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On Partitioning and Packing Products with Rectangles

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In [1] we introduced and studied for product hypergraphs $\mathscr{H}^n = \prod_{i=1}^n \mathscr{H}_i$, where $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$, the minimal size $\pi(\mathscr{H}^n)$ of a partition of $\mathscr{V}^n = \prod_{i=1}^n \mathscr{V}_i$ into sets that are elements of $\mathscr{E}^n = \prod_{i=1}^n \mathscr{E}_i$. The main result was that

$$\pi(\mathscr{H}^n) = \prod_{i=1}^n \pi(\mathscr{H}_i) \tag{1}$$

if the $\mathscr{H}_i$s are graphs with all loops included. A key step in the proof concerns the special case of complete graphs. Here we show that (1) also holds when the $\mathscr{H}_i$ are complete d-uniform hypergraphs with all loops included, subject to a condition on the sizes of the $\mathscr{V}_i$. We also present an upper bound on packing numbers.

1. Introduction

For hypergraphs $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$ $(1 \leq i \leq n)$, we define the product hypergraph $\mathscr{H}^n = (\mathscr{V}^n, \mathscr{E}^n) = \left(\prod_{i=1}^n \mathscr{V}_i, \prod_{i=1}^n \mathscr{E}_i\right)$. Edges of cardinality 1 are called loops. Special hypergraphs are graphs $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ defined by the property

$$|E| \in \{1, 2\} \text{ for all } E \in \mathscr{E}$$

and, more generally, d-uniform hypergraphs (with or without loops) that satisfy

$$|E| \in \{1, d\} \text{ for all } E \in \mathscr{E}.$$

In particular, there are d-uniform hypergraphs with all loops included, that is, $\{\{v\} : v \in \mathscr{V}\} \subset \mathscr{E}$.

When the set $\binom{\mathscr{V}}{d}$ of all vertex sets of cardinality d is contained in the edge set $\mathscr{E}$, we speak of a complete d-uniform hypergraph.

In [1], we introduced the partition number $\pi(\mathscr{H})$ as the minimal size of a partition of $\mathscr{V}$ into sets that are members of $\mathscr{E}$, if a partition exists, and ∞ otherwise. When $\mathscr{G}_i = (\mathscr{V}_i, \mathscr{E}_i)$ $(i = 1, 2, \ldots, n)$ are arbitrary finite graphs with all loops included, then we obviously have $\pi(\mathscr{G}_i) = |\mathscr{V}_i| - \nu(\mathscr{G}_i)$ for the partition number, where $\nu(\mathscr{G}_i)$ is the matching number of $\mathscr{G}_i$.

A discovery of [1] is that for the hypergraph product $\mathscr{H}^n = \mathscr{G}_1 \times \ldots \times \mathscr{G}_n$

$$\pi(\mathscr{H}^n) = \prod_{i=1}^{n} \pi(\mathscr{G}_i).$$

An important step in our proof is to show the above when all $\mathscr{G}_i$s are complete. Here we establish the following generalization.

Theorem. *For complete d-uniform hypergraphs with all loops $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$, that is, $\mathscr{E}_i = \binom{\mathscr{V}_i}{d} \cup \{\{v\} : v \in \mathscr{V}_i\}$ $(i = 1, 2, \ldots, n)$, write $|\mathscr{V}_i| = dq_i + r_i$, $0 \leq r_i < d$.*
Then for $\mathscr{H}^n = \prod_{i=1}^{n} \mathscr{H}_i$ satisfying

$$d > \prod_{i: r_i \neq 0} r_i, \tag{*}$$

we have

$$\pi(\mathscr{H}^n) = \prod_{i=1}^{n} \frac{|\mathscr{V}_i| + (d-1)r_i}{d} = \prod_{i=1}^{n} (q_i + r_i) = \prod_{i=1}^{n} \pi(\mathscr{H}_i). \tag{2}$$

(Theorem 2′ of [1] is covered by the case $d = 2$.)

We also present an upper bound on packing numbers, the maximal number of disjoint edges, in Section 5.

Remarks.

1 The condition (*) cannot be omitted. In Example 1 of [1] we have $|\mathscr{V}_i| = 7$, $n = 2$, $d = 4$, $r = 3$, $q = 1$, and thus $d < r^2$ in violation of (*). However, $\pi(\mathscr{H}^2) \leq 13 < (1+3)^2 = 16$ and thus (2) does not hold.

