On Measures of Nonnormality of Matrices

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ABSTRACT

This paper discusses several measures of nonnormality of matrices, i.e., functions $\nu : C^{n,n} \rightarrow \mathbf{R}_+$ where $\nu(A) = 0$ iff A is normal. Besides measures already in the literature, we introduce new ones and give comparisons between them at length. Some of these comparisons, e.g. (C14), (C15), and (C17) manifest well-known phenomena of ill-conditioned eigenproblems.

1. INTRODUCTION

The class of normal matrices has received some attention from numerical analysts. In particular, in connection with certain eigenvalue algorithms normal and nonnormal matrices show quite different behavior. Related to this fact is the difference in the sensitivity of the eigenvalues and eigenvectors under perturbations of the entries of the matrix [3, 4, 9, 10, 13]. For analyzing these difficulties several measures of nonnormality have appeared in the literature. We give here an overview of the measures used, introduce some new ones, and give comparisons between them. These are listed in

Theorem 2. We have done it in such a way as to give there only the best (to our knowledge) available bounds; however, we mention some weaker but earlier results during the proof.

Let $C^{n,n}$ denote the set of all $n \times n$ complex matrices, and $A \in C^{n,n}$ a fixed matrix with $n > 1$. We associate with A the following numbers and matrices:

(1) its eigenvalues $\lambda_j = \gamma_j + i\delta_j$ and singular values σ_i ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$;

(2) the matrices $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$;

(3) its polar factors H_1, H_2, i.e., the uniquely determined positive semi-definite square roots of AA^* and A^*A;

(4) its Hermitian part $F = (A + A^*)/2$ with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$;

(5) its skew Hermitian part $G = (A - A^*)/2i$ with eigenvalues $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

Here A^* is the conjugate transpose of A. A matrix A is *normal* if $AA^* = A^*A$. The sets of all normal, unitary, and diagonal matrices in $C^{n,n}$ are denoted by $\mathcal{N}$, $\mathcal{U}$, and $\mathcal{D}$ respectively. $\| \ \|_2$ and $\| \ \|_F$ are the spectral and Frobenius matrix norms. For nonsingular X, $\kappa_i(X) = \|X\|_i \|X^{-1}\|_i$, $i = 2, F$, is the condition number of X. A function ν of $C^{n,n}$ into the nonnegative real numbers is a *measure of nonnormality* if the following holds: $\nu(A) = 0$ iff $A \in \mathcal{N}$.

We make extensive use of the following relations:

$$U \in \mathcal{U}, \quad i = 2, F \quad \Rightarrow \quad \|AU\|_i = \|UA\|_i = \|A\|_i,$$

$$B, C \in C^{n,n} \quad \Rightarrow \quad \|BC\|_F \leq \|B\|_2 \|C\|_F, \quad \|BC\|_F \leq \|B\|_F \|C\|_2.$$

2. CHARACTERIZATIONS OF NORMAL MATRICES

There are quite a few characterizations for A being normal. An incomplete list is given in

THEOREM 1. *For $A \in C^{n,n}$ the following are equivalent:*

(i) *A is normal, i.e., $AA^* = A^*A$;*

(ii) *$\|Ax\| = \|A^*x\|$ for all $x \in C^n$* ($\| \ \| = $ *Euclidean vector norm*);

(iii) *$\exists V \in \mathcal{U}$ s.t. V^*AV is diagonal;*

(iv) $\sum_{i=1}^{n}|\lambda_i|^2 = \|A\|_F^2$;

(v) $|\lambda_i| = \sigma_i$, $i = 1, \ldots, n$;

(vi) $\gamma_i = \alpha_{p(i)}$, $i = 1, \ldots, n$, *for a suitable permutation* p;

(vii) $\delta_i = \beta_{q(i)}$, $i = 1, \ldots, n$, *for a suitable permutation* q;

· (viii) $H_1 = H_2$;

(ix) $F = (A + A^*)/2$ *and* $G = (A - A^*)/2i$ *commute*;

· (x) $A = X\Lambda X^{-1}$ *for some* X *such that* $\kappa_2(X) = 1$.

As these characterizations are either well known or easy consequences of Theorem 2, we refrain from giving a proof and refer to the literature [1, 8, 11, 14].

