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Analytic continuation of thermal N-point functions from imaginary to real energies

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Thermal n-point Green functions in the framework of quantum field theory at finite temperature are considered. We reanalyze how analytic continuations from imaginary to real energies relate these functions originally defined in the imaginary-time formalism to retarded and advanced real-time ones. We described a new and rather simple method which is valid to all orders of perturbation theory and which has the further advantage that it is independent of approximations often applied in actual finite-order calculations.

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I. INTRODUCTION

In quantum field theory at finite temperature and density two convenient formalisms that enable the use of conventional Feynman rules in momentum space are applied: the imaginary-time formalism (ITF) [1] and the real-time formalism (RTF) [2,3]. (For books and reviews on thermal field theory, see, e.g. [4–7].) The former one is particularly suited for the evaluation of the static or thermodynamic quantities of finite-temperature systems, while the latter is preferred for the evaluation of time-dependent quantities.

Studies of the two-point functions have a long history [1–8]. The relationship between their representations, the one obtained from ITF and the other from RTF, is well known [8]. On the other hand the interest in the three- and $n(\geq 4)$-point functions started rather recently [9–14].

An apparent difference [10] between the results for three-point functions obtained from the ITF and from the RTF has posed the following question: through analytic continuations of the Green functions evaluated in ITF, what kind of (combinations of) thermal Green functions in RTF are obtained?

Several papers have been devoted to this issue. It has been shown, either to one-loop order or in all orders of perturbation theory, that the three-point function in ITF, when analytically continued to real energies, becomes the retarded or advanced three-point Green function [11,12]. Recently, the n (≥ 4)-point functions have been investigated [13,14] and it is shown that different analytic continuations of the ITF result yield different RTF Green functions, including the retarded and advanced Green functions.

The purpose of this paper is to demonstrate a simpler and clearer derivation of the general results known from [13]. We are not claiming new relations, but show how the most straightforward analytic continuation of the nonamputated n-point Green function in ITF leads to the retarded or advanced n-point Green function.

In Sec. II, we introduce the n-point thermal Green functions with time arguments on a, to a large extent arbitrary, contour in the complex time plane. We then formulate the problem of analytic continuation of the Green function in ITF from imaginary to real energies. For the purpose of illustrating our procedure, we discuss in Sec. III the analytic continuation of the two-point function. In Sec. IV, we carry out the analytic continuation of the n-point Green functions evaluated in the ITF, and obtain the retarded and advanced Green functions. Section V is devoted to conclusions.

II. PRELIMINARIES

Throughout this paper we consider a real scalar field $\phi(x)$. Generalizations to other kinds of fields are straightforward. The thermal Green functions are defined as the statistical average of a product of Heisenberg fields:

$$\begin{aligned}G(\{t\}) &= G(t_1,t_2,\ldots,t_n)\\ &\equiv \mathrm{Tr}\{e^{-\beta H}T_c[\phi(t_1)\phi(t_2)\cdots\phi(t_n)]\}/\mathrm{Tr}e^{-\beta H}\\ &\equiv \langle T_c[\phi(t_1)\phi(t_2)\cdots\phi(t_n)]\rangle\ ,\end{aligned} \tag{1}$$

where $\beta = T^{-1}$ is the inverse temperature and H is the Hamiltonian of the system such that $\phi(t) = e^{iHt}\phi(0)e^{-iHt}$. In Eq. (1) and in the following we suppress explicit reference to the space variables. The arguments $t_1,t_2,\ldots,t_n$ lie on the contour C running from an arbitrary time t_I down to $t_I - i\beta$ in the complex time plane. The symbol T_c in (1) is the time-ordering operator along this contour C. That is, it prescribes that the operators it is applied to be arranged in the order in which their arguments lie along C, with those nearest to the beginning at t_I to the right, and those nearest to the end $t_I - i\beta$ to the left.