2 The Theorem always applies if $r_i \in \{0, 1\}$ for all i. In particular, (2) holds for complete d-uniform hypergraphs including all loops if $|\mathscr{V}| \in \{dq, dq+1\}$.

3 In the light of our former results for graphs with all loops included, it is natural to study products of hypergraphs $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$ whose edge sets $\mathscr{E}_i$ are downsets, that is, for all i and $E \in \mathscr{E}_i$

$$F \subset E \quad \text{implies} \quad F \in \mathscr{E}_i.$$

In spite of several attempts we could not even settle the case

$$\mathscr{E}_i = \{E \subset \mathscr{V}_i : |E| \leq d_i\}.$$

An immediate technical difficulty is caused by the fact that there may be non-isomorphic minimal partitions of $\mathscr{H}_i$. For example, in the case $\mathscr{V}_i = \{0, 1, \ldots, 8\}$ and $d = 4$, both, $\{\{0,1,2,3\},\{4,5,6,7\},\{8\}\}$ and $\{\{0,1,2,3\},\{4,5,6\},\{7,8\}\}$ are optimal. For products of such hypergraphs there is a lot of freedom for minimal partitions.

The paper is organized as follows. In Section 2, we prove, and present as Lemma 1, our key identity. Two simple consequences, Lemma 2 and Lemma 3, are derived in Section 3. Section 4 contains the proof of the Theorem. Finally, in Section 5 we use Lemma 3 to derive a bound on packing numbers (Corollary 1) and to determine the packing number of products of two graphs that have Hamiltonian cycles (Corollary 2).

2. An identity for packings in products of general hypergraphs

Let $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$ be hypergraphs, and consider the product $\mathscr{H}^n = (\prod_{i=1}^n \mathscr{V}_i, \prod_{i=1}^n \mathscr{E}_i) = (\mathscr{V}^n, \mathscr{E}^n)$. A packing in $\mathscr{H}^n$ is a collection of pairwise disjoint elements of $\mathscr{E}^n$. For $I \subset [n]$, we write $\mathscr{H}^I = \prod_{i\in I} \mathscr{H}_i$, $\mathscr{V}^I = \prod_{i\in I} \mathscr{V}_i$, $\mathscr{E}^I = \prod_{i\in I} \mathscr{E}_i$, and for given $E^n = \prod_{i=1}^n E_i$ and $v^n = \prod_{i=1}^n v_i$, we also write $E^I = \prod_{i\in I} E_i$ and $v^I = \prod_{i\in I} v_i$, respectively.

With a packing $\mathscr{P}$ in $\mathscr{H}^n$ we associate the support

$$\mathscr{S}(\mathscr{P}) = \bigcup_{E^n\in\mathscr{P}} E^n. \tag{3}$$

Also, for $I \subset [n]$, $v^I \in \mathscr{V}^I$ and $\mathscr{P}$, we introduce the packing of $\mathscr{H}^{I^c}$

$$\mathscr{P}_{v^I} = \{E^{I^c} : E^n \in \mathscr{P} \quad \text{with} \quad v^I \in E^I\}. \tag{4}$$

Geometrically this means taking a 'slice' from the product (corresponding to fixing some values in the coordinates I) and looking at the restrictions of all the sets in the packing to that slice.

Finally, for general functions $\varphi_i : \mathscr{E}_i \to \mathbb{R}$ $(i = 1, 2, \dots, n)$, $J \subset [n]$ and E^n, we set

$$\varphi(E^J) = \prod_{i\in J} \varphi_i(E_i), \tag{5}$$

and for $\mathscr{F} \subset \mathscr{E}^J$, we set

$$\varphi(\mathscr{F}) = \sum_{E^J\in\mathscr{F}} \varphi(E^J) \tag{6}$$

($\varphi(\emptyset) = 0$ by convention).

We are now ready to state our basic identity.

