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# The maximal length of cloud-antichains

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**Abstract**

For six natural notions of cloud-antichains in a partially ordered set $\mathscr{P}$ we determine asymptotically their maximal lengths if $\mathscr{P}$ is the family of all subsets of a finite set. Actually, in three cases we even have exact results.

## 1. Introduction

The notion of an antichain in a partially ordered set was generalized [1, 2] to the seemingly natural notion of a "cloud-antichain" $(\mathscr{A}_i)_{i=1}^N$. Whereas in antichains *elements* of a partially ordered set are compared, in cloud-antichains *sets of elements* take their role. Elements in different sets $\mathscr{A}_i$, called clouds, are required to be incomparable. Formally, for every two clouds $\mathscr{A}_i$ and $\mathscr{A}_j$ we have

$$A_i \not> A_j \text{ for } all\ A_i \in \mathscr{A}_i \text{ and } all\ A_j \in \mathscr{A}_j. \tag{1.1}$$

In [2] further notions of cloud-antichains were introduced. The logical structure of formula (1.1) suggests the idea of an antichain of *type* $(\forall, \forall)$; the new notions in [2] are of the types $(\forall, \exists)$, $(\exists, \forall)$, and $(\exists, \exists)$.

In the sequel we always consider the partially ordered set $\mathscr{P} = 2^{\Omega_n}$, the power set of $\Omega_n = \{1, 2, \dots, n\}$, with set-theoretic containment as order relation. $(\mathscr{A}_i)_{i=1}^N$ is always a family of subsets of $\mathscr{P}$. It is said to be of type $(\exists, \forall)$ if, for all $i \neq j$,

$$\text{there } exists \text{ an } A_i \in \mathscr{A}_i \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \text{ for } all\ A_j \in \mathscr{A}_j, \tag{1.2}$$

it is of type $(\forall, \exists)$ if, for all $i \neq j$,

$$\text{for } all\ A_i \in \mathscr{A}_i \text{ there } exists \text{ an } A_j \in \mathscr{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j \tag{1.3}$$

and it is of type $(\exists, \exists)$ if, for all $i \neq j$, there *exists* an $A_i \in \mathscr{A}_i$

$$\text{and there } exists \text{ an } A_j \in \mathscr{A}_j \text{ with } A_i \not\subset A_j \text{ and } A_i \not\supset A_j. \tag{1.4}$$

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The maximal cardinalities $N$ of such systems as functions of $n$ are denoted by $N_n(\exists,\forall)$, $N_n(\forall,\exists)$, and $N_n(\exists,\exists)$, respectively.

Obviously, an analogously defined quantity $N_n(\forall,\forall)$ equals $\binom{n}{\lfloor n/2\rfloor}$, because in an optimal configuration $|\mathscr{A}_i|=1$ and Sperner's classical theorem [6] applies. We also study systems with *disjoint* clouds. The maximal cardinalities are then denoted by $M_n(\exists,\forall)$, $M_n(\forall,\exists)$, and $M_n(\exists,\exists)$, respectively.

We call two functions $f:\mathbb{N}\to\mathbb{N}$ and $g:\mathbb{N}\to\mathbb{N}$ asymptotically equivalent and write $f(n)\sim g(n)$ if

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=1.$$

All the six functions measuring maximal lengths of cloud-antichains in the cases described are determined up to asymptotic equivalence. Three of the functions are even determined exactly.

## 2. The results

**Theorem 2.1.**

$$M_n(\exists,\forall)\sim 2^{n-1}.$$

**Theorem 2.2.**

$$N_n(\exists,\forall)=\binom{k}{\lfloor k/2\rfloor},\ \textit{where}\ k=\binom{n}{\lfloor n/2\rfloor}.$$

**Theorem 2.3.**

$$M_n(\forall,\exists)=\begin{cases}2 & \textit{if } n=2,\\ 2^{n-1}-1 & \textit{if } n\geqslant 3.\end{cases}$$

**Theorem 2.4.**

$$N_n(\forall,\exists)\sim 2^{2^n-2}.$$

**Theorem 2.5.**

$$M_n(\exists,\exists)=\binom{n}{\lfloor n/2\rfloor}+\left\lfloor\frac{2^n-2-\binom{n}{\lfloor n/2\rfloor}}{2}\right\rfloor.$$

**Theorem 2.6.**

$$N_n(\exists,\exists)\sim 2^{2^n}.$$

The proofs are delegated to the following sections. We begin with those for the exact estimates.