To perform the thermal trace (1), we insert a complete set of states by choosing the Hamiltonian eigenstate basis:

$$G(\{t\})\mathrm{Tr}e^{-\beta H} = \sum_{p_n} \left[\prod_{j=1}^{n-1} \theta_c(t_{\bar{j}} - t_{\overline{j+1}})\right] \sum_{l_1,l_2,\ldots,l_n} \exp[-iE_{l_1}(t_{\bar{n}} - t_{\bar{1}} - i\beta)]$$
$$\times \langle l_1|\phi(0)|l_2\rangle \prod_{j=2}^{n} \left\{\exp\left[iE_{l_j}(t_{\bar{j}} - t_{\overline{j-1}})\right] \langle l_j|\phi(0)|l_{j+1}\rangle\right\} \quad (l_{n+1} \equiv l_1) , \tag{2}$$

with θ_c the contour step function. The summation here is carried out over all permutations of n numbers p_n :

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \bar{1} & \bar{2} & \cdots & \bar{n} \end{pmatrix} . \tag{3}$$

Following common practice, we assume that the convergence of the above trace sum (2) is controlled by the exponential factors, so that it converges if, for every pair of t_i and t_j such that $\theta_c(t_i - t_j) = 1$, $\mathrm{Im}(t_i - t_j) < 0$, and $\mathrm{Im}(t_s - t_l) < \beta$ with t_s (t_l) the "smallest" ("largest") time. This condition guarantees the existence of $G(\{t\})$ [Eq. (2)] as an analytic function of $\{t\} = \{t_1, t_2, \ldots, t_n\}$. The limit of an analytic function on the boundaries of its domain of definition, where it is still continuous, is a generalized function. This implies that the thermal Green function $G(\{t\})$ is well defined for $\mathrm{Im}(t_i - t_j) \leq 0$ when $\theta_c(t_i - t_j) = 1$, and $\mathrm{Im}(t_s - t_l) \leq \beta$. This imposes the restriction on C that as a point moves along C from t_I to $t_I - i\beta$ its imaginary part is nonincreasing. Then, an analytic continuation of $G(\{t\})$ can be done by deforming the contour C with the end points t_I and $t_I - i\beta$ held fixed, keeping in mind the "nonincreasing" condition for the imaginary part of the points on C.

Among the above class of contours, we consider first a special contour C_I, a straight line from t_I down to $t_I - i\beta$, which defines the ITF. We can choose any value for t_I because of the property of time-translation invariance of (1), and we choose t_I such that $\mathrm{Re}\, t_I < 0$ and $\mathrm{Im}\, t_I = 0$ [15]. (See Fig. 1.) Next we evaluate the Fourier component of (1):

$$G(\{\omega\}) = G(\omega_1, \omega_2, \ldots, \omega_n)$$
$$= \prod_{j=1}^{n} \left(\int_{C_I} dt_j\, e^{-\omega_j t_j}\right) G(t_1, t_2, \ldots, t_n),$$
$$\omega_j = 2\pi l_j/\beta , \qquad l_j = 0, \pm 1, \pm 2, \ldots, \pm\infty . \tag{4}$$

It is to be noted that real (discrete) ω_j's here are what we call imaginary energies. By using once more the time translation invariance we rewrite (4) as

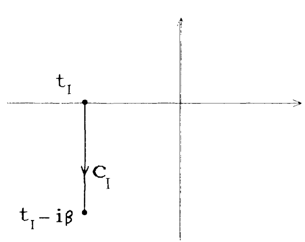

FIG. 1. The contour C_I in the complex-time plane, which defines the ITF.

$$G(\{\omega\}) = -i\beta\, \delta\left(\sum_{j=1}^{n} \omega_j \,;\, 0\right) \tilde{G}(\omega_2, \omega_3, \ldots, \omega_n) , \tag{5a}$$

$$\tilde{G}(\omega_2, \omega_3, \ldots, \omega_n)$$
$$= \prod_{j=2}^{n} \left(\int_0^{-i\beta} dt_j\, e^{-\omega_j t_j}\right) G(0, t_2, \ldots, t_n) . \tag{5b}$$

Here the integrations are performed along the imaginary-time axis. In (5a) $\delta(\cdots;\cdots)$ denotes the Kronecker δ symbol. In order to obtain (5) the Kubo-Martin-Schwinger (KMS) condition [16], which represents the invariance of the trace under the following cyclic permutation [cf. (1)],

$$\langle \phi(t_1) \ldots \phi(t_{n-1})\phi(t_n)\rangle = \langle \phi(t_n - i\beta)\phi(t_1) \ldots \phi(t_{n-1})\rangle \tag{6}$$

is used. In the ITF one calculates G or $\tilde{G}$ as defined in (5). In the following we focus our attention on how to continue (5) from imaginary to real energies.