3. MEASURES OF NONNORMALITY

Theorem 1 motivates the introduction of several measures of nonnormality. The most natural measure seems to be

$$\mu_1(A) = \min\{\|A - N\|_F : N \in \mathcal{N}\}$$

and

$$\tilde{\mu}_1(A) = \min\{\|A - N\|_2 : N \in \mathcal{N}\},$$

the distance of A from the set of normal matrices. Another quite natural measure is given by considering the matrix equation characterizing normality:

$$\mu_2(A) = \|A^*A - AA^*\|_F^{1/2},$$

$$\tilde{\mu}_2(A) = \|A^*A - AA^*\|_2^{1/2}.$$

Henrici defined in [4] for matrix norms ν the ν-departure from normality as follows:

$$\Delta_\nu(A) = \min\{\nu(M) : M \text{ strictly upper triangular},$$

$$\exists U \in \mathcal{U}, \tilde{\Lambda} \in \mathcal{D} \text{ s.t. } U^*AU = \tilde{\Lambda} + M\}.$$

We shall restrict ourselves to the spectral and Frobenius norm

$$\mu_3(A) = \Delta_F(A) = \left(\|A\|_F^2 - \sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2},$$

$$\tilde{\mu}_3(A) = \Delta_2(A).$$

Ruhe introduced in [10] the measure

$$\mu_4(A) = \max_i |\sigma_i - |\lambda_i||.$$

Besides this we consider

$$\mu_5(A) = \|H_1 - H_2\|_F;$$

$$\mu_6(A) = \min_{p,q} \left(\sum_{j=1}^{n} \left| \lambda_j - \left(\alpha_{p(j)} + i\beta_{q(j)} \right) \right|^2 \right)^{1/2},$$

$$\mu_7(A) = \min_{p} \left(\sum_{j=1}^{n} \left(\gamma_j - \alpha_{p(j)} \right)^2 \right)^{1/2},$$

$$\mu_8(A) = \min_{q} \left(\sum_{j=1}^{n} \left(\delta_j - \beta_{q(j)} \right)^2 \right)^{1/2}$$

(p, q permutations of $\{1, \ldots, n\}$);

$$\mu_9(A) = \min \left\{ \|U - V\|_F : U^*(A + A^*)U \in \mathscr{D}, \right.$$

$$\left. V^*(A - A^*)V \in \mathscr{D}, U, V \in \mathscr{U} \right\},$$

$$\tilde{\mu}_9(A) = \min \left\{ \|U - V\|_2 : U^*(A + A^*)U \in \mathscr{D}, \right.$$

$$\left. V^*(A - A^*)V \in \mathscr{D}, U, V \in \mathscr{U} \right\},$$

$$\mu_{10}(A) = \min \left\{ \|U - V\|_F : UAV^* \in \mathscr{D}, U, V \in \mathscr{U} \right\},$$

$$\tilde{\mu}_{10}(A) = \min \left\{ \|U - V\|_2 : UAV^* \in \mathscr{D}, U, V \in \mathscr{U} \right\};$$

and for A diagonalizable

$$\mu_{11}(A) = \min\{\kappa_F(X) - n : X \in C^{n,n},\ X^{-1}AX = \Lambda\},$$

$$\tilde{\mu}_{11}(A) = \min\{\kappa_2(X) - 1 : X \in C^{n,n},\ X^{-1}AX = \Lambda\}.$$

All these measures have the following invariance properties:

$$\mu(A) = \mu(\bar{A}) = \mu(A^T) = \mu(A^*) = \mu(U^*AU),$$

$$U \in \mathcal{U}, \qquad \mu \in \mathcal{M} = \{\mu_i : i = 1,\dots,11\} \cup \{\tilde{\mu}_i : i = 1,2,3,9,10,11\}$$

while the invariance with respect to shifts,

$$\mu(A) = \mu(A + \omega I), \qquad \omega \in C,$$

holds for all $\mu \in \mathcal{M} \times \{\mu_4, \mu_5, \mu_{10}, \tilde{\mu}_{10}\}$.

4. COMPARISONS BETWEEN MEASURES OF NONNORMALITY

The main results of this paper are comparisons between the abovementioned measures of nonnormality, which we summarize in

THEOREM 2. *Let $A \in C^{n,n}$. The following inequalities hold:*

(C0) $\tilde{\mu}_i \leqslant \mu_i \leqslant \sqrt{n}\,\tilde{\mu}_i,\ i = 1,2,3,9,10.$

(C1) $\mu_3 \leqslant \left(\dfrac{n^3 - n}{12}\right)^{1/4} \mu_2.$

(C2) $\mu_2^2 \leqslant \sqrt{2\left(\|A\|_F^2 + \|\Lambda\|_F^2\right)}\,\mu_3 \leqslant 2\|A\|_F\mu_3.$

