ON SETS OF WORDS WITH PAIRWISE COMMON LETTER IN DIFFERENT POSITIONS

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1. Introduction and results[1]

For a finite (or infinite) alphabet $X_\alpha = \{1, 2, \ldots, \alpha\}$ we consider the set of words of length n

$$X_\alpha^n = \{x^n = (x_1, x_2, \ldots, x_n) : x_t \in X_\alpha \ \text{ for } \ t = 1, 2, \ldots, n\} \tag{1.1}$$

and also its subset W_α^n of words without repitition of letters, that is,

$$W_\alpha^n = \{x^n = (x_1, x_2, \ldots, x_n) | x_t \in X_\alpha \ \text{ and } \ x_s \neq x_t \text{ for } s \neq t\} \ . \tag{1.2}$$

We describe now an extremal problem, which was mentioned by Mullin at the meeting Designs and Codes, Oberwolfach April 1990, and was said to be of interest to computer scientists.

The words x^n and y^n are in "good relation", if for some $s \neq t \quad x_s = y_t$. For this relation we write $x^n \swarrow\!\searrow y^n$.

The set $F \subset X_\alpha^n$ is good, if for all $x^n, y^n \in F \quad x^n \swarrow\!\searrow y^n$.

Denoting the family of all good sets in W_α^n by $\mathcal{F}_\alpha^n$ the quantity of interest is

$$f_\alpha^n = \max\{|F| : F \in \mathcal{F}_\alpha^n\} \ . \tag{1.3}$$

Generally speaking the determination of this number constitutes an extremal problem in a (growing) class of similar problems whose prototype or historically first candidate is the intersection problem of [1].

Clearly, it is certainly also meaningful to study $\mathcal{G}_\alpha^n$, the family of all good sets in X_α^n , and the quantity

$$g_\alpha^n = \max\{|E| : E \in \mathcal{G}_\alpha^n\} \ . \tag{1.4}$$

The functions f_α^n and g_α^n are rather complex. We present here results for the first two non–trivial configurations of the parameters α and n , namely the cases $n = \alpha - 1$ and $n = 3$. Also, we have a limit theorem for α tending to infinity.

Specifically, we have the following results.

Theorem 1.

$f_\alpha^{\alpha-1} = \frac{1}{2}|W_\alpha^{\alpha-1}| = \frac{1}{2}\alpha!$.

Moreover, we determine all optimal configurations.

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[1] After these results were obtained we received a preprint "Sets of properly separated permutations" by R.C. Mullin, D.R. Stinson, and W.D. Wallis. The relation to our work is this:

Our Theorem 2 and their Theorem 2.2 are identical. Our Theorem 1 goes beyond their Theorem 1.1, because we determine all optimal configurations. Otherwise the papers have no overlap. In particular, our Theorems 3 and 4 are new.

Theorem 2. [2]

$f^3_\alpha = f^3_\infty = f^3_4 = 12$ *for* $\alpha \geq 4$.

Theorem 3.

$g^3_\alpha = 3\alpha + 7$ *for* $3 \leq \alpha < \infty$.

Theorem 4.

$\lim_{\alpha\to\infty} \frac{g^n_\alpha}{\binom{\alpha-1}{n-2}} = \binom{n}{2}(n-2)!$ *or, equivalently,* $g^n_\alpha = \alpha^{n-2}\left(\binom{n}{2} + o(1)\right)$ *as* $\alpha \to \infty$.

2. Proof of Theorem 1[3]

We present first some elementary facts in Lemma 1 and a well known fact in Lemma 2. Then we introduce the candidates for optimal configurations and establish some properties in Lemma 3. Then comes the proof of Theorem 1.

Lemma 1.

(i) $f^n_\alpha \leq f^{n'}_\alpha$ *for* $n < n' \leq \alpha$

(ii) $f^n_\alpha \leq f^n_{\alpha'} \leq f^n_\infty$ *for* $\alpha < \alpha'$

(iii) $f^\alpha_\alpha = |W^\alpha_\alpha| = \alpha!$

(iv) $f^2_\alpha = 3$ *for* $3 \leq \alpha \leq +\infty$.

Proof: Since (i)–(iii) are obviously true, we have to verify only (iv). For $E \in \mathcal{F}^2_\alpha$ with $|E| \geq 3$ necessarily for distinct letters a, b, c $ab, bc \in E$. In good relation with these two words is only the word ca .

Lemma 2. *For* $\tau_1, \tau_2 \in \Sigma_n$ *, the group of permutations on* $\{1, 2, \dots, n\}$ *, there is a sequence of transpositions* $\pi_1, \dots, \pi_t$ *with* $\tau_1 = \pi_t \circ \cdots \circ \pi_1 \circ \tau_2$ *. Moreover, if* $\tau_1 = \pi'_{t'} \circ \cdots \circ \pi'_1 \circ \tau_2 = \pi_t \circ \cdots \circ \pi_1 \circ \tau_2$ *, then* $t \equiv t' \mod 2$.