III. TWO-POINT THERMAL GREEN FUNCTION

Although the relation between the ITF and RTF is well understood [8] for two-point thermal Green functions, we start with the thermal two-point Green function in the ITF for the purpose of illustrating our procedure:

$$G(\omega_1, \omega_2)$$
$$= \int_{C_I} dt_1 \int_{C_I} dt_2\, e^{-(\omega_1 t_1 + \omega_2 t_2)} \langle T_{C_I}[\phi(t_1)\phi(t_2)]\rangle . \tag{7}$$

As explained in Sec. II, the integrand of (7) is an analytic function of t_1 and t_2 in the strip, $-\beta < \mathrm{Im}\, t_j < 0$ ($j = 1, 2$) with $t_1 \neq t_2$, and we may deform the contour C_I keeping the property as mentioned above after (3). In this way we may obtain the contour $C_R = C_1 \oplus C_2 \oplus C_3$ as depicted in Fig. 2: the contour runs along the real axis from t_I to t_F, returns back to t_I along the real

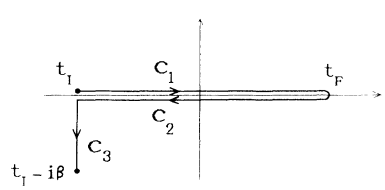

FIG. 2. The contour $C_R = C_1 \oplus C_2 \oplus C_3$ in the complex-time plane; the segments C_1 and C_2 lie on the real axis.

axis, and ends at $t_I - i\beta$. For a technical reason [7,15] it is a common practice to give the contours C_1 and C_2 infinitesimal downward slopes.

Because of time translation invariance $G(\omega_1,\omega_2)$ is nonvanishing only for $\omega_1+\omega_2=0$ [cf. (5a)], and the integrand of (7), on the path C_R, is a function of t_2-t_1. This, together with the above-mentioned analyticity property of $G(t_1,t_2)$, allows us to evaluate $G(\omega_1,\omega_2)$ by fixing t_1 on any point on the contour $C_R = C_1 \oplus C_2 \oplus C_3$. We first fix t_1 on C_1, to obtain

$$G(\omega_1,\omega_2) = -i\beta\,\delta(\omega_1;-\omega_2)\left[\int_{C_1} dt_2\, e^{-\omega_2(t_2-t_1)}\langle T_{C_1}[\phi(t_1)\phi(t_2)]\rangle + \int_{C_2\oplus C_3} dt_2\, e^{-\omega_2(t_2-t_1)}\langle\phi(t_2)\phi(t_1)\rangle\right] \quad (8a)$$

$$= -i\beta\,\delta(\omega_1;-\omega_2)\int_{t_I}^{t_F} dt_2\, e^{-\omega_2(t_2-t_1)}\left[\langle T[\phi(t_1)\phi(t_2)]\rangle - \langle\phi(t_2)\phi(t_1)\rangle\right] + (\text{contribution from } C_3)\ . \quad (8b)$$

The symbol T is the ordinary time-ordering symbol. Equation (8) is now well suited for the purpose of an analytic continuation of $G(\omega_1,\omega_2)$ to real energies. In order to arrive at the RTF with real energy $p_{20}(=-p_{10})$ we take the limit $t_I \to -\infty$ and $t_F \to +\infty$, taking into account that t_I may be chosen arbitrarily.

We realize (cf. [6]) that the term in the square brackets in (8b) may be written as

$$\langle T[\phi(t_1)\phi(t_2)]\rangle - \langle\phi(t_2)\phi(t_1)\rangle = \theta(t_1-t_2)\langle[\phi(t_1),\phi(t_2)]\rangle\ . \quad (9)$$

Then, from (8) and (9), we learn that the above limit, $t_I \to -\infty$ and $t_F \to +\infty$, can be taken if we continue to the real energy as follows:

$$\omega_2 \to -i(p_{20} - i\epsilon)\ , \quad (10a)$$

or equivalently,

$$\omega_1 \to -i(p_{10} + i\epsilon)\ , \quad (10b)$$

with ϵ an infinitesimal positive number.

