Nonnegative Matrices, Zero Patterns, and Spectral Inequalities

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ABSTRACT

We give an inequality for the spectral radius of positive linear combinations of tuples of nonnegative matrices linked in a certain way. From this several observations are deduced, including the perturbation inequality $\rho(A + t\,\text{sgn}(A)) \geqslant \rho((A) + t\,\text{sgn}(\rho(A)))$, in which sgn denotes the signum function, applied componentwise. This was motivated in part by our work on characterizing the functions f such that $\rho(f(A_1, \ldots, Ap)) \leqslant f(\rho(A_1), \ldots, \rho(Ap))$, in which f is applied componentwise on the left, to be published elsewhere. One of our observations, the question of studying the combinatorial structure of the cone (algebra) of nonnegative matrices with given left and right Perron vectors.

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1. INTRODUCTION

We consider throughout n-by-n, componentwise nonnegative matrices. If A is such a matrix, we denote its spectral radius, or Perron eigenvalue, by $\rho(A)$. We say that a componentwise nonnegative (nonzero) vector $x = (x_1, \ldots, x_n)^T$ is a right Perron eigenvector of A if

$$Ax = \rho(A)x,$$

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and that a componentwise nonnegative (nonzero) vector $y = (y_1, \ldots, y_n)^T$ is a left Perron eigenvector of A if

$$A^T y = \rho(A)y.$$

Of interest will be the Hadamard product

$$z = x \circ y$$

of a left and right eigenvector, in which

$$z = (z_1, \ldots, z_n)^T = (x_1 y_1, \ldots, x_n y_n)^T.$$

If z is nonzero, we shall typically take it to be normalized so that

$$z^T e = 1,$$

in which $e = (1, 1, \ldots, 1)^T$.

We shall be interested in spectral inequalities involving the Perron root of one or more nonnegative matrices. Of the several rather different appearing inequalities we present, the most central one involves the spectral radius of a weighted sum of nonnegative matrices. It should be noted that in general there is no relationship between the spectral radius of a sum of nonnegative matrices and the sum of the spectral radii; for example, for nonnegative n-by-n A and B, $\rho(A + B)$ can be larger than, less than, or equal to $\rho(A) + \rho(B)$. Of course, if A and B share a right or a left Perron eigenvector, then $\rho(A + B) = \rho(A) + \rho(B)$. We show that if several irreducible nonnegative matrices $A_1, \ldots, A_k$ share something much less, namely a common z-vector as defined above, then

$$\rho(A_1 + \cdots + A_k) \geqslant \rho(A_1) + \cdots + \rho(A_k)$$

at least. Furthermore, there is a certain converse to this statement. These ideas are developed in Section 2.

The inquiry that motivated discovery of many of the results here was the characterization of the functions $f: R_+^p \to R_+$ such that

$$\rho(f(A_1, \ldots, A_p)) \leqslant f(\rho(A_1), \ldots, \rho(A_p)).$$

Here, $A_1, \ldots, A_p$ are arbitrary n-by-n nonnegative matrices and f is applied to $A_1, \ldots, A_p$ componentwise.

One function f that enjoys this functional property for $p = 1$ turns out to be the f defined by

$$f(x) = \max\{0, x - a\}$$

for a fixed $a \in \mathbb{R}^+$. Equivalent to the case $a = 1$ is a fact that appears to be of interest by itself. Consider the signum function defined on $\mathbb{R}^+$ by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Extend this function to nonnegative matrices componentwise, so that $\operatorname{sgn}(A)$ has 1's where A has positive entries and 0's where A has 0's. We then have the inequality

$$\rho(A + \operatorname{sgn}(A)) \geq \rho(A) + \operatorname{sgn}(\rho(A)). \tag{1.1}$$

One view of this inequality is that it generalizes a certain inequality for nonnegative real numbers. Let $a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$ be two sets of n nonnegative numbers. We then have the known inequality

$$[(a_1 + b_1) \cdots (a_n + b_n)]^{1/n} \geq (a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n}, \tag{1.2}$$

which is equivalent to its special case in which $b_i = \operatorname{sgn}(a_i)$. This is easily seen to be the special case of (1.1) in which

$$A = \begin{bmatrix} 0 & a_1 & 0 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & 0 \\ & 0 & & & & \cdot & a_{n-1} \\ a_n & & & & & & 0 \end{bmatrix}.$$

The inequality (1.1) and, in fact, stronger variants are developed in Section 3.