(C3) $\mu_2^2 \leqslant 4\|A\|_2\mu_1.$

(C4) $\mu_2^2 \leqslant 4\|A\|_F\tilde{\mu}_1 + \sqrt{2n}\,\tilde{\mu}_1^2.$

(C5) $\mu_1 \leqslant \mu_3.$

(C6) $\mu_4 \leqslant \tilde{\mu}_3.$

(C7) $\mu_3^2 \leqslant 2\sqrt{n}\,\|A\|_F\mu_4 \leqslant 2n\|A\|_2\mu_4.$

(C8) $\|A^+\|_2^{-1}\mu_5 \leqslant \mu_2^2 \leqslant 2\|A\|_2\mu_5.$

(C9) $\mu_6^2 \leqslant \mu_3^2 \leqslant 2\|A\|_F\mu_6 - \mu_6^2.$

(C10) $\mu_7^2 \leqslant \tfrac{1}{2}\mu_3^2 \leqslant 2\|F\|_F\mu_7 - \mu_7^2.$

(C11) $\mu_8^2 \leqslant \frac{1}{2}\mu_3^2 \leqslant 2\|G\|_F\mu_8 - \mu_8^2$.

(C12) $\tilde{\mu}_2^2 \leqslant 8\|A\|_2^2\tilde{\mu}_9$, $\mu_2^2 \leqslant 8\|A\|_2^2\mu_9$.

(C13) $\tilde{\mu}_2^2 \leqslant 2\|A\|_2^2\tilde{\mu}_{10}$, $\mu_2^2 \leqslant 2\|A\|_2^2\mu_{10}$.

(C14) If $\mu_2^2 < 2\delta\tilde{\delta}$, where

$$\delta = \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j|, \quad \tilde{\delta} = \min_{\beta_i \neq \beta_j} |\beta_i - \beta_j|,$$

then

$$\mu_9 \leqslant \frac{\sqrt{2}\,\mu_2^2}{2\delta\tilde{\delta} - \mu_2^2}.$$

(C15) $\mu_{10} \leqslant \dfrac{\sqrt{2}}{\tau}\mu_5$, where $\tau = \min\{|\sigma_i - \sigma_j|: \sigma_i \neq \sigma_j\}$ and $\tau = 1$ if all σ_i are equal.

If, in addition, A is diagonalizable, then:

(C16) $\dfrac{\tilde{\mu}_{11}^2}{1 + \tilde{\mu}_{11}} \leqslant \mu_{11} \leqslant \dfrac{n}{2}\dfrac{\tilde{\mu}_{11}^2}{1 + \tilde{\mu}_{11}}$.

(C17) If all eigenvalues of A are simple and

$$\delta_j = \min\{|\lambda_i - \lambda_j|: i \neq j\}, \qquad j = 1,\ldots,n,$$

then

$$\mu_{11} \leqslant \sum_{j=1}^n \left[\left\{ 1 + \frac{\mu_3^2}{\delta_j^2(n-1)} \right\}^{(n-1)/2} - 1 \right].$$

(C18) $\mu_2^2 \leqslant 2\|A\|_2\|A\|_F\tilde{\mu}_{11}(2 + \tilde{\mu}_{11}) \leqslant 2\|A\|_F^2\tilde{\mu}_{11}(2 + \tilde{\mu}_{11})$.

(C19) $\mu_3^2 \leqslant \dfrac{\|A\|_F^2\tilde{\mu}_{11}(2 + \tilde{\mu}_{11})}{(1 + \tilde{\mu}_{11})^2}$.

(C20) $\mu_4 \leqslant \|A\|_2\tilde{\mu}_{11}$.

Here we have used the abbreviation μ_i for $\mu_i(A)$, $i = 1,\ldots,11$, and $\tilde{\mu}_i$ for $\tilde{\mu}_i(A)$, $i = 1,2,3,9,10,11$.

