Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series B

Vol. 28, No. 2, April 1980
Printed in Belgium

Simple Hypergraphs with Maximal Number of Adjacent Pairs of Edges

RUDOLF AHLSWEDE

Fakultät für Mathematik der Universität Bielefeld Universitätsstrasse 1, 4800 Bielefeld 1, German Federal Republic

Communicated by the Managing Editors

Received October 6, 1976

1. THE RESULT

Let $X = \{x_i : 1 \leqslant i \leqslant n\}$ be a finite set, and let $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant N\}$ be a set of N nonempty subsets of X. The couple $H_n{}^N = (X, \mathscr{E})$ is called a (simple) hypergraph (see [5]) if $\bigcup_{i=1}^N E_i = X$.

The x_i's are called vertices and the E_i's are called edges. Two distinct edges are said to be *adjacent* if their intersection is not empty.

Let $g(H_n{}^N) = \sum_{i=1}^{N-1} |\{j : j > i, E_j \cap E_i \neq \varnothing\}|$ count the number of adjacent pairs of edges in $H_n{}^N$.

A hypergraph $H_n{}^N$ is of the type $S_n{}^N$ if its edge set $\mathscr{E}$ is of the form

$$\mathscr{E} = \{E : E \subset X, |E| \geqslant m + 1\} \cup \{E_1, ..., E_{N_m}\}, \tag{1}$$

where $|E_i| = m$ for $i = 1, ..., N_m$ and m and N_m are uniquely defined by

$$N = \sum_{t=m+1}^{n} \binom{n}{t} + N_m, \qquad 0 \leqslant N_m < \binom{n}{m}.$$

THEOREM. *For all natural numbers n, N with $1 \leqslant N \leqslant 2^n$:*

$$\max_{H_n{}^N} g(H_n{}^N) = \max_{S_n{}^N} g(S_n{}^N).$$

2. PROOF OF THE THEOREM

We first give a few definitions and then we proceed by proving two lemmas. Lemma 1 is needed in the proof of Lemma 2 only. The theorem easily follows by iteratively applying Lemma 2. For $\mathscr{A} \subset \mathscr{P}(X)$, the power set of X, let $\mathscr{A}_t = \{E : E \in \mathscr{A}, |E| = t\}$ and $\bar{\mathscr{A}_t} = \mathscr{P}(X)_t \backslash \mathscr{A}_t$.

0095-8956/80/020164-04$02.00/0

For $\mathscr{A}, \mathscr{B} \subset \mathscr{P}(X)$ define

$$s(\mathscr{A}, \mathscr{B}) = |\{(A, B): A \in \mathscr{A}, B \in \mathscr{B}, A \cap B = \varnothing\}|$$

and use the abbreviation $s(\mathscr{A})$ for $s(\mathscr{A}, \mathscr{A})$.

LEMMA 1. *Let $H_n^N = (X, \mathscr{E})$ be a hypergraph and let m, l be integers such that $n \geqslant m \geqslant l$. If*

$$|\mathscr{E}_m| = |\mathscr{E}_l|, \tag{2}$$

then

$$s(\mathscr{E}_l, \bar{\mathscr{E}}_m) \geqslant s(\mathscr{E}_m, \bar{\mathscr{E}}_l).$$

Strict inequality holds if and only if $m > l$ and $n > l + m$.

Proof. One has

$$s(\mathscr{E}_l, \mathscr{E}_m \cup \bar{\mathscr{E}}_m) = |\mathscr{E}_l| \binom{n-l}{m},$$

$$s(\mathscr{E}_m, \mathscr{E}_l \cup \bar{\mathscr{E}}_l) = |\mathscr{E}_m| \binom{n-m}{l},$$

and therefore by (2)

$$\begin{aligned} s(\mathscr{E}_l, \bar{\mathscr{E}}_m) - s(\mathscr{E}_m, \bar{\mathscr{E}}_l) &= |\mathscr{E}_l| \binom{n-l}{m} - |\mathscr{E}_m| \binom{n-m}{l} \\ &= |\mathscr{E}_l| \left[\binom{n-l}{n-m-l} - \binom{n-m}{n-m-l}\right] \geqslant 0, \end{aligned}$$

because $l \leqslant m$. Clearly, $\binom{n-l}{n-m-l} - \binom{n-m}{n-m-l} \geqslant 1$ if and only if $m > l$ and $n > m + l$.

LEMMA 2. *For a hypergraph $H_n^N = (X, \mathscr{E})$ define m, l, and R by*

$$m = \max\{t: \bar{\mathscr{E}}_t \neq \varnothing\},$$

$$l = \min\{t: \mathscr{E}_t \neq \varnothing\},$$

$$R = \min(|\mathscr{E}_l|, |\bar{\mathscr{E}}_m|).$$

If for any $\mathscr{E}_l^0 \subset \mathscr{E}_l$, $\mathscr{E}_m^0 \subset \bar{\mathscr{E}}_m$ with $|\mathscr{E}_l^0| = |\mathscr{E}_m^0| = R$ we define

$$\tilde{\mathscr{E}} = (\mathscr{E} - \mathscr{E}_l^0) \cup \mathscr{E}_m^0,$$

then

$$s(\tilde{\mathscr{E}}) \leqslant s(\mathscr{E}),$$

provided that $m \geqslant l + 1$.