Lemma 1. *For every packing $\mathscr{P}$ of the product hypergraph $\mathscr{H}^n$ and every function φ defined by (5) and (6)*

$$\sum_{I\subset[n]} \sum_{v^I} \varphi(\mathscr{P}_{v^I}) = \sum_{E^n\in\mathscr{P}} \prod_{\ell=1}^n \big(|E_\ell| + \varphi_\ell(E_\ell)\big). \tag{7}$$

Proof.

$$\begin{aligned}
\sum_{I\subset[n]} \sum_{v^I} \varphi(\mathscr{P}_{v^I}) &= \sum_{I\subset[n]} \sum_{v^I} \sum_{E^{I^c}\in\mathscr{P}_{v^I}} \varphi(E^{I^c}) = \sum_{I\subset[n]} \sum_{E^n\in\mathscr{P}} \sum_{v^I\in E^I} \varphi(E^{I^c}) \\
&= \sum_{I\subset[n]} \sum_{E^n\in\mathscr{P}} |E^I|\varphi(E^{I^c}) = \sum_{E^n\in\mathscr{P}} \sum_{I\subset[n]} \left(\prod_{i\in I} |E_i|\right) \left(\prod_{j\in I^c} \varphi_j(E_j)\right) \\
&= \sum_{E^n\in\mathscr{P}} \prod_{\ell=1}^n \big(|E_\ell| + \varphi_\ell(E_\ell)\big) \quad \text{(by the Binomial formula)}. \qquad \square
\end{aligned}$$

In the next section we derive two consequences of Lemma 1. The first is used in the proof of the Theorem. The second is used for our results on packings in Section 5.

In [2] we have developed a more general machinery to study partitions of products of hypergraphs. There we employ Rota's theory of Möbius transforms for posets (see [3] or [4]). However, the best concrete results there are also covered by [1] and the present result.

3. Two consequences of Lemma 1

Lemma 2. *Let $\mathscr{H}^n$ be a product of d-uniform hypergraphs with loops, then for a partition $\mathscr{P}$ of $\mathscr{H}^n$,*

$$d^n|\mathscr{P}| = \sum_{I \subsetneq [n], v^I \in \mathscr{V}^I} (d-1)^{n-|I|} J_{v^I} + \prod_{i=1}^{n} |\mathscr{V}_i|, \tag{8}$$

where J_{v^I} is the number of elements with size 1 in the partition $\mathscr{P}_{v^I}$ of $\mathscr{V}^{I^c}$.

Lemma 3. *For a packing $\mathscr{P}$ in $\mathscr{H}^n = \prod_{i=1}^n \mathscr{H}_i$, where $\mathscr{H}_i$ is a d_i-uniform hypergraph without loops,*

$$|\mathscr{P}| = \frac{\sum_{0<|I|<n} \sum_{v^I} |\mathscr{P}_{v^I}|}{\prod_{i=1}^n (d_i+1) - \left(1 + \prod_{i=1}^n d_i\right)}. \tag{9}$$

Proof of Lemma 2. We have $|E_i| = d$ for all non-singletons E_i. Choose $\varphi_i(E_i) = d - |E_i|$. Then the left-hand side of (7) is

$$\sum_{I \subset [n]} \sum_{v^I} \varphi(\mathscr{P}_{v^I}) = \sum_{I \subset [n]} \sum_{v^I \in \mathscr{V}^I} (d-1)^{n-|I|} J_{v^I},$$

where J_{v^I} is the number of singletons in $\mathscr{P}_{v^I}$, and the right-hand side of (7) is

$$\sum_{E^n \in \mathscr{P}} \prod_{\ell=1}^{n} \left(|E_\ell| + d - |E_\ell|\right) = d^n|\mathscr{P}|.$$

Since $\mathscr{P}$ is a partition, we have $\sum_{v^{[n]}} J_{v^{[n]}} = \prod_{i=1}^n |\mathscr{V}_i|$ and thus (8). □

Proof of Lemma 3. Now choose $\varphi_i(E_i) = 1$ for all $E_i \in \mathscr{E}_i$ and $i = 1, 2, \dots, n$. Then (7) becomes

$$\sum_{I \subset [n]} \sum_{v^I} |\mathscr{P}_{v^I}| = \prod_{i=1}^{n} (d_i + 1)|\mathscr{P}|.$$

Since $\mathscr{P}_\emptyset = \mathscr{P}$ and $|\mathscr{P}_{b^n}| = 1$ if $b^n \in \mathscr{S}(\mathscr{P})$ and equal to 0 otherwise, we conclude

$$\sum_{0<|I|<n} \sum_{v^I} |\mathscr{P}_{v^I}| = \prod_{i=1}^{n} (d_i + 1)|\mathscr{P}| - |\mathscr{P}| - |\mathscr{S}(\mathscr{P})|.$$

As $|\mathscr{S}(\mathscr{P})| = \prod_{i=1}^n d_i |\mathscr{P}|$, we have established the result. □

4. Proof of Theorem

We show first that condition (*) implies

$$J_{v^I} \geq \prod_{j \in I^c} r_j \quad \text{for} \quad I \subset [n]. \tag{10}$$