With every $x^{\alpha-1} \in W^{\alpha-1}_\alpha$ we associate next a partition $\{M(x^{\alpha-1}, 0), M(x^{\alpha-1}, 1)\}$ of $W^{\alpha-1}_\alpha$ into two sets. They are both in $\mathcal{F}^{\alpha-1}_\alpha$ and have the cardinalities $f^{\alpha-1}_\alpha$. Moreover, we show that they are the only optimal configurations.

<u>Definition of $M(x^{\alpha-1}, i)$ and some properties</u>

It is convenient to introduce first

$$B(x^n) = \{x : \text{ for some } t \ x_t = x\}, \tag{2.1}$$

[2]Since our proof is rather technical and since there is the duplication mentioned in footnote 1, we omitted it here. However, it can be found in [3].
[3]A significant simplification of this proof was provided by Gyula Katona and Vu Ha Van (Personal Communication May 1992).

the set of letters in x^n . For $n = \alpha - 1$ and $x^{\alpha-1} \in W_\alpha^{\alpha-1}$ $|B(x^{\alpha-1})| = \alpha - 1$ and apparently either

$$B(x^{\alpha-1}) = B(y^{\alpha-1}) \text{ or } |B(x^{\alpha-1}) \bigtriangleup B(y^{\alpha-1})| = 2 \tag{2.2}$$

for all $x^{\alpha-1}, y^{\alpha-1} \in W_\alpha^{\alpha-1}$. In the second case we denote by $x^{\alpha-1}/y^{\alpha-1}$ the word $x'^{\alpha-1}$ with $B(x'^{\alpha-1}) = B(y^{\alpha-1})$, obtained by changing exactly one component of $x^{\alpha-1}$.

For the definition of $M(x^{\alpha-1}, i)$ we need a map $\mu : W_\alpha^{\alpha-1} \times W_\alpha^{\alpha-1} \to \{0,1\}$ defined as follows:

If $B(x^{\alpha-1}) = B(y^{\alpha-1})$ and if there is a sequence of transpositions $\pi_1, \dots, \pi_t$ with $x^{\alpha-1} = \pi_t \circ \dots \circ \pi_1(y^{\alpha-1})$, then set

$$\mu(x^{\alpha-1}, y^{\alpha-1}) \equiv t \mod 2,$$

and if $|B(x^{\alpha-1}) \bigtriangleup B(y^{\alpha-1})| = 2$, then set

$$\mu(x^{\alpha-1}, y^{\alpha-1}) \equiv \mu(x^{\alpha-1}/y^{\alpha-1}, y^{\alpha-1}) + 1 \mod 2 .$$

Lemma 2 guarantees that μ is well–defined.

Also, immediately from the definition we conclude that for all $x^{\alpha-1}, y^{\alpha-1} \in W_\alpha^{\alpha-1}$

$$\mu(x^{\alpha-1}, y^{\alpha-1}) = \mu(y^{\alpha-1}, x^{\alpha-1}) . \tag{2.3}$$

Finally we define for $x^{\alpha-1} \in W_\alpha^{\alpha-1}$ and $i = 0, 1$

$$M(x^{\alpha-1}, i) = \{z^{\alpha-1} : \mu(x^{\alpha-1}, z^{\alpha-1}) = i\} . \tag{2.4}$$

Lemma 3. *For all* $x^{\alpha-1} \in W_\alpha^{\alpha-1}$

(i) $M(x^{\alpha-1}, 0) \cap M(x^{\alpha-1}, 1) = \varnothing$

(ii) $M(x^{\alpha-1}, 0) \cup M(x^{\alpha-1}, 1) = W_\alpha^{\alpha-1}$

(iii) The partition $\{M(x^{\alpha-1}, 0), M(x^{\alpha-1}, 1)\}$ *is independent of* $x^{\alpha-1}$.

(iv) $|M(x^{\alpha-1}, i)| = \frac{1}{2}\alpha!$ *for* $i = 0, 1$.

Proof: The disjointness holds, because for $z^{\alpha-1} \in W_\alpha^{\alpha-1}$ $\mu(x^{\alpha-1}, z^{\alpha-1})$ is well-defined, that is, takes one value. Since it takes a value from $\{0,1\}$ also (ii) holds.

In order to verify (iii), we establish first the identity

$$\mu(x^{\alpha-1}, z^{\alpha-1}) \equiv \mu(x^{\alpha-1}, y^{\alpha-1}) + \mu(y^{\alpha-1}, z^{\alpha-1}) \mod 2 \tag{2.5}$$

for all $x^{\alpha-1}, y^{\alpha-1}$, and $z^{\alpha-1} \in W_\alpha^{\alpha-1}$.

