A Note on the Variation of Permanents

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ABSTRACT

It is shown that for any two n-by-n complex matrices A, B the inequality

$$|\operatorname{per}(A) - \operatorname{per}(B)| \leqslant n\|A - B\| \max(\|A\|, \|B\|)^{n-1}$$

holds, if $\| \ \|$ is either the row-sum or the column-sum norm. It is conjectured that this result holds for any operator norm.

In [1], R. Bhatia proved that for any two n-by-n matrices A, B the inequality

$$|\operatorname{per}(A) - \operatorname{per}(B)| \leqslant n\|A - B\|_2 \max(\|A\|_2, \|B\|_2)^{n-1} \tag{1}$$

holds. Here $\| \ \|_2$ denotes the spectral norm.

The purpose of this note is to prove an analogous result for the row-sum and the column-sum norm. We recall that

$$\|A\|_\infty = \max_i \sum_k |a_{ik}|,$$

$$\|A\|_1 = \max_k \sum_i |a_{ik}|$$

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are the operator norms for the vector norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|x\|_\infty = \max_i |x_i|$$

respectively, where $A = (a_{ik})$, $x = (x_1, \dots, x_n)^T$. We show

$$|\mathrm{per}(A) - \mathrm{per}(B)| \leqslant n\|A - B\|_p \max(\|A\|_p, \|B\|_p)^{n-1}, \qquad p = 1, \infty. \quad (2)$$

As $\|A\|_1 = \|A^T\|_\infty$, it suffices to prove the case $p = 1$. We make use of the obvious inequality (see e.g. [3, p. 113])

$$|\mathrm{per}\, A| \leqslant \|a_1\|_1 \|a_2\|_1 \cdots \|a_n\|_1,$$

where $A = (a_1, \dots, a_n)$ and a_i denotes the ith column of A. If $B = (b_1, \dots, b_n)$, define

$$A_k = (a_1, a_2, \dots, a_k, b_{k+1}, \dots, b_n), \qquad k = 1, \dots, n-1,$$

$A_0 = B$, $A_n = A$. Then

$$|\mathrm{per}(A_i) - \mathrm{per}(A_{i-1})| = |\mathrm{per}(a_1, \dots, a_{i-1}, a_i - b_i, b_{i+1}, \dots, b_n)|$$
$$\leqslant \|a_i - b_i\|_1 \prod_{j<i} \|a_j\|_1 \prod_{j>i} \|b_j\|_1.$$

Hence

$$|\mathrm{per}(A) - \mathrm{per}(B)| \leqslant \sum_{i=1}^{n} |\mathrm{per}(A_i) - \mathrm{per}(A_{i-1})|$$
$$\leqslant n \max_i \|a_i - b_i\|_1 \max_i \left(\prod_{j<i} \|a_j\|_1 \prod_{j>i} \|b_j\|_1 \right)$$
$$\leqslant n\|A - B\|_1 \max(\|A\|_1, \|B\|_1)^{n-1}.$$

This establishes (2).

We remark that this proof is completely elementary, while that in [1] uses nontrivial tools from multilinear algebra.

Also, (1) and (2) are not comparable, as different operator norms are not comparable.

It is tempting to state the

CONJECTURE. If $\| \ \|$ denotes any operator norm for n-by-n matrices, then

$$|\operatorname{per}(A) - \operatorname{per}(B)| \leqslant n\|A - B\| \max(\|A\|, \|B\|)^{n-1}. \tag{3}$$

We remark finally that S. Friedland has proved (3) for the determinant function instead of the permanent [2].

Note added in proof. S. Friedland has shown (private communication) the following related result, which implies (1) and is near to the conjecture: For any operator norm $\| \ \|$

$$|\operatorname{per}(A) - \operatorname{per}(B)| \leqslant \frac{n}{2}\Big[\|A - B\| \max(\|A\|, \|B\|)^{n-1} + \|A^* - B^*\| \max(\|A^*\|, \|B^*\|)^{n-1}\Big].$$

REFERENCES

1 R. Bhatia, Variation of symmetric tensor products and permanents, *Linear Algebra Appl.* 62:269–276 (1984).

2 S. Friedland, Variation of tensor powers and spectra, *Linear and Multilinear Algebra* 12:81–98 (1982).

3 H. Minc, Permanents, in *Encyclopedia of Mathematics and Its Applications*, Vol. 6, Addison-Wesley, 1978.

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