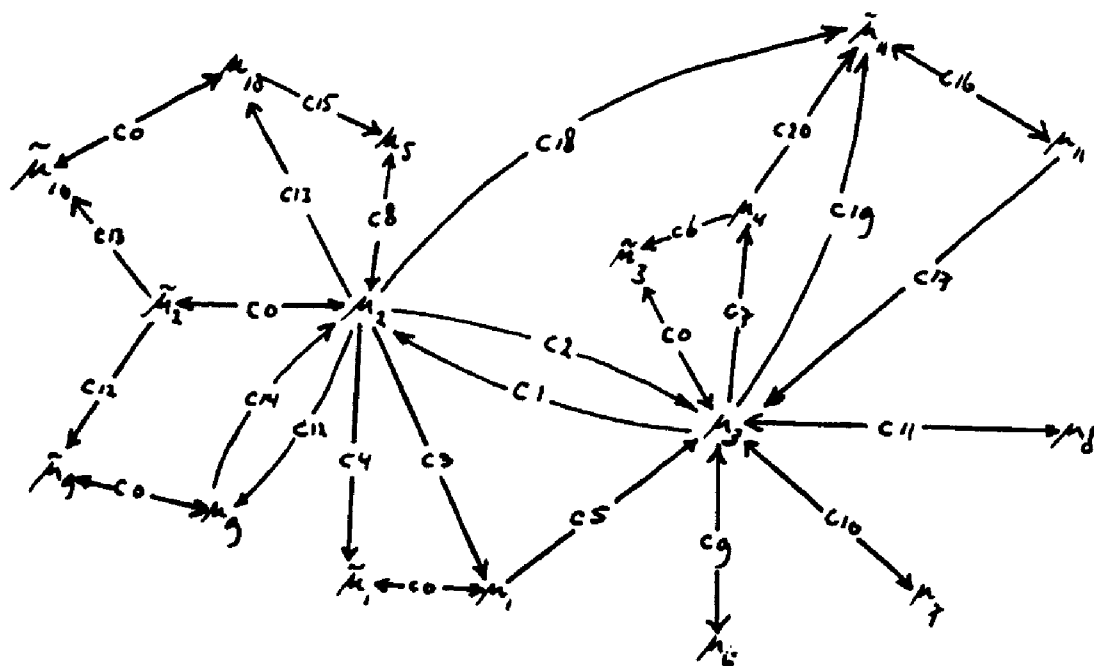In particular, the functions $\mu \in \mathcal{M}$ are measures of nonnormality.

FIG. 1.

REMARK. Theorem 2 can be interpreted by saying that for *certain* $\mu, \nu \in \mathcal{M}$ there is a function φ, depending on ν, μ and on A, $\varphi(0) = 0$, continuous, monotonic, s.t.

$$\nu(A) \leqslant \varphi(\mu(A)) \tag{4.1}$$

for all A. If we consider a directed graph with the elements of $\mathcal{M}$ as nodes and edges from ν to μ, if (4.1) holds, then this graph is strongly connected [this is the reason why the trivial inequalities (C0) are included]. Hence for *any* $\nu, \mu \in \mathcal{M}$ a relation (4.1) holds, as (4.1) is transitive. See Figure 1. In this sense all measures of nonnormality considered here are equivalent. The measures $\mu_1, \ldots, \mu_8, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3$ except for μ_5 are equivalent in a stricter sense, namely that in (4.1) φ depends only on ν, μ and norms of A and A^*, but not on the eigenvalues of A.

Proof of Theorem 2. (C0) is an easy consequence of the relation

$$\|B\|_2 \leqslant \|B\|_F \leqslant \sqrt{n}\,\|B\|_2 \qquad \text{for any} \quad B \in C^{n,n}.$$

(C1) is a result of Henrici [4]. The first inequality in (C2) is a rearrangement

of the inequality

$$\left(\sum_{i=1}^{n} |\lambda_i|^2 \right)^2 \leqslant \|A\|_F^4 - \tfrac{1}{2}\mu_2^4 \tag{4.2}$$

established by Kress, de Vries, and Wegmann in [6]; the second one is a consequence of

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \|A\|_F^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{4.2'}$$

also known as "Schur's lemma" [11].

It should be remarked that the first result of the form (C2) was given by Eberlein, who showed in [2] that

$$\mu_2^2 \leqslant \sqrt{6}\,\|A\|_F \mu_3.$$

For the proof of (C3) and (C4) we use that for N normal the equation

$$A^*A - AA^* = A^*(A-N) + (A-N)^*N - (A-N)N^* - A(A-N)^* \tag{4.3}$$

$$= A^*(A-N) + (A-N)^*A - (A-N)A^* - A(A-N)^*$$
$$+ (A-N)(A-N)^* - (A-N)^*(A-N) \tag{4.4}$$

holds. Hence for any $N \in \mathcal{N}$ we get from (4.3)

$$\mu_2^2 \leqslant 2(\|A\|_2 + \|N\|_2)\|A - N\|_F. \tag{4.5}$$

If N is such that $\mu_1 = \|A - N\|_F$, and $U \in \mathcal{U}$ such that $U^*NU = D \in \mathcal{D}$, then it is obvious from

$$\mu_1 = \|A - N\|_F = \|U^*AU - D\|_F = \mathrm{Min}\{\|U^*AU - D\|_F : D \in \mathcal{D},\ U \in \mathcal{U}\} \tag{4.6}$$

that D is the diagonal of U^*AU. In particular

$$\|N\|_2 = \|D\|_2 \leqslant \|A\|_2. \tag{4.7}$$

This together with (4.5) yields (C3).