In the light of (2.3) and since we add $\mod 2$ the identity holding for $x^{\alpha-1}, y^{\alpha-1}$ and $z^{\alpha-1}$ in the order specified in (2.5) implies that it also holds in any other order. We are therefore left with the 2 cases

$$B(x^{\alpha-1}) = B(y^{\alpha-1}) = B(z^{\alpha-1}) \text{ and } B(x^{\alpha-1}), B(y^{\alpha-1}) \neq B(z^{\alpha-1}),$$

In the first case just use the product of the two products of transpositions. In the second case we have from the identity in the first case

$$\mu(x^{\alpha-1}/z^{\alpha-1}, z^{\alpha-1}) \equiv \mu(x^{\alpha-1}/z^{\alpha-1}, y^{\alpha-1}/z^{\alpha-1}) + \mu(y^{\alpha-1}/z^{\alpha-1}, z^{\alpha-1}) \mod 2$$

and by adding a 1 on both sides

$$\mu(x^{\alpha-1}, z^{\alpha-1}) \equiv \mu(x^{\alpha-1}/z^{\alpha-1}, y^{\alpha-1}/z^{\alpha-1}) + \mu(y^{\alpha-1}, z^{\alpha-1}) \mod 2 .$$

If $B(x^{\alpha-1}) = B(y^{\alpha-1})$, then by relabelling

$$\mu(x^{\alpha-1}/z^{\alpha-1}, y^{\alpha-1}/z^{\alpha-1}) = \mu(x^{\alpha-1}, y^{\alpha-1}) \tag{2.6}$$

and thus (2.5). Finally, if $B(x^{\alpha-1}) \neq B(y^{\alpha-1})$ then $B(x^{\alpha-1}) = X_\alpha \setminus \{x\}$, $B(y^{\alpha-1}) = X_\alpha \setminus \{y\}$ and $B(z^{\alpha-1}) = X_\alpha \setminus \{z\}$.

For letter $i \in X_\alpha$ let a_i be its position in $x^{\alpha-1}$, let b_i be its position in $y^{\alpha-1}$, let a_i' be its position in $x^{\alpha-1}/z^{\alpha-1}$, and let b_i' be its position in $y^{\alpha-1}/z^{\alpha-1}$. (It is assumed that i occurs in the respective words.)

Let now $\tau^{\alpha-1} x^{\alpha-1}/y^{\alpha-1} = y^{\alpha-1}$.

This transformation changes the position of y in $x^{\alpha-1}$ to that of x in $y^{\alpha-1}$. We indicate this by $a_y \to b_x$, and similarly $a_z \to b_z$.

Now verify $a_y' = a_y, a_x' = a_z,\ b_y' = b_z, b_x' = b_x$.

Furthermore, $\tau^{\alpha-1}$ causes on $x^{\alpha-1}/z^{\alpha-1}$ $a_y' \to b_x'$ and $a_x' \to b_y'$. We get

$$\pi_{x,y} \circ \tau^{\alpha-1}\ x^{\alpha-1}/z^{\alpha-1} = y^{\alpha-1}/z^{\alpha-1}, \text{ if } \pi_{xy}$$

exchanges x and y.

We have shown

$$\begin{aligned} &\mu(x^{\alpha-1}, y^{\alpha-1}) \equiv \mu(x^{\alpha-1}/z^{\alpha-1}, y^{\alpha-1}) + 1 \\ &\equiv \mu(x^{\alpha-1}/z^{\alpha-1}, y^{\alpha-1}/z^{\alpha-1}) + 1 + 1 \mod 2 \end{aligned}$$

and therefore again (2.6).

Since $\mu(x^{\alpha-1}, x^{\alpha-1}) = 0$ and by (2.3) and (2.5) we have an equivalence relation

$$x^{\alpha-1} \sim y^{\alpha-1} \text{ iff } \mu(x^{\alpha-1}, y^{\alpha-1}) = 0 \ . \tag{2.7}$$

Since for $y^{\alpha-1}, z^{\alpha-1} \in M(x^{\alpha-1}, 0)$ $y^{\alpha-1} \sim z^{\alpha-1}$ and also for $y^{\alpha-1}, z^{\alpha-1} \in M(x^{\alpha-1}, 1)$ $y^{\alpha-1} \sim z^{\alpha-1}$ (iii) holds. Furthermore for all $x^{\alpha-1} \in W_\alpha^{\alpha-1}$ $|M(x^{\alpha-1}, 0)| = |M(x^{\alpha-1}, 1)| = \frac{1}{2}|W_\alpha^{\alpha-1}| = \frac{1}{2}\alpha!$, because changing the last component of $x^{\alpha-1} \in W_\alpha^{\alpha-1}$ transforms $M(x^{\alpha-1}, i)$ into $M(x^{\alpha-1}, (i+1) \mod 2)$.

We give now the complete technical formulation of the Theorem

Theorem 1.

$$f_\alpha^{\alpha-1} = \frac{1}{2}|W_\alpha^{\alpha-1}| = \frac{1}{2}\alpha! \tag{2.8}$$

For any $x^{\alpha-1} \in W_\alpha^{\alpha-1}$

$$M(x^{\alpha-1}, 0),\ M(x^{\alpha-1}, 1) \in \mathcal{F}_\alpha^{\alpha-1} \tag{2.9}$$

and for every $F \in \mathcal{F}_\alpha^{\alpha-1}$ *with* $|F| = f_\alpha^{\alpha-1}$

$$F \in \left\{M(x^{\alpha-1}, 0), M(x^{\alpha-1}, 1)\right\} \ .$$

Proof: It is convenient to use the language of graphs. We define $\mathcal{W}_\alpha^{\alpha-1} = (W_\alpha^{\alpha-1}, \mathcal{E})$, where $(x^{\alpha-1}, y^{\alpha-1}) \in \mathcal{E}$, that is, vertices $x^{\alpha-1}$ and $y^{\alpha-1}$ are connected, exactly if $x^{\alpha-1} \nearrow\!\!\!\nwarrow y^{\alpha-1}$, that is, the vertices are not in good relation.