Throughout the paper we use a representation of the partially ordered set $(\mathscr{P}, \subset)$ as sequence space $(\{0,1\}^n, \prec)$, where $A \in \mathscr{P}$ corresponds to $S(A)=(a_1, a_2, \ldots, a_n)$ with

$$a_t = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A, \end{cases}$$

and the inclusion $A \subset B$ translates into $S(A) \prec S(B) = (b_1, b_2, \ldots, b_n)$, which means that $a_t \leqslant b_t$ for $t=1,2,\ldots,n$.

## 3. Proof of Theorem 2.2

We view the cloud-antichain $\{\mathscr{A}_i\}_{i=1}^N$ of type $(\exists, \forall)$ in $\{0,1\}^n$. For $x \in \{0,1\}^n$ let the weight $w(x)$ be the number of 1's in $x$. Let $m$ be the maximal weight of members of $\bigcup_{i=1}^N \mathscr{A}_i$ and let $\{v_1, v_2, \ldots, v_t\}$ be the set of members of $\bigcup_{i=1}^N \mathscr{A}_i$ with weight $m$. We assume first that $m > \lfloor n/2 \rfloor$. It is known that there exist different sequences $v'_1, v'_2, \ldots, v'_t$ of weight $m-1$ and the following property for corresponding sequences:

$$v'_j \leqslant v_j \quad \text{for } j=1,2,\ldots,t. \tag{3.1}$$

For every $i$ $(i=1,\ldots,N)$ we replace all members of $\{v_1, v_2, \ldots, v_t\}$ in $\mathscr{A}_i$ by the corresponding members of $\{v'_1, v'_2, \ldots, v'_t\}$ and call the new cloud $\mathscr{A}'_i$.

One readily verifies that $\{A'_i\}_{i=1}^N$ has again the $(\exists, \forall)$-property. Symmetrically, one can perform a transformation of the clouds via sequences of smallest weight if it is smaller than $\lfloor n/2 \rfloor$. Iteration of these two kinds of transformation results in a cloud-antichain $\{\mathscr{A}_i^*]_{i=1}^N$ with the $(\exists, \forall)$-property involving only sequences of weight $\lfloor n/2 \rfloor$. There are $k = \binom{n}{\lfloor n/2 \rfloor}$ such sequences and every $\mathscr{A}_i^*$ can be represented via the usual incidence relation as a binary vector $u_i$ of length $k$.

Now observe that the $(\exists, \forall)$-property is equivalent to the following one: $u_i \not\succ u_j$ for all $i \neq j$. Sperner's theorem [6] implies $N \leqslant \binom{k}{\lfloor k/2 \rfloor}$.

Conversely, by choosing all clouds consisting of $\lfloor k/2 \rfloor$ sets with $\lfloor n/2 \rfloor$ elements each, we achieve this bound.

## 4. Proof of Theorem 2.3

We make use of an auxiliary result. For $X \subset \{0,1\}^n$ let $\mathscr{C}_n(X)$ be the set of elements of $\{0,1\}^n$ which are comparable with at least one element in $X$.

**Lemma 4.1.** *If $X$ is an (ordinary) antichain in $\{0,1\}^n$, $n \geqslant 4$, then* $|\mathscr{C}_n(X)| \geqslant 2|X|+3$.

**Proof.** Suppose $\alpha \in X$ with $w(\alpha)=1$ (or $w(\alpha)=n-1$). Then necessarily $|\mathscr{C}_n(\{\alpha\}) \setminus \{(0,\ldots,0), (1,\ldots,1)\}| = 2^{n-1}-1$ and $\mathscr{C}_n(\{\alpha\}) \cap (X \setminus \{\alpha\}) = \emptyset$, which implies $|\mathscr{C}_n(X) \setminus \{(0,\ldots,0), (1,\ldots,1)\}| \geqslant |\mathscr{C}_n(\{\alpha\}) \setminus \{(0,\ldots,0), (1,\ldots,1)\}| + |X| - 1 = 2^{n-1} - 2 + |X|$. Now, $2^{n-1}-2+|X| > 2|X|$ holds for $n \geqslant 5$, because there $2^{n-1}-2 > \binom{n}{\lfloor n/2 \rfloor} \geqslant |X|$, and for $n=4$, because here $|X| \leqslant 4$ under the supposition $w(\alpha)=1$ for $\alpha \in X$.