We use now that for any $G \in C^{n,n}$

$$\|GG^* - G^*G\|_F^2 = 2\|G^*G\|_F^2 - 2\|G^2\|_F^2 \leqslant 2\|G\|_F^4 \qquad (4.8)$$

(see e.g., Eberlein [2]) and get from (4.4)

$$\mu_2^2 \leqslant 4\|A\|_F\|A - N\|_2 + \sqrt{2}\,\|A - N\|_F^2 \leqslant \big(4\|A\|_F + \sqrt{2n}\,\|A - N\|_2\big)\|A - N\|_2$$

for any $N \in \mathcal{N}$, which implies (C4).

To prove (C5) we assume that U is unitary and $U^*AU = \Lambda + M$, M strictly upper triangular. Then obviously $\mu_3^2 = \|M\|_F^2 = \|U^*AU - \Lambda\|_F^2$ $= \|A - U\Lambda U^*\|_F^2 \geqslant \mu_1^2$, as $U\Lambda U^*$ is normal.

(C6) is in Ruhe [10], as well as the second of the reverse inequalities (C7). However, we can do a little better:

$$\mu_3^2 = \sum_{i=1}^n \big(\sigma_i^2 - |\lambda_i|^2\big) = \sum_{i=1}^n \big(\sigma_i + |\lambda_i|\big)\big(\sigma_i - |\lambda_i|\big)$$

$$\leqslant \mu_4 \sum_{i=1}^n \big(\sigma_i + |\lambda_i|\big) \leqslant \mu_4\sqrt{n}\left\{\Big(\sum \sigma_i^2\Big)^{1/2} + \Big(\sum |\lambda_i|^2\Big)^{1/2}\right\}$$

$$\leqslant 2\sqrt{n}\,\|A\|_F \mu_4,$$

i.e. (C7), first inequality.

The second inequality of (C8) is an immediate consequence of

$$A^*A - AA^* = H_2^2 - H_1^2 = H_2(H_2 - H_1) + (H_2 - H_1)H_1$$

and $\|H_1\|_2 = \|H_2\|_2 = \|A\|_2$.

We now make use of the singular value decomposition (SVD)

$$A = W\Sigma V^* \qquad (4.9)$$

where $W, V \in \mathcal{U}$, $\Sigma = \mathrm{diag}(\sigma_i)$. In terms of the SVD we get

$$H_1 = W\Sigma W^*, \qquad H_2 = V\Sigma V^* \qquad (4.10)$$

and

$$A = H_1 U = UH_2, \qquad (4.11)$$

where $U = WV^*$. Define $Y = W^*V = (y_{ij}) \in \mathscr{U}$. Then we get

$$H_1 - H_2 = W(\Sigma Y - Y\Sigma)V^*,$$

$$AA^* - A^*A = W(\Sigma^2 Y - Y\Sigma^2)V^*,$$

and hence

$$\mu_5^2 = \|H_1 - H_2\|_F^2 = \sum_{i,j} |y_{ij}|^2 (\sigma_i - \sigma_j)^2, \qquad (4.12)$$

$$\mu_2^4 = \|AA^* - A^*A\|_F^2 = \sum_{i,j} |y_{ij}|^2 (\sigma_i^2 - \sigma_j^2). \qquad (4.13)$$

If A^+ denotes the Moore-Penrose inverse of A, then

$$\|A^+\|_2 = \max\{\sigma_i^{-1} : \sigma_i > 0, \ i = 1, \ldots, n\},$$

and we have

$$|\sigma_i - \sigma_j| \leq \|A^+\|_2 |\sigma_i^2 - \sigma_j^2|.$$

This implies via (4.12) and (4.13) the first inequality of (C8). We remark that also the second inequality of (C8) can be proved via (4.12), (4.13).