Now $F \in \mathcal{F}_\alpha^{\alpha-1}$ iff F is an independent set of vertices in $\mathcal{W}_\alpha^{\alpha-1}$. Therefore $f_\alpha^{\alpha-1} = \psi(\mathcal{W}_\alpha^{\alpha-1})$, the independence number of $\mathcal{W}_\alpha^{\alpha-1}$. We show first

$\underline{f_\alpha^{\alpha-1} \leq \frac{1}{2}\alpha! :}$

For distinct positions $s, t \in \{1, 2, \ldots, \alpha - 1\}$ we define a partition $\mathcal{P}(s,t)$ of $W_\alpha^{\alpha-1}$. This definition can be understood from the example $\mathcal{P}(\alpha-2, \alpha-1) = \left\{P(x^{\alpha-3}) : x^{\alpha-3} \in W_\alpha^{\alpha-3}\right\}$ with $P(x^{\alpha-3}) = \left\{x^{\alpha-3}yz : y, z \in X_\alpha \smallsetminus B(x^{\alpha-3}), y \neq z\right\}$.

Since there are 6 words $yz (y \neq z)$ with y, z from a 3 elements set, we have $|P(x^{\alpha-3})| = 6$. Actually the subgraph of $\mathcal{W}_\alpha^{\alpha-1}$ induced on $P(x^{\alpha-3})$ is a sexangle with the edges $\{xy, xz\}$, $\{xz, yz\}$, $\{yz, yx\}$, $\{yx, zx\}$, $\{zx, zy\}$, $\{zy, xy\}$. It has independence number 3 (see also Lemma 1(iv)).

Therefore for any independent set of vertices F , $|F \cap P(x^{\alpha-3})| \leq 3$, and thus

$$|F| = \sum_{x^{\alpha-3}} |F \cap P(x^{\alpha-3})| \leq \frac{1}{2}|W_\alpha^{\alpha-1}| = \frac{1}{2}\alpha! \ .$$

$\underline{f_\alpha^{\alpha-1} \geq \frac{1}{2}\alpha! :}$

We actually show that $M(x^{\alpha-1}, i) \in \mathcal{F}_\alpha^{\alpha-1}$. Since the equation $\mu(x^{\alpha-1}, y^{\alpha-1}) \equiv 0 \mod 2$ defines an equivalence relation, it suffices to show that it implies $x^{\alpha-1} \swarrow\!\searrow y^{\alpha-1}$.

(The converse implication is not true!)

If $B(x^{\alpha-1}) = B(y^{\alpha-1})$ this is obvious and otherwise we have

$$\mu(x^{\alpha-1}/y^{\alpha-1}, y^{\alpha-1}) \equiv 1 \mod 2 . \tag{2.10}$$

If now $x^{\alpha-1}$ and $y^{\alpha-1}$ don't have a letter in different positions, then they agree in all but one position, where they differ: $x^{\alpha-1} \nearrow\!\nwarrow y^{\alpha-1}$ iff exists t with

$$x_t \neq y_t \text{ and } x_s = y_s \text{ for all } s \neq t . \tag{2.11}$$

This contradicts (2.10).

<u>Uniqueness:</u>

We have just seen that the graph $W_\alpha^{\alpha-1}$ is bipartite with vertex partition $\big\{M(x^{\alpha-1}, 0),$

$M(x^{\alpha-1}, 1)\big\}$. The graph is also connected as can be seen inductively in α by using (2.11). If $W_{\alpha-1}^{\alpha-2}$ is connected, then there is a path between $x^{\alpha-2}$ and $y^{\alpha-2}$ and also between $\alpha x^{\alpha-2}$ and $\alpha y^{\alpha-2}$. If we want to connect $\alpha x^{\alpha-2}$ and $yy^{\alpha-2}$ with $y \notin B(y^{\alpha-2})$, then connect $\alpha x^{\alpha-2}$ with $\alpha y^{\alpha-2}$ and this in turn with $yy^{\alpha-2}$.

