEOREM 1.   For $A, B \in C^{n,n}$

$$s_A(B) \leqslant (\|A\| + \|B\|)^{1-1/n}\|A - B\|^{1/n}. \qquad (4)$$

Proof.   Let j be such that $s_A(B) = \min_i |\lambda_i - \mu_j|$. We may assume, by eventually applying a unitary similarity transformation, that $Be_1 = \mu_j e_1$. Then

$$s_A(B)^n = \min_i |\lambda_i - \mu_j|^n \leqslant \prod_{i=1}^{n} |\lambda_i - \mu_j| = \left| \det(A - \mu_j I) \right| \qquad (5)$$

$$\leqslant \left\|(A - \mu_j I)e_1\right\| \cdots \left\|(A - \mu_j I)e_n\right\| \qquad \text{(Hadamard's inequality)}$$

$$= \left\|(A - B)e_1\right\|\left\|(A - \mu_j I)e_2\right\| \cdots \left\|(A - \mu_j I)e_n\right\| \qquad (6)$$

$$\leqslant \|A - B\|(\|A\| + \|B\|)^{n-1}, \qquad (7)$$

as $\|(A - \mu_j I)e_k\| \leqslant \|Ae_k\| + |\mu_j| \leqslant \|A\| + \|B\|.$ ∎

We describe now the cases in which equality holds in (4).

THEOREM 2.   For $A, B \in C^{n,n}$, $A \neq B$, $B \neq 0$ the following are equivalent:

(i) $s_A(B) = (\|A\| + \|B\|)^{1-1/n}\|A - B\|^{1/n}$,

(ii) $\exists \varepsilon \in C$, $|\varepsilon| = 1$, such that $A = \varepsilon\|A\|I$, and B has eigenvalue $-\varepsilon\|B\|$.

Proof. Assume that A, B satisfy (ii). Then obviously $s_A(B) = \|A\| + \|B\|$. As $\epsilon(\|A\| + \|B\|)$ is an eigenvalue of $A - B$, we have $\|A - B\| = \|A\| + \|B\|$. Hence (i) holds.

Assume now that (i) is satisfied. Then equality holds in (5),(6),(7). From (7), $\|(A - \mu_j I)e_k\| = \|A\| + \|B\|$, and we have $\mu_j = -\epsilon\|B\|$ for some $\epsilon \in C$, $|\epsilon| = 1$, and $Ae_k = \epsilon\|A\|e_k$ $(k = 2,\ldots,n)$. Equality in Hadamard's inequality holds only for orthogonal columns; hence from (6), $(A - \mu_j I)e_1 = \alpha e_1$; hence $Ae_1 = \lambda_s e_1$ for some s. But as by (5) all $\lambda_i - \mu_j$ have the same modulus, we have $\lambda_s = \epsilon\|A\|$ as well.   ∎

3. CONCLUDING REMARKS

1. In [3] Friedland established (3) for any operator norm $\| \ \|$ and hence for any submultiplicative matrix norm. It is easy to see that (4) holds for all norms $\| \ \|_{T,p}$ where T nonsingular, $1 \leqslant p \leqslant \infty$, and

$$\|A\|_{T,p} = \max_{x \neq 0} \frac{\|TAx\|_p}{\|Tx\|_p},$$

$\| \ \|_p$ being the l_p norm. I conjecture that this is true for all operator norms. Observe that the implication (ii) → (i) of Theorem 2 holds also for operator norms.

2. In Theorem 2 we have excluded the case $B = 0$. It is obvious that in this case (i) holds iff $\max_i |\lambda_i| = \|A\|$.

3. Using the inequality [2, Theorem 1]

$$v(A, B) \leqslant a_n \operatorname{Max}(h_A(B), h_B(A)),   \tag{8}$$

where

$$h_A(B) = \max_{0 \leqslant t \leqslant 1} s_A(tA + (1-t)B) \quad \text{and} \quad a_n = \begin{cases} n, & n \text{ odd}, \\ n-1, & n \text{ even}, \end{cases}$$

we get from (4)

$$v(A, B) \leqslant a_n (2M)^{1 - 1/n} \|A - B\|^{1/n}, \qquad M = \operatorname{Max}(\|A\|, \|B\|).   \tag{9}$$

The reasoning in the proof of (i) → (ii) of Theorem 2 can be extended to show

that

$$s_A(B) \geq (1 - \varepsilon)(2M)^{1 - 1/n}\|A - B\|^{1/n}$$

implies

$$v(A, B) \leq (2M)^{1 - 1/n}\|A - B\|^{1/n}\left[1 + O(\varepsilon)\right].$$

This together with (8) and (9) shows that in (9) a_n can be replaced by a somewhat smaller number. This is in accordance with Friedland's conjecture [3] that $v(A, B) \leq C(2M)^{1-1/n}\|A - B\|^{1/n}$ with some C independent of n.

REFERENCES

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3 S. Friedland, Variation of tensor powers and spectra, *Linear and Multilinear Algebra* 12:81–98 (1982).