Another inequality [which turns out to be closely related to (1.1)] is noted in Section 4. Given nonnegative A with left and right Perron eigenvectors y

and x and signum matrix S, we have

$$y^T x \leqslant y^T S x \qquad (1.3)$$

(in fact a stronger result is given). If we consider the cone of all nonnegative matrices with a left Perron eigenvector y and a right Perron eigenvector x, it is then clear that there are restrictions upon the zero pattern of matrices in this cone. A natural question for further research is to fully understand such restrictions.

2. THE SPECTRAL RADIUS OF A WEIGHTED SUM

We shall need the following result:

LEMMA 2.1.   *Let A be an n-by-n matrix, $A \geqslant 0$, irreducible with spectral radius $\rho(A) = \rho$, and associated right and left Perron eigenvectors x, y:*

$$Ax = \rho x, \qquad A^T y = \rho y, \qquad (2.1)$$

normalized, such that $z = x \circ y$ satisfies $z^T e = 1$. Then

$$\rho(A) = y^T A x \leqslant \prod \left[\frac{(Av)_i}{v_i} \right]^{z_i} \leqslant u^T A v$$

$$for \quad u > 0, \quad v > 0, \quad u \circ v = z. \qquad (2.2)$$

If $w > 0$, $w^T e = 1$, $w \neq z$, then there exist $u > 0$, $v > 0$ such that

$$u \circ v = w, \qquad u^T A v < y^T A x = \rho(A). \qquad (2.3)$$

The proof of (2.2) can be found in [5]; different proofs are in [2] and [1]. The weaker result $\rho(A) \leqslant u^T A v$, known to Fiedler since 1970, was published in [4]. (2.3) is in [7].

Let us point out in passing that the inequality $y^T A x \leqslant u^T A v$ in (2.2) has a nice geometric interpretation: Consider the set S of all positive vectors η such that $\text{diag}(\eta_i) - A$ is a (possibly singular) M-matrix. Obviously $\rho e \in \partial S$. Then the hyperplane $H = \{\xi : \xi^T z = \rho(A)\}$ is the unique supporting hyper-

plane of S at the point ρe. This can be seen as follows: For $\eta \in S$ there exists $w > 0$ such that $Aw \leqslant \mathrm{diag}(\eta_i)\, w$. But then

$$\eta^T z = \sum_{i=1}^{n} \eta_i z_i \geqslant \sum_{i=1}^{n} \frac{(Aw)_i}{w_i} z_i \geqslant \rho(A)$$

by (2.2) with equality for $\eta = \rho e$. Equation (2.3) gives the uniqueness. We now prove the first main result:

THEOREM 2.1. *Let A_i be irreducible n-by-n nonnegative matrices and*

$$A_i x^{(i)} = \rho(A_i) x^{(i)}, \qquad A_i^T y^{(i)} = \rho(A_i) y^{(i)}, \tag{2.4}$$

$i = 1, \ldots, p,$ *such that*

$$x^{(i)} \circ y^{(i)} = x^{(j)} \circ y^{(j)} = z, \qquad i, j = 1, \ldots, p, \tag{2.5}$$

and $e^T z = 1$. Then for $t_i \geqslant 0$, $i = 1, \ldots, p$,

$$\rho\left(\sum_{i=1}^{p} t_i A_i \right) \geqslant \sum_{i=1}^{p} t_i \rho(A_i). \tag{2.6}$$

Proof. Let $g > 0$ be a right Perron eigenvector of $\sum_{i=1}^{p} t_i A_i$:

$$\left(\sum_{i=1}^{p} t_i A_i \right) g = \rho\left(\sum_{i=1}^{p} t_i A_i \right) g,$$

and $h > 0$ such that $h \circ g = z$. We infer from (2.2) and (2.5)

$$h^T A_i g \geqslant y^{(i)T} A_i x^{(i)} = \rho(A_i). \tag{2.7}$$

Multiplying (2.7) by $t_i \geqslant 0$ and summing gives

$$\rho\left(\sum_{i=1}^{p} t_i A_i \right) = h^T\left(\sum_{i=1}^{p} t_i A_i \right) g \geqslant \sum_{i=1}^{p} t_i \rho(A_i).$$

We note that the special case in which $A_1 = A$, $A_2 = B^T$ and $x^{(1)} = y^{(2)}$, $x^{(2)} = y^{(1)}$ may be found as Theorem 3 of [1]. Observe that this also includes

Levinger's inequality

$$\rho\left(tA + (1 - t)A^T\right) \geqslant \rho(A), \qquad 0 \leqslant t \leqslant 1. \tag{2.8}$$

The condition (2.5) seems to be very strong. However, the next theorem, which can be viewed as a converse to Theorem 2.1, shows that a somewhat stronger version of (2.6) implies (2.5). Only the case $p = 2$ is treated.