The time path for G consists of two real-time segments, namely C_1 from $-\infty$ to $+\infty$ and C_2 from $+\infty$ to $-\infty$, and of C_3 from $-\infty$ to $-\infty - i\beta$, the standard contour which is employed in the literature [5–7]. The computation of (8) with (10) by perturbative methods, follows the standard rules known from perturbative real-time thermal field theory [3,5–7]: As far as the thermal Green functions [such as (8) with (10)] with their finite time arguments lying on C_1 and/or C_2 are concerned, the contribution from C_3 can be ignored, provided that $|p_0|$ is chosen as the argument of the statistical factors [7,17] (see also [18]).

The analytic continuation of the energy conservation δ function in (8) is given by the prescription [7]: $\delta(\omega_1;-\omega_2) \to 2\pi i\beta^{-1}\delta(p_{10}+p_{20})$.

Thus we obtain the analytically continued G with real energies p_{10} and p_{20},

$$\lim_{\epsilon\to+0} G(p_{10}+i\epsilon,\, p_{20}-i\epsilon) = G_{++} - G_{+-} \quad (11a)$$

$$= 2\pi\delta(p_{10}+p_{20}) \times \lim_{\epsilon\to+0}\int_{-\infty}^{\infty} dt\, e^{i(p_{20}-i\epsilon)t}\,\theta(-t)\langle[\phi(0),\phi(t)]\rangle\ , \quad (11b)$$

where we follow, e.g., [6] and introduce

$$G_{\alpha\beta}(p_{10},p_{20}) = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, e^{i(p_{10}t_1+p_{20}t_2)}\langle A_{\alpha\beta}(t_1,t_2)\rangle\ , \quad (12)$$

with the definitions

$$\begin{aligned} A_{++}(t_1,t_2) &= \theta(t_1-t_2)\phi(t_1)\phi(t_2) + \theta(t_2-t_1)\phi(t_2)\phi(t_1),\\ A_{+-}(t_1,t_2) &= \phi(t_2)\phi(t_1),\\ A_{--}(t_1,t_2) &= \theta(t_1-t_2)\phi(t_2)\phi(t_1) + \theta(t_2-t_1)\phi(t_1)\phi(t_2),\\ A_{-+}(t_1,t_2) &= \phi(t_1)\phi(t_2)\ . \end{aligned} \quad (13)$$

The functions $G_{\alpha\beta}$ $(\alpha,\beta=+,-)$ in (12) denote Green functions in the two-component real-time thermal field theory, applying the closed time-path formalism introduced by Schwinger and Keldysh [3]. It is to be noted, in passing, that the $A_{\alpha\beta}$'s are not independent since they satisfy the relation [6]

$$A_{++}(t_1,t_2) + A_{--}(t_1,t_2) - A_{+-}(t_1,t_2) - A_{-+}(t_1,t_2) = 0\ . \quad (14)$$

It is important to note that the continued function G in (11b) is identical with the retarded Green function. In this way the "physical" representation as discussed in [6] is established, where this function is denoted by $G = iG_{21}(p_{10},p_{20})$; however, the primary quantities that are calculated directly in real-time thermal field theory are $G_{\alpha\beta}$ in (12).

When we fix t_2, instead of t_1, on C_1, and repeating similar steps as above, we obtain the retarded Green function *with respect to* $\phi(t_2)$, i.e., the advanced Green function with respect to $\phi(t_1)$. Fixing the time variable as $t_1 \in C_2$ leads to the same result as above, (11), and likewise for the choice $t_2 \in C_2$. Fixing t_1 on C_3, $t_1 \in C_3$, is not suitable for analytic continuations under consideration; this is also the case for the choice $t_2 \in C_3$.

Finally we may deform the contour C_I in (7) to the one that is mirror symmetric to the one of Fig. 2. Changing the time variable t_j as $t_j \equiv \mathrm{Re}\,t_j + i\mathrm{Im}\,t_j \to -\mathrm{Re}\,t_j + i\mathrm{Im}\,t_j$, we get back the time path $C_1 \oplus C_2 \oplus C_3$ of Fig. 2. Then, fixing t_1 on the upper path on the real axis in the complex time plane, we deduce

$$\lim_{\epsilon\to+0} G(p_{10}-i\epsilon,\, p_{20}+i\epsilon) = -G_{++} + G_{-+} \tag{15a}$$

$$= -2\pi\delta(p_{10}+p_{20}) \times \lim_{\epsilon\to+0}\int_{-\infty}^{\infty} dt\, e^{i(p_{20}+i\epsilon)t}\,\theta(t)\langle[\phi(0),\phi(t)]\rangle\ , \tag{15b}$$

i.e., the advanced Green function.