We prove now (C10). We may assume $A = \Lambda + M$, M strictly upper triangular. Then

$$F = \frac{A + A^*}{2} = \frac{\Lambda + \Lambda^*}{2} + \frac{M + M^*}{2}. \qquad (4.14)$$

The theorem of Hoffman and Wielandt [5] gives the first inequality of (C10). Considering the Frobenius norm in (4.14), we get

$$\sum_i \alpha_i^2 = \|F\|_F^2 = \sum_i \gamma_i^2 + \tfrac{1}{2}\mu_3^2. \qquad (4.15)$$

Hence

$$\tfrac{1}{2}\mu_3^2 = \sum_{i=1}^n \left(\alpha_i^2 - \gamma_i^2\right) = \sum (\alpha_{p(i)} - \gamma_i)(\alpha_{p(i)} + \gamma_i)$$

for any permutation p. Taking p such that $\mu_7^2 = \Sigma(\gamma_i - \alpha_{p(i)})^2$, we have by

the Schwarz inequality

$$\tfrac{1}{2}\mu_3^2 \le \mu_7 \left[\left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2} + \left(\sum_{i=1}^{n} \gamma_i^2 \right)^{1/2} \right],$$

and by (4.15)

$$\tfrac{1}{2}\mu_3^2 \le \mu_7 \left[\|F\|_F + \sqrt{\|F\|_F^2 - \tfrac{1}{2}\mu_3^2} \right]. \tag{4.16}$$

The second inequality of (C10) is just a rearrangement of (4.16):

$$\tfrac{1}{2}\mu_3^2 \le \max\left\{ \mu_7\|F\|_F, 2\mu_7\|F\|_F - \mu_7^2 \right\} = \mu_7(2\|F\|_F - \mu_7),$$

as follows from (4.15).

The proof of (C11) is analogous, using instead of (4.14) the relation

$$G = \frac{A - A^*}{2i} = \frac{\Lambda - \Lambda^*}{2i} + \frac{M - M^*}{2i}. \tag{4.17}$$

(C9) follows from (C10) and (C11) by observing that

$$\mu_6^2 = \mu_7^2 + \mu_8^2 \quad \text{and} \quad \|A\|_F^2 = \left\| \frac{A + A^*}{2} \right\|_F^2 + \left\| \frac{A - A^*}{2i} \right\|_F^2.$$

For the proof of (C12) we assume that $U, V \in \mathcal{U}$ and

$$\tilde{M} = U^* \left(\frac{A + A^*}{2} \right) U \in \mathcal{D}, \qquad \tilde{N} = V^* \left(\frac{A - A^*}{2} \right) V \in \mathcal{D}.$$

An easy calculation (using $\tilde{M}\tilde{N} = \tilde{N}\tilde{M}$) gives

$$\tfrac{1}{2}(A^*A - AA^*) = U\tilde{M}U^* V\tilde{N}V^* - V\tilde{N}V^* U\tilde{M}U^*$$

$$= (U - V)\tilde{M}U^* V\tilde{N}V^* + V\tilde{M}(U - V)^* V\tilde{N}V^*$$

$$- V\tilde{N}V^*(U - V)\tilde{M}V^* - V\tilde{N}V^* U\tilde{M}(U - V)^*.$$

Taking norms on both sides and using $\|\tilde{N}\|_2 \le \|A\|_2$, $\|\tilde{M}\|_2 \le \|A\|_2$ gives (C12).

(C13) is proved in a similar way. If $UAV = D \in \mathscr{D}$, then

$$A^*A - AA^* = (V - U)^* D^* DV + U^* D^* D(V - U),$$

from which, by taking norms on both sides, (C13) follows.

For the proofs of (C14) and (C15) we need the following

LEMMA. Let $Y = (Y_{ij})_{i,j=1,\ldots,k} \in \mathscr{U}$ be a block matrix with $Y_{ij} \in C^{n_i,n_j}$, $\Sigma n_j = n$, and Y_{ii} positive semidefinite, $i = 1,\ldots,k$. Then

$$\|Y - I\|_F^2 \leqslant 2 \sum_{i \neq j} \|Y_{ij}\|_F^2. \tag{4.18}$$

Proof of the lemma. Let $\hat{Y} = \mathrm{diag}(Y_{ii})$. Then we have

$$\|\hat{Y}\|_F^2 + \|Y - \hat{Y}\|_F^2 = \|Y\|_F^2 = n, \tag{4.19}$$

as $Y \in \mathscr{U}$ and

$$\|I - \hat{Y}\|_F^2 \leqslant n - \|\hat{Y}\|_F^2 = \|Y - \hat{Y}\|_F^2, \tag{4.20}$$

which can be established most easily by considering the eigenvalues μ of $\hat{Y}$ (satisfying $0 \leqslant \mu \leqslant 1$) and (4.19). Then

$$\|Y - I\|_F^2 = \|\hat{Y} - I\|_F^2 + \|\hat{Y} - Y\|_F^2 \leqslant 2\|\hat{Y} - Y\|_F^2$$

by (4.20).   ∎

We turn now to the proof of (C14). By eventually replacing A by U^*AU with a suitable $U \in \mathscr{U}$ we may assume that