Suppose that for some I the inequality does not hold: $\prod_{j\in I^c} r_j - J_{v^I} > 0$. Since every $E^n \in \prod_{i=1}^n \{\{v\} : v \in \mathscr{V}_i\}$ has size 1 and every $E^n \in \prod_{i=1}^n (\mathscr{E}_i \cup \{\{v\} : v \in \mathscr{V}_i\}) \setminus \prod_{i=1}^n \{\{v\} : v \in \mathscr{V}_i\}$ has a size divisible by d, we can write

$$\prod_{j\in I^c} |\mathscr{V}_j| = Ad + \prod_{j\in I^c} r_j = Bd + J_{v^I},$$

where A and B are positive integers, and therefore $d | \left(\prod_{j\in I^c} r_j - J_{v^I}\right)$, in contradiction to our supposition and (*).

Now (10) and Lemma 2 imply

$$\begin{aligned} d^n|\mathscr{P}| &\geq \sum_{k=0}^{n-1}(d-1)^{n-k} \sum_{I:|I|=k} \prod_{i\in I} |\mathscr{V}_i| \prod_{j\in I^c} r_j + \prod_{i=1}^{n} |\mathscr{V}_i| \\ &= \prod_{i=1}^{n} \left[|\mathscr{V}_i| + (d-1)r_i\right] \quad \text{(by the Binomial formula)}, \end{aligned}$$

and hence $|\mathscr{P}| \geq \prod_{i=1}^n (q_i + r_i)$.

The opposite inequality follows from the fact that the product of partitions is a partition.

5. A consequence of Lemma 2

Denote by $p(\mathscr{H})$ the packing number of $\mathscr{H}$.

Corollary 1. *For d_i-uniform hypergraphs $\mathscr{H}_i = (\mathscr{V}_i, \mathscr{E}_i)$ $(i = 1, 2)$*

$$p(\mathscr{H}_1 \times \mathscr{H}_2) \leq \left\lfloor \frac{|\mathscr{V}_1|\; p(\mathscr{H}_2) + |\mathscr{V}_2|\; p(\mathscr{H}_1)}{d_1 + d_2} \right\rfloor . \tag{11}$$

Finally, we present a striking example, which shows that (11) is surprisingly sharp.

Corollary 2. *For two graphs $\mathscr{G}_i = (\mathscr{V}_i, \mathscr{E}_i)$ $(i = 1, 2)$ with $|\mathscr{V}_1| = |\mathscr{V}_2| = u$ both having Hamiltonian cycles,*

$$p(\mathscr{G}_1 \times \mathscr{G}_2) = \left\lfloor \frac{u \left\lfloor \frac{u}{2} \right\rfloor}{2} \right\rfloor .$$

Proof. The upper bound on $p(\mathscr{G}_1 \times \mathscr{G}_2)$ follows from Corollary 1. The opposite inequality is trivial for even m and follows for odd m by inspection (notice that deleting edges can only make the packings worse!) of configurations, which we now describe.

We label the edges of m-cycles C_m by $0, 1, \ldots, m-1$, such that any two connected edges have labels with difference 1 (mod m), and denote by $\oplus$ and $\ominus$ the addition and subtraction in the modulo m group.

Case 1: $m = 4k + 1 (k \geq 1)$. Here (11) yields $P(C^2_{4k+1}) \leq k(4k+1)$. Tightness of this bound follows from consideration of the packing

$$\mathscr{P} = \left\{(i, 2i \oplus 4j) : 0 \leq i \leq 4k,\ 0 \leq j \leq k-1\right\}.$$

Case 2: $m = 4k + 3$. Here (11) implies $P(C_{k+3}) \leq 4k^2 + 5k + 1$ and tightness of this bound follows from the consideration of the construction $\mathscr{P}' = \bigcup_{i=0}^{m-1} \mathscr{P}_i$, where

$$\mathscr{P}_i = \begin{cases} \{(i, 2j \ominus \frac{i}{2}) : j = 0, 1, \ldots, k\}, & \text{if } i \text{ is even and } \quad i \neq 4k+2, \\ \{(i, 2j \ominus \frac{i}{2}) : j = 0, 1, \ldots, k-1\}, & \text{if } \quad i = 4k+2, \\ \{(i, 2j \ominus \frac{i-1}{2}) : j = k+1, \ldots, 2k\}, & \text{if } i \text{ is odd}. \end{cases}$$

□

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