Let now $F \in \mathcal{F}_\alpha^{\alpha-1}$ be maximal. If we can show that

$$x^{\alpha-1} \in F, x^{\alpha-1} \nearrow\!\nwarrow y^{\alpha-1} \nearrow\!\nwarrow z^{\alpha-1} \text{ implies } z^{\alpha-1} \in F \tag{2.12}$$

then $M(x^{\alpha-1}, 0) \subset F$, because by the structure of the graph the vertices in $M(x^{\alpha-1}, 0)$ are connected with $x^{\alpha-1}$ by paths of even length and by (2.12) they are all in F . By maximilaty of $M(x^{\alpha-1}, 0)$ necessarily $F = M(x^{\alpha-1}, 0)$. To see that (2.12) holds we apply (2.11) to the hypotheses in (2.12). They say that for some $i, j \in \{1, \dots, \alpha-1\}$, $i \neq j$, $x_i \neq y_i$ and $x_k = y_k$ for all $k \neq i$; $y_j \neq z_j$ and $y_\ell = z_\ell$ for all $\ell \neq j$. This means that $x^{\alpha-1}, y^{\alpha-1}$, and $z^{\alpha-1}$ are in the same sexangle of the partition $P(i,j)$ and that $(x^{\alpha-1}, y^{\alpha-1})$ and $(y^{\alpha-1}, z^{\alpha-1})$ are two of its neighbooring edges. Since $x^{\alpha-1} \in F$, only if also $z^{\alpha-1} \in F$ F has a chance to contain 3 vertices of $P(i,j)$, which it has to do in order to be maximal!

3. Proof of Theorem 3

Here and in the last section we use the following notation.

For a set $F \subset \mathcal{F}_\infty^3$ and $A \subset X_\infty$ we define for $t = 1, 2, 3$

$$F_A^t = \big\{(x_1, x_2, x_3) \in F : x_t \in A\big\} \tag{3.1}$$

and the set $\hat{F_A^t}$ as the set of pairs obtained from the triples in F_A^t by omitting the t'th letter. We also use the abbreviation

$$B(\hat{F_A^t}) = \bigcup_{y^2 \in \hat{F_A^t}} B(y^2), \tag{3.2}$$

that is, the set of all letters occuring.

Again we begin with some elementary facts.

Lemma 4.

(i) $g_\alpha^n \le g_\alpha^{n'}$ *for* $n < n'$

(ii) $g_\alpha^n \le g_{\alpha'}^n \le g_\infty^n$ *for* $\alpha < \alpha'$

(iii) $f_\alpha^n \le g_\alpha^n$

(iv) $g_\alpha^2 = 3$ *for* $3 \le \alpha \le \infty$ *and for* $G \in \mathcal{G}_\alpha^2$ *with* $|G| = 3$ *either* $G \in \mathcal{F}_\alpha^2$ *or* $G = \{(a,a),(a,b),(c,a)\}$ *for some* $a \ne b, c$ *.*

(v) $g_2^n = 2^n - 1$

(vi) $g_3^3 = 16$

Proof: (i) – (iv) are trivial. Since $\{0,0,\dots,0) \nearrow\!\!\nwarrow (1,1,\dots,1)$ and since $X_2^n \setminus \{(1,1,\dots,1)\} \in \mathcal{G}_2^n$, (v) follows.

For the verification of (vi) use that at most one word from each of the following sets can be used in $G \in \mathcal{G}_3^3$:

$\{111,222,333\},\{112,332\},\{121,323\},\{211,233\},\{113,223\},\{131,232\},\{211,233\}$, $\{221,331\},\{212,313\},\{122,133\}$
and that $f_3^3 = 3! = 6$.

We are left with all but the first case in Theorem 3.

We present first a set $G(\alpha) \in \mathcal{G}_\alpha^3$ of cardinality $3\alpha + 7$, namely,
$\{(1,2,3),(1,4,2),(1,3,4),(2,1,3),(3,1,4),(4,1,2),(2,3,1),(3,4,1),(4,2,1),$
$(1,1,1)\}\cup\{(1,1,x) : x \in X_\alpha \setminus \{1\}\}\cup\{(1,x,1) : x \in X_\alpha \setminus \{1\}\}\cup\{(x,1,1) = x \in X_\alpha \setminus \{1\}\}$.

It remains to be seen that there is no larger set in $\mathcal{G}_\alpha^3$, that is, that

$$g_\alpha^3 \le 3\,\alpha + 7, \quad 4 \le \alpha < \infty \tag{3.3}$$

Proof: We first partition $X_\alpha^3 \setminus W_\alpha^3$ into 2 parts, say Y_2 and Y_3 , such that

$$Y_s = \big\{x^3 : |\{t : x_t = x\}| = s \ \text{ for some } \ x \in B(x^3)\big\} \ . \tag{3.4}$$

Consider now $G \in \mathcal{G}_\alpha^3$ for $\alpha \ge 4$ and define

$$\mathcal{L}_2(G) = \big\{B(x^3) : x^3 \in Y_2 \cap G\big\} \ . \tag{3.5}$$

This is an Erdös–Ko–Rado family ([1]) of sets with 2 elements.

Case 1: There is a common point, say 1, in all $S \in \mathcal{L}_2(G)$. Let $F = G \cap W_\alpha^3$, $F \supset F_1 = \{x^3 \in F : 1 \in B(x^3)\}$, and let $F_1^c = F \setminus F_1$.