Proof. Since

$$s(\mathscr{E}) = 2s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_l^0) + s(\mathscr{E}_l^0) + s(\mathscr{E} - \mathscr{E}_l^0)$$
$$\geqslant 2s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_l^0) + s(\mathscr{E} - \mathscr{E}_l^0)$$

and since

$$s(\tilde{\mathscr{E}}) = 2s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_m^0) + s(\mathscr{E}_m^0) + s(\mathscr{E} - \mathscr{E}_l^0)$$
$$\leqslant 2s(\tilde{\mathscr{E}}, \mathscr{E}_m^0) + s(\mathscr{E} - \mathscr{E}_l^0)$$

it suffices to show that

$$s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_l^0) - s(\tilde{\mathscr{E}}, \mathscr{E}_m^0) \geqslant 0.$$

Now

$$s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_l^0) \geqslant R \sum_{t=m+1}^{n} \binom{n-l}{t} + s(\mathscr{E}_m, \mathscr{E}_l^0) \tag{3}$$

and

$$s(\tilde{\mathscr{E}}, \mathscr{E}_m^0) \leqslant R \sum_{t=l+1}^{n} \binom{n-m}{t} + s(\mathscr{E}_l - \mathscr{E}_l^0, \mathscr{E}_m^0). \tag{4}$$

Therefore

$$s(\mathscr{E} - \mathscr{E}_l^0, \mathscr{E}_l^0) - s(\tilde{\mathscr{E}}, \mathscr{E}_m^0)$$
$$\geqslant R \sum_{t=0}^{n-m-l-1} \left[\binom{n-l}{t} - \binom{n-m}{t}\right] + [s(\mathscr{E}_m, \mathscr{E}_l^0) - s(\mathscr{E}_l - \mathscr{E}_l^0, \mathscr{E}_m^0)].$$

The first summand is nonnegative, because $n - l \geqslant n - m$. The second summand is also nonnegative. This is obvious in the case $\mathscr{E}_l = \mathscr{E}_l^0$, and in the case $\mathscr{E}_l \neq \mathscr{E}_l^0$ we have $|\mathscr{E}_l^0| = |\mathscr{E}_m^0|$, $\mathscr{E}_m^0 = \bar{\mathscr{E}}_m$ and now Lemma 1 applies.

Q.E.D.

3. Open Problems

It was shown in [1] that the maximal number of pairwise nondisjoint k-tupels in an n-set is $\binom{n-1}{k-1}$. This result has been generalized in various directions (see [3, 4]); however, the following questions are still not answered.

(1) Given N, $\binom{n-1}{k-1} < N \leqslant \binom{n}{k}$, what is the *maximal* number of pairs of nondisjoint k-tuples one can have for a set of N k-tuples chosen from an n-set?

(2) Under the same conditions as those in (1), what is the *minimal* number?

For $k = 2$ question 1 was answered in [2]. There is hope that the techniques used there are suited also for solving the problem for general k. This solution would allow one to specify the "boundary" of an optimal hypergraph S_n^N in the theorem. Problem (2) is easy for $k = 2$ and 3, but seems hard for larger values of k. It is intimately connected with the next problem.

(3) For which hypergraphs is $\min_{H_n^N} g(H_n^N)$ assumed?

(4) Let $g_d(H_n^N)$ for $H_n^N = (X, \mathscr{E})$ count the pairs of sets from $\mathscr{E}$ whose intersection has cardinality at least d. Find hypergraphs for which $\max_{H_n^N} g_d(H_n^N)$ is assumed. Already for $d = 2$ there are values of n and N for which no S_n^N is optimal.

Note added in proof. The problem solved in this paper has been solved independently by P. Frankel in his paper "On the Minimum Number of Disjoint Pairs in a Family of Finite sets," *J. Combinatorial Theory Ser. A 22* (1977), 249–251. However, our method of solution is different.

References

1. P. Erdös, Chao Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* **12** (1961), 313–320.
2. R. Ahlswede and G. O. H. Katona, Graphs with maximal number of adjacent pairs of edges, *Acta Math. Acad. Sci. Hung.* **32** (1978), 97–120.
3. C. Greene and D. J. Kleitman, "Proof Techniques in the Theory of Finite Sets," MIT Lecture Notes.
4. G. O. H. Katona, Extremal problems for hypergraphs, Survey article, *in* "Proceedings, Advanced Study Institute on Combinatorics, Nijenrode Castle Brenkelen, The Netherlands, July 8–20, 1974," pp. 13–42, Math. Center Tracts No. 56, Math. Center, Amsterdam, 1974.

Printed by the St. Catherine Press Ltd., Tempelhof 37, Bruges, Belgium