It remains to consider the case where $2 \leqslant w(\alpha) \leqslant n-2$ for all $\alpha \in X$. Define now $X^* = \{(a_1, a_2, \ldots, a_{n-1}, \bar{a}_n) \mid (a_1, a_2, \ldots, a_n) \in X\}$ and notice that $X, X^* \subset \mathscr{C}_n(X) \setminus \{(0,\ldots,0), (1,\ldots,1)\}$ and that $X^* \cap X = \emptyset$, because $X$ is an antichain.

Moreover, for every $(a_1, \ldots, a_n) \in X$ there is some $i \in [1, n-1]$ with $a_i = 1$ and thus $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathscr{C}_n(X)$, $e_i \notin X \cup X^*$. Therefore

$$|\mathscr{C}_n(X) \setminus \{(0,\ldots,0),(1,\ldots,1)\}| \geqslant |X| + |X^*| + 1 = 2|X| + 1. \qquad \square$$

Now Theorem 2.3 is readily established. Suppose first that $1 = |\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_s| < 2 \leqslant |\mathscr{A}_{s+1}| \leqslant \cdots \leqslant |\mathscr{A}_N|$ with $1 \leqslant s \leqslant \binom{n}{\lfloor n/2 \rfloor}$.

Define then $T = \mathscr{C}_n(\bigcup_{i=1}^s \mathscr{A}_i) \setminus (\bigcup_{i=1}^s \mathscr{A}_i)$ and conclude with Lemma 4.1 that $|T| \geqslant 2s + 3 - s$. Since by the $(\forall, \exists)$-property $T \cap (\bigcup_{i=1}^N \mathscr{A}_i) = \emptyset$, we have

$$2^n \geqslant \sum_{i=1}^{N} |\mathscr{A}_i| + |T| \geqslant s + 2(N-s) + s + 3$$

and thus

$$N \leqslant 2^{n-1} - 2 \quad \text{for } n \geqslant 4. \tag{4.1}$$

Furthermore, since $\{(0,\ldots,0), (1,\ldots,1)\} \cap \bigcup_{i=1}^N \mathscr{A}_i = \emptyset$ we have in the remaining case $2 \leqslant |\mathscr{A}_1| \leqslant \cdots \leqslant |\mathscr{A}_N|$, $N \leqslant \frac{1}{2}(2^n - 2)$, and thus $N \leqslant 2^{n-1} - 1$.

On the other hand, we have a simple construction: every $\mathscr{A}_i$ consists of a sequence $\alpha_i \neq (0, \ldots, 0), (1, \ldots, 1)$ and its complement $\bar{\alpha}_i$. There are $2^{n-1} - 1$ such clouds. The $(\forall, \exists)$-property holds. Finally, the cases $n = 2, 3$ go by inspection.

In the case $n = 2$ the only optimal configuration has clouds of cardinality 1. For $n = 3$ there is (up to isomorphics) also the solution $\{\{110\}, \{101\}, \{011\}\}$ with clouds of cardinality 1 only. Furthermore, there are three nonisomorphic solutions, for instance $\{\{110, 001\}, \{101, 010\}, \{011, 100\}\}$, $\{\{110, 010\}, \{101, 001\}, \{011, 100\}\}$, and $\{\{110, 010\}, \{101, 100\}, \{011, 010\}\}$, which clouds of cardinality 2.

Actually, for $n \geqslant 4$ our construction is unique, i.e. every cloud $\mathscr{A}$ is of the form $\mathscr{A} = \{a, \bar{a}\}$. Since by the previous arguments in an optimal configuration all clouds have cardinality 2, it remains to look at a cloud $\mathscr{A} = \{a, b\}$ with $b \neq \bar{a}$. Then $a$ and $b$ have a component value in common, say 0 in the first component. But then $(0, 1, \ldots, 1)$ cannot be in any other cloud, it has to be in $\mathscr{A}$ and equal, say $a$. If now $w(b) \leqslant n-3$, then there is a $c$ with $w(c) = w(b) + 1$, $c \prec a$, $c \succ b$, and $c \notin \bigcup_{i=1}^N \mathscr{A}_i$.