IV. THERMAL N-POINT GREEN FUNCTION

We proceed in a similar manner as in Sec. III. Starting with the thermal n-point function in the ITF, Eq. (4), we deform the contour C_I to C_R as depicted in Fig. 2. Under the constraint $\sum_j \omega_j = 0$ [cf. (5a)], we evaluate (4) with C_R for C_I by fixing t_1 on C_1, $t_1 \in C_1$, which is suitable for continuation to real energies. In place of Eq. (8) we now have

$$G(\{\omega\}) = -i\beta\,\delta\left(\sum_j \omega_j;0\right)\prod_{j=2}^{n}\left(\int_{C_1\oplus C_2} dt_j\, e^{-\omega_j(t_j-t_1)}\right) \times \langle T_{C_1\oplus C_2}[\phi(t_1)\cdots\phi(t_{n-1})\phi(t_n)]\rangle + \cdots \qquad (t_1\in C_1)\ , \tag{16a}$$

$$= -i\beta\,\delta\left(\sum_j \omega_j;0\right)\prod_{j=2}^{n}\left(\int_{t_I}^{t_F} dt_j\, e^{-\omega_j(t_j-t_1)}\right) \times \left[\sum_{\alpha_2,\alpha_3,\cdots,\alpha_n=+,-}(-)^s\langle T_{C_1\oplus C_2}[\phi_+(t_1)\phi_{\alpha_2}(t_2)\cdots\phi_{\alpha_n}(t_n)]\rangle\right] + \cdots\ , \tag{16b}$$

where

$$s = \frac{1}{2}\sum_{j=2}^{n}(1-\alpha_j)\ , \tag{17}$$

and the notation of [6] is used: for $n > 2$ it becomes more transparent to make the subscripts $+,-$ explicit, which indicate that the times t_i assume values on either the "positive" time path C_1 or on the "negative" one C_2; e.g., this implies for all $+$ subscripts time ordering, whereas for all $-$ subscripts antitime ordering. This generalizes the expressions in (13) to the $n \geq 3$ cases (for n=3 explicit expressions are given in [6]). In Eq. (16) the dots indicate the contributions when some of the t's among $t_2,\ldots,t_n$ are on C_3; this portion may be ignored as in Sec. III.

Next we transform to the "physical" representation. By applying the procedure described in [6], we express the term in the square brackets in (16b) as

$$\sum_{p_{n-1}}\theta(1,\overline{2},\ldots,\overline{n})\langle[\ldots[\phi(t_1),\,\phi(t_{\overline{2}})],\,\phi(t_{\overline{3}})],\ldots],\,\phi(t_{\overline{n}})]\rangle\ , \tag{18}$$

where θ is the multistep function defined by $\theta(1,2,\ldots,n) = \theta(1,2)\ \theta(2,3)\ \ldots\ \theta(n-1,n)$ with $\theta(1,2) = \theta(t_1-t_2)$. Equation (18) is the generalization of the two-point function case Eq. (9). The summation here is carried out over all permutations of $n-1$ numbers p_{n-1}:

$$\begin{pmatrix} 2 & 3 & \cdots & n \\ \overline{2} & \overline{3} & \cdots & \overline{n}\end{pmatrix}\ . \tag{19}$$

Equation (18) tells us that t_1 is the largest time; therefore, we are allowed to take the limit $t_I \to -\infty$ and $t_F \to +\infty$ if we continue ω_j in (16b) to real energies as

$$\omega_j \to -i(p_{j0}-i\epsilon)\ ,\qquad j=2,3,\ldots,n, \tag{20a}$$

and

$$\omega_1 \to -i[p_{10}+i(n-1)\epsilon]\ . \tag{20b}$$

Then we arrive at the Green function in the RTF:

$$\lim_{\epsilon\to+0} G[p_{10}+i(n-1)\epsilon,\ \{p_{j0}-i\epsilon;\ j=2,3,\ldots,n\}] = \sum_{\alpha_2,\ldots,\alpha_n=+,-}(-)^s G_{+,\alpha_2,\ldots,\alpha_n} \tag{21a}$$