THEOREM 2.2. *Let* A_1, A_2 *be two nonneggative irreducible n-by-n matrices*

$$A_i x^{(i)} = \rho(A_i)x^{(i)}, \qquad A_i^T y^{(i)} = \rho(A_i)y^{(i)}, \qquad i = 1, 2, \tag{2.9}$$

where the eigenvectors have been normalized so that $x^{(i)T}y^{(i)} = 1$, $i = 1, 2$. *Suppose that for* $t \geqslant 0$ *and all positive diagonal matrices* D *we have*

$$\rho\left(A_1 + tDA_2D^{-1}\right) \geqslant \rho(A_1) + t\rho(A_2). \tag{2.10}$$

Then

$$x^{(1)} \circ y^{(1)} = x^{(2)} \circ y^{(2)}. \tag{2.11}$$

Proof. Denoting for fixed D the right-hand side of the inequality (2.10) by $\varphi(t)$ and the left-hand side by $\psi(t)$, we have $\psi(t) \geqslant \varphi(t)$ and $\psi(0) = \varphi(0)$. This implies

$$\left.\frac{d}{dt}\psi(t)\right|_{t=0} \geqslant \left.\frac{d}{dt}\varphi(t)\right|_{t=0},$$

or

$$y^{(1)T}DA_2D^{-1}x^{(1)} \geqslant \rho(A_2). \tag{2.12}$$

If (2.11) fails to hold, then (2.3) applied to A_2 shows that there exists a positive diagonal matrix D such that (2.12) is violated. ∎

We present now a theorem, which is a generalization of Theorem 2.1 and Theorem 4 in [1].

THEOREM 2.3. *Under the assumptions of Theorem* 2.1 *let* $\Delta_j = \text{diag}(\Delta_1^j, \ldots, \Delta_n^j)$, $\Delta_i^j > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, p$. *Then*

$$\rho\left(\sum_{j=1}^p \Delta_j A_j\right) \geq \sum_{j=1}^p \rho(A_j) \prod_{i=1}^n \left(\Delta_i^j\right)^{z_i}. \tag{2.13}$$

Proof. Let $u > 0$, $v > 0$ be positive vectors, and let A be any of the matrices $A_i (i = 1, \ldots, p)$. Then

$$u^T A v = \sum_{i=1}^p \frac{(Av)_i}{v_i} \frac{v_i u_i}{z_i} z_i \geq \prod_{i=1}^n \left[\frac{(Av)_i}{v_i} \frac{v_i u_i}{z_i}\right]^{z_i}$$

$$\geq \rho(A) \prod_{i=1}^n \left(\frac{u_i v_i}{z_i}\right)^{z_i} \quad \text{by (2.2).} \tag{2.14}$$

If we define now $h > 0$, $g > 0$ by

$$\left(\sum_{i=1}^p \Delta_i A_i\right) g = \rho\left(\sum_{i=1}^p \Delta_i A_i\right) g, \qquad g \circ h = z,$$

then by using (2.14) we get

$$h^T \Delta_j A_j g \geq \rho(A_j) \sum_{i=1}^n \left(\frac{h_i g_k \Delta_i^j}{z_i}\right)^{z_i} = \rho(A_j) \prod_{i=1}^n \left(\Delta_i^j\right)^{z_i}.$$

Summing over j gives (2.13). ■

3. PERTURBATION BY THE SIGNUM MATRIX

We are now in a position to prove the inequalities (1.1) addressed in the introduction.