$$F = \frac{A + A^*}{2} = \mathrm{diag}(\tilde{\alpha}_i I_{n_i}), \qquad \tilde{\alpha}_i \neq \tilde{\alpha}_j \text{ for } i \neq j \quad i, j = 1,\ldots,k,$$

and

$$G = \frac{A - A^*}{2i} = (G_{ij})_{i,j=1,\ldots,k} \qquad (G_{ij} \in C^{n_i,n_j})$$

$$= \tilde{\Gamma} + \tilde{G}$$

where $\tilde{\Gamma} = \mathrm{diag}(G_{ii}) \in \mathscr{D}$. Observe that F and G have the same block

decomposition and that the notation of the eigenvalues of F differs from that in the introduction. From

$$\tfrac{1}{4}\|A^{*}A - AA^{*}\|_{F}^{2} = \|GF - FG\|_{F}^{2} = \sum_{i \neq j} (\tilde{\alpha}_{i} - \tilde{\alpha}_{j})^{2}\|G_{ij}\|_{F}^{2}$$

we have

$$\|\tilde{G}\|_{F}^{2} \leqslant \frac{\mu_{2}^{4}}{4\delta^{2}} = \varepsilon^{2}, \tag{4.21}$$

and according to the theorem of Hoffman and Wielandt [5] we get

$$\sum_{i=1}^{n} (\beta_{i} - \gamma_{i})^{2} \leqslant \varepsilon^{2}, \tag{4.22}$$

where β_{i} and γ_{i} are the eigenvalues of G and $\tilde{\Gamma}$ numbered in descending order. Define a permutation P such that

$$P\tilde{\Gamma}P^{T} = \Gamma = \mathrm{diag}(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}).$$

The eigenvalues β_{i} of G define a new block decomposition

$$B = \mathrm{diag}(\beta_{i}) = \mathrm{diag}(\tilde{\beta}_{i} I_{r_{i}})_{i=1,\ldots,s},$$

and $\tilde{\beta}_{i} \neq \tilde{\beta}_{j}$, $i \neq j$. There is a unitary Y satisfying

$$PGP^{T}Y = YB, \tag{4.23}$$

and subdividing Y accordingly, we may also assume that the diagonal blocks Y_{ii} of Y are positive semidefinite $(i = 1, \ldots, s)$, for B is invariant under unitarily block diagonal transformations:

$$\Gamma Y - YB = P(\tilde{\Gamma} - G)P^{T}Y = -P\tilde{G}P^{T}Y. \tag{4.24}$$

If γ_{j} is not in the same block as $\tilde{\beta}_{i}$, then by (4.22)

$$|\tilde{\beta}_{i} - \gamma_{j}| \geqslant |\tilde{\beta}_{i} - \beta_{j}| - |\beta_{j} - \gamma_{j}| \geqslant \delta - \varepsilon = \Delta$$

and $\Delta > 0$ as $\mu_2^2 < 2\delta\tilde{\delta}$. Hence we have from (4.24)

$$\sum_{i \neq j} \|Y_{ij}\|_F^2 \Delta^2 \leqslant \|\tilde{G}\|_F^2 \leqslant \epsilon^2.$$

By the lemma we have

$$\|Y - I\|_F^2 \leqslant \frac{2\epsilon^2}{\Delta^2} = \frac{2\mu_2^4}{\left(2\delta\tilde{\delta} - \mu_2^2\right)^2}. \tag{4.25}$$

But $P^T Y$ diagonalizes G and P^T diagonalizes F. Hence

$$\mu_9 \leqslant \|Y - I\|_F,$$

and by (4.25), (C14) is proved.

For the proof of (C15) we start from the SVD

$$A = W\Sigma V^* \tag{4.26}$$

of A. Writing $\Sigma = \text{diag}(\tilde{\sigma}_i I_{n_i})$, $i = 1, \ldots, k$, $\tilde{\sigma}_i \neq \tilde{\sigma}_j$ for $i \neq j$, a block decomposition is defined. If $k = 1$, then A is normal and (C15) is satisfied. So we assume $k > 1$. Let

$$Y = W^*V = \left(Y_{ij}\right)$$

be decomposed accordingly. By considering the polar decomposition of Y_{ii} we get V_i, D_i unitary, D_i diagonal, such that $V_i Y_{ii} V_i^* D_i$ is positive semidefinite. Replacing V^* by $\text{diag}(\bar{D}_i V_i)V^*$ and W by $W\text{diag}(V_i^*)$, we get

$$A = W\tilde{D}V^*, \tag{4.27}$$

where $\tilde{D} = \Sigma \text{diag}(D_i)$, $|\tilde{D}| = \Sigma$, and the diagonal blocks of $Y = W^*V$ are definite.