Moreover, let

$U = \big\{(1,1,x) : x \in X_\alpha \setminus \{1\}\big\} \cup \big\{(1,x,1) : x \in X_\alpha \setminus \{1\}\big\} \cup \big\{(x,1,1) : x \in X_\alpha \setminus \{1\}\big\} \subset Y_2$, let
$V_1 = \big\{(1,x,x) : x \in X_\alpha \setminus \{1\}\big\}$, $V_2 = \big\{(x,1,x) : x \in X_\alpha \setminus \{1\}\big\}$, $V_3 = \big\{(x,x,1) : x \in X_\alpha \setminus \{1\}\big\}$, and let $V = \bigcup_{t=1}^3 V_t \subset Y_2$.

By these definitions

$$G \cap Y_2 = (G \cap U) \cup (G \cap V) \ . \tag{3.6}$$

Clearly

$$|F_1 \cup (G \cap V)| = |\bigcup_{t=1}^{3} F_{\{1\}}^t \cup (V_t \cap G)| \leq 9, \tag{3.7}$$

because by (IV) in Lemma 4 $|F_{\{1\}}^t \cup (V_t \cap G)| \leq g_\alpha^2 = 3$. For all $x^3 \in W_\alpha^3$ and $S \subset W_\alpha^3$ define now

$$M(x^3) = \{y^3 \in U : y^3 \nearrow\!\!\!\nwarrow x^3\}, M(S) = \bigcup_{x^3 \in S} M(x^3) \ . \tag{3.8}$$

Then $M\big((x_1,x_2,x_3)\big) = \big\{(1,1,x) : x \in X_\alpha \setminus \{1,x_1,x_2\}\big\} \cup \big\{(1,x,1) : x \in X_\alpha \setminus \{1,x_1,x_3\}\big\} \cup \big\{(x,1,1) : x \in X_\alpha \setminus \{1,x_2,x_3\}$.

In particular for $(x_1,x_2,x_3) \in F_1^c$

$$|M\big((x_1,x_2,x_3)\big)| = 3(\alpha - 3) \ . \tag{3.9}$$

Since obviously $|G \cap V_t| \leq 1$, we also have

$$|G \cap V| \leq 3, \tag{3.10}$$

and finally we state the inequality

$$|G \cap Y_3| \leq 1, \tag{3.11}$$

which always holds.

We are now prepared to complete the proof in this case through 2 subcases.

Subcase 1a: $F_1^c = \varnothing$.

$|G| = |F_1 \cup (G \cap Y_2) \cup (G \cap Y_3)| \leq |F_1 \cup (G \cap V)| + |U| + |G \cap Y_3|$, (by (3.6)) and since $|U| = 3(\alpha - 1)$ by (3.7) and (3.11) $|G| \leq 9 + 3(\alpha - 1) + 1 = 3\alpha + 7$.

Subcase 1b: $F_1^c \neq \varnothing$.

We start with

$$G \cap U \subset U \setminus M(F_1^c) \subset U \setminus M(x^3) \text{ for } x^3 \in F_1^c . \tag{3.12}$$

This, (3.6), (3.9), and (3.10) imply

$$|G \cap Y_2| \leq |U| - 3(\alpha - 3) + 3 = 3(\alpha - 1) - 3(\alpha - 3) + 3 = 9 . \tag{3.13}$$

We know from Theorem 2 that

$$|G \cap W_\alpha^3| = |F| \leq 12 . \tag{3.14}$$

Summarizing, we have from (3.11), (3.13), and (3.14) $|G| = |G \cap Y_3| + |G \cap Y_2| + |G \cap W_\alpha^3| \leq 1 + 9 + 12 = 22 \leq 3\alpha + 7$ for $\alpha \geq 5$, and $\alpha = 4$ needs special consideration. Here $|F_1^c| = |\{x^3 \in F : B(x^3) = \{2,3,4\}| \leq 6$ and $|F_1^c| = 6$ implies $M(F_1^c) = U$. Since $|U| = 3(4-1) = 9$, we have

$$|F_1^c| \leq M(F_1^c) . \tag{3.15}$$

Moreover, for $\alpha = 4$ and $x^3 = (x_1, x_2, x_3) \in F_1^c$ we have $M(x^3) = \{(1,1,x_3), (1,x_2,1), (x_1,1,1)\}$ and for all $x^3, x'^3 \in F_1^c$ $|M(x^3) \cap M(x'^3)| \leq 1$.

Therefore $|M(F_1^c)| \geq 3|F_1^c| - \binom{|F_1^c|}{2}$ and again (3.15) holds for $|F_1^c| \leq 5$. Thus $|G| = |G \cap Y_3| + |F_1 \cup (G \cap V)| + |F_1^c| + |G \cap U| \leq 10 + |F_1^c| + |U| - |M(F_1^c)| \leq 10 + |U| = 19 = 3 \cdot 4 + 7$ (Using (3.6), (3.7), (3.11), (3.12), and (3.15)).