$$= 2\pi\,\delta\left(\sum_{j=1}^{n}p_{j0}\right)\lim_{\epsilon\to+0}\prod_{j=2}^{n}\left(\int_{-\infty}^{\infty}dt_j e^{i(p_{j0}-i\epsilon)(t_j-t_1)}\right) \times \sum_{p_{n-1}}\theta(1,\overline{2},\ldots,\overline{n})\langle[\cdots[\phi(t_1),\,\phi(t_{\overline{2}})],\,\phi(t_{\overline{3}})],\ldots],\,\phi(t_{\overline{n}})]\rangle \tag{21b}$$

$$\equiv i^{n-1}\tilde{G}_{211\cdots1}(p_{10},\ldots,p_{n0})\ . \tag{21c}$$

where s is given in (17). Thus we have derived the Green function (21c) in the "physical" representation, denoted by $\tilde{G}_{211\cdots 1}$ [6], which is the thermal n-point retarded function as seen in (21b). In (21a), $G_{+,\alpha_2,\ldots,\alpha_n}$ is a Green function in the RTF and it is defined [6] analogously to (12) and (13):

$$G_{\alpha_1,\alpha_2,\ldots,\alpha_n} = \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} dt_j e^{ip_{j0}t_j} \right) \times \langle T_{C_1 \oplus C_2} [\phi_{\alpha_1}(t_1)\phi_{\alpha_2}(t_2)\cdots\phi_{\alpha_n}(t_n)] \rangle . \quad (22)$$

As in the two-point function case, (14), there is one identity:

$$\sum_{\alpha_1,\ldots,\alpha_n=+,-} (-)^{s'} G_{\alpha_1,\alpha_2,\ldots,\alpha_n} = 0 \quad (23)$$

where

$$s' = \frac{1}{2}\sum_{j=1}^{n}(1-\alpha_j) . \quad (24)$$

It is worth mentioning that the primary quantities that are evaluated in real-time thermal field theory are $G_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ defined in (22), through which the retarded Green function G in (21b) is obtained.

We have developed the derivation by fixing t_1 on C_1. Of course, we may proceed in a similar manner by fixing other t_j ($2 \le j \le n$) on C_1: $n-1$ different retarded Green functions are the result.

In case Eq. (4) is evaluated with C_R for C_I by fixing t_1 on C_2, we obtain the same result as above, (21); $t_1 \in C_3$ is not suitable for analytic continuations.

When we deform the contour C_I in (4) to the one that is mirror symmetric to the one of Fig. 2, and fixing t_1 on the upper path on the real axis in the complex-time plane, i.e., the counterpart of C_1 in Fig. 2, we derive the advanced Green function

$$\lim_{\epsilon\to+0} G(p_{10} - i(n-1)\epsilon, \{p_{j0} + i\epsilon;\ j = 2,3,\ldots,n\}) = -2\pi\,\delta\left(\sum_{j=1}^{n} p_{j0}\right) \lim_{\epsilon\to+0} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} dt_j\, e^{i(p_{j0}+i\epsilon)(t_j-t_1)} \right) \times \sum_{P_{n-1}} \theta(\overline{n},\ldots,\overline{2},1)\langle [\cdots[\phi(t_1),\, \phi(t_{\overline{2}})],\, \phi(t_{\overline{3}})],\cdots],\, \phi(t_{\overline{n}})] \rangle . \quad (25)$$

V. CONCLUSIONS

In this paper we have addressed the question of what kind of thermal functions in RTF emerge by analytic continuations of the n-point thermal Green functions defined in the ITF. This amounts to performing analytic continuations in the energies of the external legs from the discrete imaginary values to real continuous ones.

The thermal Green functions are defined on a path in the complex-time plane, a path which is to a large extent arbitrary. On the basis of this observation, we have carried out the above-mentioned continuations in the most straightforward and familiar manner by deforming the contour, starting from the one that defines ITF to the one defining RTF. In this way, we show that ITF n-point Green functions become retarded or advanced thermal Green functions. Although the results obtained in this paper, being exact and valid independent of the approximation used in actual, mainly perturbative, calculations, are already known from [13], the simpler derivation presented here helps to enlighten the relation between the ITF and RTF.

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