This contradicts the equality $\bigcup_{i=1}^N \mathscr{A}_i = \{0, 1\}^n \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}$. If, on the other hand, $w(b) = n-2 \geqslant 2$ (since $n \geqslant 4$), then some $d$ with $w(d) = w(b) - 1$ and $d \prec b \prec a$ is not in $\bigcup_{i=1}^N \mathscr{A}_i$.

## 5. Proof of Theorem 2.5

There are at most $\binom{n}{\lfloor n/2 \rfloor}$ clouds with one member, and the sequences $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ can be eliminated from all clouds. Therefore

$$N \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} + \left\lfloor 2^{-1} \left( 2^n - 2 - \binom{n}{\lfloor \frac{n}{2} \rfloor} \right) \right\rfloor.$$

We abbreviate the right-hand-side expression by $R$ and now construct $R$ clouds with the $(\exists, \exists)$-property.

*Case* $n=2l$: For $i=1,\dots,\binom{n}{l}$ choose $\mathscr{A}_i=\{a_i\}$ with $w(a_i)=l$. For $i=\binom{n}{l}+1,\dots,R$ choose $\mathscr{A}_i=\{b_i,\bar{b}_i\}$ with $1\leqslant w(b_i)<l$.

*Case* $n=2l+1$: For $n=3$ the choice $\mathscr{A}_1=\{100\}$, $\mathscr{A}_2=\{010\}$, $\mathscr{A}_3=\{001\}$, $\mathscr{A}_4=\{011, 101, 110\}$ works. For $n>3$ there exists a partition of vectors of weight $l+1$ into $\lfloor\binom{2l+1}{l+1}/2\rfloor$ disjoint pairs $\mathscr{A}_i=\{c_i,d_i\}$ with Hamming distance $d_{\mathrm{H}}(c_i,d_i)\geqslant 4$. Further, for the next $\binom{2l+1}{l}$ indices we define $\mathscr{A}_i=\{a_i\}$ with $w(a_i)=l$, and for all the remaining indices we set $\mathscr{A}_i=\{b_i,\bar{b}_i\}$ with $1\leqslant w(b_i)<l$. The $(\exists, \exists)$-property is readily verified.

## 6. Proof of Theorem 2.1

Since $M_n(\forall,\exists)\geqslant M_n(\exists,\forall)$ we conclude from Theorem 2.3 that $M_n(\exists,\forall)\leqslant 2^{n-1}-1$. The issue is to construct a cloud-antichain satisfying this bound asymptotically. We make use of the *general form of Baranyai's theorem*: Let $n_1,\dots,n_t$ be natural numbers such that $\sum_{i=1}^{t} n_i=\binom{n}{k}$; then $\binom{\Omega_n}{k}$ can be partitioned into disjoint sets $P_1,\dots,P_t$ such that $|P_i|=n_i$ and each $l\in\Omega_n$ is contained in exactly $\lceil n_i\cdot k/n\rceil$ or $\lfloor n_i\cdot k/n\rfloor$ members of $P_i$.

Our main auxiliary result is the following lemma.

**Lemma 6.1.** *For positive integers* $n,k,\lambda$ *with* $2k-n\leqslant\lambda<k$, $\binom{\Omega_n}{k}$ *has a partition* $P(n,k,\lambda)=\{P_1,P_2,\dots,P_{\lfloor\frac{1}{2}\binom{n}{k}\rfloor}\}$ *with* $P_i=\{a_i,b_i\}$, $|a_i\cap b_i|=\lambda$.

**Proof.** If $\lambda=0$ or $\lambda=2k-n$, then the statement follows from Baranyai's theorem. We proceed by induction. If at least one of the numbers $\binom{n-1}{k}$, $\binom{n-1}{k-1}$ is even, then we see that

$$|P(n,k,\lambda)|=|P(n-1,k,\lambda)|+|P(n-1,k-1,\lambda-1)|.$$

If $\binom{n-1}{k}\equiv\binom{n-1}{k-1}\equiv 1 \bmod 2$, then there remain two sets $v=\binom{\Omega_{n-1}}{k}\setminus P(n-1,k,\lambda)$, $u=\binom{\Omega_{n-1}}{k-1}\setminus P(n-1,k-1,\lambda-1)$. Using symmetry we can assume that $|v\cap u|=\lambda$. $\square$