Case 2: There is no common point $x \in X_\alpha$ in all $S \in \mathcal{L}_2(G)$. Then necessarily $|\mathcal{L}_2(G)| = 3$ and w.l.o.g. we can assume $\mathcal{L}_2(G) = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$. Since at most three vectors in each of the sets $\big\{(x,x,y) : y \in \{1,2,3\} \setminus \{x\}\big\}$, $\big\{(x,y,x) : y \in \{1,2,3\} \setminus \{x\}\big\}$, and $\big\{(y,x,x) : y \in \{1,2,3\} \setminus \{x\}\big\}$ may be in G ,

$$|G \cap Y_2| \leq 9 . \tag{3.16}$$

Moreover, in this case we always have

$$G \cap Y_3 = \varnothing . \tag{3.17}$$

Therefore (3.14), (3.16), and (3.17) yield

$$|G| \le 12 + 9 = 21 < 3\alpha + 7 \quad \text{for} \quad \alpha \ge 5 \; . \tag{3.18}$$

In the smallest case $\alpha = 4$, as before, there is a little bit more work. If $|G \cap Y_2| \le 7$, the previous argument gives $|G| \le 12 + 7 = 19 = 3 \cdot 4 + 7$ and we are done. The cases

$$8 \le |G \cap Y_2| \le 9 \tag{3.19}$$

remain to be analysed. Here at least two of the cardinalities $|G \cap \{(x,x,y) : y \ne x\}|$, $|G \cap \{(x,y,x) : y \ne x\}|$ and $|G \cap \{(y,x,x) : y \ne x\}|$ must be 3. W.l.o.g. assume that $|G \cap \{(y,x,x) : y \ne x\}| = 3$, i.e. there are $x_i \in X_\alpha (i = 1,2,3)$ such that $(i, x_i, x_i) \in G$ and $x_i \ne i$ for $i = 1,2,3$.

$\hat{F}{}^{1}_{\{i\}} \cup \{(x_i, x_i)\} \subseteq G_4^2$ for $i = 1,2,3$, therefore $|F^1_{\{i\}}| \le 2$ for $i = 1,2,3$.
Thus we have

$$|F| = \Big|\bigcup_{i=1}^{4} F^1_{\{i\}}\Big| \le 6 + |F^1_{\{4\}}| = 9 \tag{3.20}$$

and by (3.19), (3.20), and (3.17) $|G| \le 18 < 19$.

This settles the case $\alpha = 4$.

4. Proof of Theorem 4

The main idea in the forgoing arguments is to partition X_α^n according to frequency patterns of letters in the words. Recall that a partition of integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \dots, \lambda_r$ with $\sum_{i=1}^r \lambda_i = n$. Denote by $\mathcal{P}(n)$ the set of partitions of n . We partition now X_α^n according to $\mathcal{P}(n)$ as follows. For $\Lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}(n)$ set $\mathcal{Y}(\Lambda) = \{x^n \in X_\alpha^n : \exists \theta_1, \dots, \theta_r \in X_\alpha$ such that θ_i occurs in $x^n = (x_1, \dots, x_n)$ exactly λ_i times$\}$.

For example, if $\Lambda_0 = (1, \dots, 1)$, then $\mathcal{Y}(\Lambda_0) = W_\alpha^n$. In another direction, we have $\mathcal{P}(3) = \{(1,1,1), (2,1), (3)\}$ and sets used in the proof of Theorem 3 are $W_\alpha^3 = \mathcal{Y}((1,1,1)), Y_2 = \mathcal{Y}((2,1)),$ and $Y_3 = \mathcal{Y}((3))$.

They suggested to consider the partition $\{\mathcal{Y}(\Lambda) : \Lambda \in \mathcal{P}(n)\}$. With their help we derive first an upper bound for g_α^n (Lemma 5) and then a lower bound (Lemma 6). Finally we show how the gap between these bounds can be closed. The lower bound is tight. To fix ideas we begin with a rough bound.

Lemma 5.

$$\overline{\lim}_{\alpha\to\infty} \frac{g_\alpha^n}{\binom{\alpha-1}{n-2}} \le \binom{n}{2}(n-1)\ !$$

Proof: We subdivide $\{\mathcal{Y}(\Lambda) : \Lambda \in \mathcal{P}(\Lambda)\}$ into 3 classes. The class 0 consists of $\mathcal{Y}(\Lambda_0)$, where $\Lambda_0 = (1,\dots,1)$, the class 1 consists of $\mathcal{Y}(\Lambda_1)$ for $\Lambda_1 = (2,1,\dots,1)$ and the remaining sets belong the class 2.

We are going to show that the major part of $G \in \mathcal{G}_\alpha^n$ falls into class 1.

First of all, for all $G \in \mathcal{G}_\alpha^n$ by our definitions $|G \cap \mathcal{Y}(\Lambda_0)| \le f_\alpha^n$.

It is easy to see that $f_\alpha^n \le (n!)^2$, a bound independent of α . In fact, a simple induction argument works. Choose any $(x_1,\dots,x_n) \in F \in \mathcal{F}_\infty^n$.

For all $y^n \in F$ $B(y^n) \cap \{x_1, x_2, \dots, x_n\} \ne \varnothing$ and, on the other hand, for fixed j and i $|F \cap \{y^n : y_j = x_i\}| \le f_\infty^{n-1}$. This implies $f_\infty^n \le n^2\ f_\infty^{n-1}$ and clearly $f_\alpha^n \le f_\infty^n$. Therefore

$$|G \cap \mathcal{Y}(\lambda_0)| \le (n!)^2, \tag{4.1}$$

Next we consider class 2.