THEOREM 3.1. *Let* $A \geq 0$ *be an n-by-n matrix. Then*

$$\rho(A + \text{sgn}(A)) \geq \rho(A) + \text{sgn}(\rho(A)). \tag{3.1}$$

Proof. Considering the reducible normal form of A [6], we see that it is sufficient to prove (3.1) for the 1×1 zero matrix (for which it is trivial) and for irreducible A. Hence consider the case in which A is irreducible. If D is a positive diagonal matrix, and $t > 0$ such that

$$\operatorname{sgn}(A) \geqslant tDAD^{-1}, \tag{3.2}$$

then by (3.2) and (2.6) we have

$$\rho(A + \operatorname{sgn}(A)) \geqslant \rho(A + tDAD^{-1}) \geqslant (1 + t)\rho(A). \tag{3.3}$$

The maximal t in (3.2) is obviously given by $t = t_0$, where

$$t_0^{-1} = \min_{d_1, \dots, d_n > 0} \ \max_{1 \leqslant i, k \leqslant n} \left(a_{ik} \frac{d_i}{d_k} \right) := \mu(A).$$

This well-known quantity appears in several applications; for an overview on different characterizations of $\mu(A)$ see [3]. In particular it is known that $(1/n)\rho(A) \leqslant \mu(A) \leqslant \rho(A)$. Hence we have by (3.2) and (3.3)

$$\rho(A + \operatorname{sgn}(A)) \geqslant (1 + t_0)\rho(A) = \rho(A) + \frac{\rho(A)}{\mu(A)} \geqslant \rho(A) + 1. \qquad \blacksquare$$

We see from the proof that we have actually

$$\rho(A + \operatorname{sgn}(A)) \geqslant \rho(A) + \frac{\rho(A)}{\mu(A)} \tag{3.4}$$

for A irreducible. Also, upon replacing A by $t^{-1}A$ ($t > 0$) and multiplying by t, we get

$$\rho(A + t\operatorname{sgn}(A)) \geqslant \rho(A) + t\operatorname{sgn}(\rho(A))\frac{\rho(A)}{\mu(A)}. \tag{3.5}$$

We observe that (since $\rho(A)/\mu(A) \geqslant 1$) (3.1) and (3.5) imply

$$\rho(g(A)) \geqslant g(\rho(A))$$

where $g: x \rightarrow x + t \operatorname{sgn}(x)$ and $g(A)$ is understood elementwise as in the introduction.

A consequence of (3.1) is the following: If $f(x) = \max(0, x - 1)$ then

$$\rho(f(A)) \leqslant f(\rho(A)). \tag{3.6}$$

For a proof, consider first the case $\rho(A) \leqslant 1$. In this case all cyclic products of A are less than or equal to 1; hence in each cycle there is at least one element $\leqslant 1$. But then all cyclic products of $f(A)$ vanish, i.e. $\rho(f(A)) = 0$, and (3.6) holds.

Consider now the remaining case $\rho(A) > 1$, $\rho(f(A)) > 0$. But then, as $f(A) + \operatorname{sgn} f(A) \leqslant A$, we infer

$$\rho(A) \geqslant \rho(f(A) + \operatorname{sgn} f(A)) \geqslant \rho(f(A)) + \operatorname{sgn} \rho(f(A)) = \rho(f(A)) + 1$$

and

$$f(\rho(A)) = \rho(A) - 1 \geqslant \rho(f(A)).$$

4. THE BILINEAR FORM OF THE SIGNUM MATRIX

Our goal in this section is to observe

THEOREM 4.1. *Let A be an n-by-n nonnegative matrix, and let y and x be, respectively, left and right Perron eigenvectors of A. If $S = \operatorname{sgn}(A)$, we have*

$$\mu(A)y^{T}Sx \geqslant \rho(A)y^{T}x. \tag{4.1}$$

In the event that A is irreducible, the inequality (4.1) may be proven by differentiating (3.5) and evaluating at $t = 0$. However, in general, especially when the left and right Perron eigenspaces are not one-dimensional, it is not clear how to adapt such a proof for arbitrary nonnegative eigenvectors. For this reason, we need a lemma that strengthens a portion of Lemma 2.1.

LEMMA 4.1. *Let A be an n-by-n nonnegative matrix, and suppose that x and y are, respectively, right and left Perron eigenvectors of A such that*

$y^T x = 1$. *Then for any nonnegative vectors u, v such that $u \circ v = x \circ y = z$ we have*

$$u^T A v \geqslant \prod_{i: z_i \neq 0} \left[\frac{(Av)_i}{v_i} \right]^{z_i} \geqslant y^T A x = \rho(A). \tag{4.2}$$