Now $H_1 - H_2 = (AA^*)^{1/2} - (A^*A)^{1/2} = W\Sigma W^* - V\Sigma V^*$, and hence

$$\mu_5^2 = \|H_1 - H_2\|_F^2 = \|Y\Sigma - \Sigma Y\|_F^2 \geqslant \tau^2 \sum_{i \neq j} \|Y_{ij}\|_F^2.$$

This and the lemma give

$$\|W - V\|_F^2 = \|Y - I\|_F^2 \leqslant 2 \sum_{i \neq j} \|Y_{ij}\|_F^2 \leqslant \frac{2\mu_5^2}{\tau^2},$$

and (C15) is proved.

We come now to the comparisons involving the spectral condition numbers κ_2, κ_F.

(C16) is just a rearrangement of Smith's result [12]

$$n - 2 + \kappa_2 + \kappa_2^{-1} \leqslant \kappa_F \leqslant \frac{n}{2}\left(\kappa_2 + \kappa_2^{-1}\right),$$

and (C17) is nothing else than the inequality

$$\kappa_F \leqslant \sum_{j=1}^{n} \left(1 + \frac{\mu_3^2}{(n-1)\delta_j^2}\right)^{(n-1)/2}$$

in [12].

For the proof of (C18) we need the following facts:

(a) If S is Hermitian and X nonsingular, then

$$\|S\|_F \leqslant \|X^{-1}SX\|_F, \tag{4.28}$$

as can be seen from Schur's lemma (4.2′) applied to $X^{-1}SX$.

(b) If Y is positive definite and $G \in C^{n,n}$, then

$$\|Y^{-1}GY - G\|_F \leqslant \left(\kappa_2(Y) - 1\right)\|G\|_F. \tag{4.29}$$

This can be shown by writing the linear operator

$$L : C^{n,n} \rightarrow C^{n,n}, \qquad L(G) = Y^{-1}GY - G$$

in the usual vectorized form (see e.g. [8, p. 9])

$$\text{vec}(L(G)) = \left(Y \otimes Y^{-1} - I_n \otimes I_n\right)\text{vec}(G) = \tilde{L}\,\text{vec}(G),$$

where $\otimes$ denotes the Kronecker product. As $\tilde{L}$ is Hermitian and has

eigenvalues $\eta_i/\eta_j - 1$, $i, j = 1, \ldots, n$, where the η_i are the eigenvalues of Y, and $\|G\|_F$ is the usual Euclidean norm of the vector $vec(G) \in C^{n^2}$, (4.29) follows.

Assume $A = X_1 \Lambda X_1^{-1}$ and $\tilde{\mu}_{11} = \kappa_2(X_1) - 1$. Then

$$X_1^{-1}(A^*A - AA^*)X_1 = (Y^{-1}\Lambda^*Y - \Lambda^*)\Lambda - \Lambda(Y^{-1}\Lambda^*Y - \Lambda^*),$$

where $Y = X_1^*X_1$ is positive definite. Using (4.28) and (4.29) yields

$$\mu_2^2 \leqslant \|X_1(A^*A - AA^*)X_1^{-1}\|_F = 2\|\Lambda\|_2\|Y^{-1}\Lambda^*Y - \Lambda^*\|_F$$

$$\leqslant 2\|\Lambda\|_2\|\Lambda\|_F[\kappa_2(Y) - 1] = 2\|\Lambda\|_2\|\Lambda\|_F[\kappa_2^2(X_1) - 1]$$

$$= 2\|\Lambda\|_2\|\Lambda\|_F\tilde{\mu}_{11}(2 + \tilde{\mu}_{11}), \quad \text{i.e. (C18)}.$$

For the proof of (C19) we observe that $A = X_1\Lambda X_1^{-1}$ yields

$$\|A\|_F^2 \leqslant \|\Lambda\|_F^2\kappa_2^2(X_1)$$

and hence

$$\mu_3^2 = \|A\|_F^2 - \|\Lambda\|_F^2 \leqslant \|A\|_F^2[1 - \kappa_2(X_1)^{-2}],$$

which is just (C19). Observe that (C19) strengthens a result of Loizou [7],

$$\mu_3^2 \leqslant \|A\|_F^2\tilde{\mu}_{11}(2 + \tilde{\mu}_{11}). \tag{4.30}$$

(C20) is proved by Ruhe [10].   ∎

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