For all $x^n, y^n \in G$, $B(x^n) \cap B(y^n) \ne \varnothing$, so $\{B(x^n) : x^n \in G \cap \mathcal{Y}(\Lambda)\}$ forms an EKR–system with r–element sets, if $\Lambda = (\lambda_1, \dots, \lambda_r)$. For a $\mathcal{Y}(\Lambda)$ in the class 2 Λ partitions n into $r \le n-2$ parts and for all $x^n \in \mathcal{Y}(\Lambda)$

$$|\{y^n : B(y^n) = B(x^n)\}| \le r^n \le (n-2)^n\ . \tag{4.2}$$

This leads to the estimate

$$|G \cap (\bigcup_{\Lambda \ne (\lambda_0, \lambda_1)} \mathcal{Y}(\Lambda))| \le |\mathcal{P}(n)| \binom{\alpha-1}{n-3}(n-2)^n\ . \tag{4.3}$$

For the major part we have for all $x^n \in \mathcal{Y}(\Lambda_1)$

$$|\{y^n : B(y^n) = B(x^n)\}| = \binom{n}{2}(n-1)\ ! \tag{4.4}$$

Now Lemma 5 follows from (4.1), (4.3), (4.4) and the EKR–Theorem, which accounts for the factor $\binom{\alpha-1}{n-2}$.

Lemma 6.

$$\underline{\lim}_{\alpha\to\infty} \frac{g_\alpha^n}{\binom{\alpha-1}{n-2}} \ge \binom{n}{2}(n-2)\ !$$

Proof: Define $G_0 = \{x^n \in X_\alpha^n : |B(x^n)| = n-1$ and 1 occurs exactly twice in $x^n\}$. Obviously $G_0 \in \mathcal{G}_\alpha^n$, $|G_0| = \binom{n}{2}(n-2)!\binom{\alpha-1}{n-2}$, and thus the claim follows.

Proof of Theorem 4:

It suffices to show that for $G \in \mathcal{G}_\alpha^n$

$$\overline{\lim}_{\alpha \to \infty} |G \cap \mathcal{Y}(\Lambda_1)| \binom{\alpha - 1}{n-2}^{-1} \leq \binom{n}{2} (n-2)\,! \tag{4.5}$$

To do this, we have to refine our partition, more precisely, we partition $\mathcal{Y}(\Lambda_1) \cap G$ into a few subparts. First of all we can assume that the EKR–system $\{B(x^n) : x^n \in G \cap \mathcal{Y}(\Lambda_1)\}$ is not a 2–intersecting family, because otherwise $|G \cap \mathcal{Y}(\Lambda_1)| \leq \binom{\alpha-2}{n-3}\binom{n}{2}(n-1)! \sim \alpha^{n-3}$ by (4.4), which would imply (4.5).

This assumption means that there are $a^n, b^n \in G \cap \mathcal{Y}(\Lambda_1)$ with $|B(a^n) \cap B(b^n)| = 1$. W.l.o.g. let $B(a^n) = \{1, 2, \dots, n-1\}$ and $B(b^n) = \{1, n, n+1, \dots, 2n-3\}$. Denote by $\mathcal{Z} = \{x^n \in \mathcal{Y}(\Lambda_1) \cap G : 1 \notin B(x^n)\}$ the set of x^n in which 1 does not appear. Since for all $x^n \in \mathcal{Z}$ $B(x^n) \cap B(a^n) \neq \varnothing$ and $B(x^n) \cap B(b^n) \neq \varnothing$ $|\{B(x^n) : x^n \in \mathcal{Z}\}| < 2^{2(n-2)}\binom{\alpha-2n+2}{n-3}$.

Consequently, by (4.4)

$$|\mathcal{Z}| < 2^{2(n-1)} \binom{n}{2} (n-1)! \binom{\alpha - 2n + 2}{n-3} . \tag{4.6}$$

Let now $C_i = \{(c_1, \dots, c_n) \in \mathcal{Y}(\Lambda_1) : c_i = 1, c_j \neq 1 \text{ for } j \neq i\}$ for $i = 1, 2, \dots, n$. Then

$$\mathcal{Y}(\Lambda_1) \cap G = (G \cap G_0) \,\dot\cup\, \mathcal{Z} \,\dot\cup\, (C_1 \cap G) \,\dot\cup \dots \dot\cup\, (C_n \cap G) \tag{4.7}$$

where G_0 is defined in the proof of Lemma 6.

Because $\{(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n) : (c_1, \dots, c_{i-1}, 1, c_{i+1}, \dots, c_n) \in C_i \cap G\} \in \mathcal{G}_\alpha^{n-1}$, we have

$$|C_i \cap G| = O(\alpha^{n-3}) \text{ (as } \alpha \to \infty) \tag{4.8}$$

by Lemma 5.

Finally,

$$|G_0 \cap G| \leq |G_0| = \binom{n}{2} (n-2)! \binom{\alpha - 1}{n-2} \tag{4.9}$$

and (4.6) – (4.9) imply (4.5).

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