Proof. As (4.2) is true for $\rho(A) = 0$, we assume $\rho(A) = 1$, so that $Ax = x$, $y^T A = y^T$. For a nonnegative vector $w = (w_1, \ldots, w_n)^T$ define $J_w = \{i: w_i > 0\}$. Introduce the partition $\bigcup_{i=1}^4 I_i = \{1, \ldots, n\}$ in the following manner: $I_1 = J_z$, $I_2 = J_x \div J_z$, $I_3 = J_y \div J_z$, $I_4 = \{1, \ldots, n\} \div J_{x+y}$. This leads to a partition of any vector w: $w = (w_1^T, w_2^T, w_3^T, w_4^T)^T$, and we have in particular $x_1 > 0$, $x_2 > 0$, $x_3 = 0$, $x_4 = 0$ and $y_1 > 0$, $y_2 = 0$, $y_3 > 0$, $y_4 = 0$. If $A = (A_{ij})_{i,j=1}^4$ is the associated block partition of A, we have

$$A_{11}x_1 + A_{12}x_2 = x_1, \qquad A_{31}x_1 + A_{32}x_2 = x_3 = 0,$$

$$y_1^T A_{11} + y_3^T A_{31} = y_1^T, \qquad y_1^T A_{12} + y_3^T A_{32} = y_2^T = 0,$$

which yields $A_{31}x_1 = 0$ and, as $x_1 > 0$, $A_{31} = 0$. Similarly $A_{32} = 0$, $A_{12} = 0$, and hence

$$A_{11}x_1 = x_1, \qquad y_1^T A_{11} = y_1^T.$$

From [6, Theorem 7, p. 78] we infer that (up to a permutation)

$$A_{11} = A_1 \otimes \cdots \otimes A_q, \tag{4.3}$$

where the A_i are irreducible and $\rho(A_i) = \rho(A_{11}) = 1$. Having established this result, (4.2) is now a simple consequence of (2.2): Subdivide $u_1, v_1, x_1, y_1, z_1, I_1$ according to (4.3), e.g. $u_1 = (u_{1,1}^T, u_{1,2}^T, \ldots, u_{1,q}^T)^T$, $I_1 = \bigcup_{i=1}^q I_{1,i}$. We now have

$$u^T A v \geqslant \sum_{i \in I_1} u_i (Av)_i = \sum_{i \in I_1} z_i \frac{(Av)_i}{v_i} \geqslant \prod_{i \in I_1} \left[\frac{(Av)_i}{v_i} \right]^{z_i} \tag{4.4}$$

by the weighted arithmetic–geometric-mean inequality. The product in (4.4)

can be written in the form $\prod_{j-1}^{q} P_j$, where

$$P_j = \prod_{i \in I_{1,j}} \left[\frac{(A_j v_{1,j})_i}{(v_{1,j})_i} \right]^{z_i}.$$

As $A_j x_{1,j} = x_{1,j}$, $y_{1,j}^T A_j = y_{1,j}^T$, and $v_{1,j} \circ u_{1,j} = y_{1,j} \circ x_{1,j} = z_{1,j}$, (2.2) applied to A_j gives $P_j \geqslant 1$. This together with (4.4) shows the inequality (4.2). ∎

The proof of Theorem 4.1 may now be completed easily. Since $\rho(A) = 0$ exactly when $\mu(A) = 0$, (4.1) clearly holds for $\rho(A) = 0$; so assume $\rho(A) > 0$. In this event, choose a positive diagonal matrix D so that

$$\mu(A)S \geqslant D^{-1}AD$$

componentwise. Then, using Lemma 4.1,

$$\mu(A)y^T Sx \geqslant y^T D^{-1}ADx \geqslant y^T Ax = \rho(A)y^T x,$$

which verifies Theorem 4.1.

Since $\mu(A) \leqslant \rho(A)$, we have

COROLLARY 4.1. *Under the circumstances of Theorem 4.1, we have*

$$y^T \mathrm{sgn}(A) x \geqslant \mathrm{sgn}(\rho(A)) y^T x.$$

The corollary makes it clear that there are nontrivial relationships between certain cones of nonnegative matrices. For a given directed graph G on n vertices, let C_G be the cone of all nonnegative matrices $A = (a_{ij})$ such that $a_{ij} > 0$ implies (i, j) is an edge of G. For nonnegative n-vectors x, y, let $C_{x,y}$ be the cone of all n-by-n nonnegative matrices A such that x is a right Perron eigenvector and y is a left Perron eigenvector of A. Let S be the incidence matrix of the undirected graph G. The corollary provides a condition for $C_{x,y}$ and C_G to intersect nontrivially, namely,

$$y^T Sx \geqslant y^T x.$$

For example, if $x^T = y^T = (1,3)$, S could not be $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Further investigation of such combinatorial conditions upon the cones $C_{x,y}$ seems warranted.

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