For even $n=2l$ as well as for odd $n=2l+1$ we define the cloud-antichain

$$P=\bigcup_{s=l-\lfloor(l-1)/7\rfloor}^{s=l+\lfloor(l-1)/7\rfloor} P\left(n,s,l-s+3\left\lfloor\frac{l-1}{7}\right\rfloor\right)$$

and calculate

$$|P|=\sum_{i=-\lfloor(l-1)/7\rfloor}^{\lfloor(l-1)/7\rfloor}\left\lfloor\frac{\binom{n}{l+i}}{2}\right\rfloor\sim\tfrac{1}{2}2^n.$$

It remains to be seen that $P$ has the $(\exists,\forall)$-property. For this, consider two clouds $\{a,b\}$ and $\{a',b'\}$ with $|a|=|b|=s$, $|a'|=|b'|=s'$ and w.l.o.g. $s<s'$ and $a\subset a'$. We claim

that $b \not\subset a'$, because otherwise $a \cup b \subset a'$, in contradiction to

$$|a \cup b| = 2s - \left(l - s + 3\left\lfloor\frac{l-1}{7}\right\rfloor\right) = 3s - 3\left\lfloor\frac{l-1}{7}\right\rfloor - l \geqslant 3\left(l - \left\lfloor\frac{l-1}{7}\right\rfloor\right) - 3\left\lfloor\frac{l-1}{7}\right\rfloor - l$$

$$= 2l - 6\left\lfloor\frac{l-1}{7}\right\rfloor > l + \left\lfloor\frac{l-1}{7}\right\rfloor \geqslant s'.$$

We claim also that $b \not\subset b'$, because otherwise $a \cap b \subset a' \cap b'$, in contradiction to $|a \cap b| = l - s + 3\lfloor (l-1)/7 \rfloor > l - s' + 3\lfloor (l-1)/7 \rfloor$. $b' \not\subset b$ *and* $a' \not\subset b$ obviously hold, because $|a'| = |b'| = s' > s = |b|$. Finally, we claim that $a \not\subset b'$, because otherwise $a \subset a' \cap b'$, in contradiction to $|a| > |a \cap b| > |a' \cap b'|$. We have shown that $\{a, b\}$ and $\{a', b'\}$ are not comparable in the sense $(\exists, \forall)$.

**Remark 6.2.** Herwig [5] was the first to show that $\liminf_{n \to \infty} Mn(\forall, \exists) 2^{-n} = c > 0$. By arguments based on the marriage theorem he actually proved that $c \geqslant \frac{1}{18}$.

## 7. Proof of Theorem 2.4

Since necessarily $(0, 0, \ldots, 0), (1, 1, \ldots, 1) \notin \bigcup_{i=1}^{N} \mathscr{A}_i$, we have $\{A_i\}_{i=1}^{N} \subset \Omega' \triangleq \mathscr{P}(\{0,1\}^n \setminus \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\})$ and thus $N \leqslant 2^{2^n - 2}$. On the other hand, let us consider $\{\mathscr{A}_i\}_{i=1}^{N^*} \subset \Omega'$, where each $\mathscr{A}_i$ contains a subset $\{\alpha, \bar{\alpha}\}$ and $N^*$ is maximal. The $(\forall, \exists)$-property holds.

There are $2^{n-1} - 1$ sets $\{\alpha, \bar{\alpha}\}$ and therefore

$$|\Omega'| - N^* = \sum_{k=0}^{2^{n-1}-1} \binom{2^{n-1}-1}{k} \cdot 2^k = 3^{2^{n-1}}.$$

This implies $N^* = 2^{2^n - 2} - 3^{2^{n-1}} \sim 2^{2^n - 2}$.

## 8. Proof of Theorem 2.6

Consider all clouds containing at least two sequences of weight $\lfloor n/2 \rfloor$. This defines a cloud-antichain of type $(\exists, \exists)$ and length

$$N = 2^{2^n} - 2^{2^n - \binom{n}{\lfloor n/2 \rfloor}} \left( \binom{n}{\lfloor \frac{n}{2} \rfloor} + 1 \right) \sim 2^{2^n}.$$

Clearly $N_n(\exists, \exists) \leqslant 2^{2^n}$.

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