Bounds for the Variation of the Roots of a Polynomial and the Eigenvalues of a Matrix

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ABSTRACT

We derive some new bounds for the distance between the roots of two polynomials in terms of their coefficients and for the distance between the eigenvalues of two matrices in terms of the norm of their difference.

1. INTRODUCTION

Let f and g be two monic polynomials of degree n with complex coefficients $a_1,\dots,a_n$ and $b_1,\dots,b_n$ respectively. Let $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$ be their respective roots:

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n=\prod_{i=1}^{n}(z-\alpha_i), \tag{1}$$

$$g(z)=z^n+b_1z^{n-1}+\cdots+b_n=\prod_{i=1}^{n}(z-\beta_i). \tag{2}$$

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One of the basic problems of interest in perturbation theory and numerical analysis is to estimate the distance between the roots α_i and β_i in terms of the coefficients a_i and b_i. Let

$$\Gamma = \max_{1 \leqslant k \leqslant n} \left(|a_k|^{1/k}, |b_k|^{1/k} \right), \qquad \gamma = 2\Gamma. \tag{3}$$

A celebrated result of Ostrowski [12, p. 276] states that the roots of f and g can be enumerated as $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ respectively in such a way that

$$\max_i |\alpha_i - \beta_i| \leqslant (2n-1) \left\{ \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k} \right\}^{1/n}. \tag{4}$$

This result was improved by Elsner [6], who showed that the factor $2n-1$ occurring in the right hand side of (4) can be replaced by $n-1$ when n is even and by n when n is odd. See also [1, p. 91].

Our first result gives a significant improvement of the above by replacing the factor $2n-1$ in (4) by a constant smaller than 4 for all n. We have:

THEOREM 1. *Let f, g, and γ be as defined in* (1), (2), *and* (3) *above. Then the roots of f and g can be enumerated as $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ in such a way that*

$$\max_i |\alpha_i - \beta_i| \leqslant 4 \times 2^{-1/n} \left\{ \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k} \right\}^{1/n}. \tag{5}$$

Next consider the space $\mathscr{B}(\mathbb{C}^n)$ of linear operators on $\mathbb{C}^n$, identified as usual with the space $\mathbb{M}(n)$ of $n \times n$ complex matrices. A problem similar to the one stated above is that of estimating the distance between the eigenvalues of two matrices in terms of that between the matrices themselves.

Let $\|\cdot\|$ be any norm on the vector space $\mathbb{C}^n$, and let $\|\cdot\|$ also denote the induced operator norm on $\mathscr{B}(\mathbb{C}^n)$, i.e., for $A \in \mathscr{B}(\mathbb{C}^n)$ define $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Of particular interest is the case of the usual Euclidean norm $\|\cdot\|_2$ on $\mathbb{C}^n$ and the operator norm $\|\cdot\|_2$ on $\mathscr{B}(\mathbb{C}^n)$. The latter is often referred to as the *spectral norm* in the numerical analysis literature.

Let A, B be any two $n\times n$ matrices with eigenvalues $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_n$ respectively. It is well known that these eigenvalues can be enumerated in such a way that

$$\max_i|\lambda_i-\mu_i|\leqslant c(n)(\|A\|_2+\|B\|_2)^{1-1/n}\|A-B\|_2^{1/n}, \tag{6}$$

where $c(n)$ is a constant growing with the dimension n. See [1, Chapter 5] for a survey of such results and for reference to earlier work. The best result of this type was obtained by Elsner [7], who showed that (6) is true with $c(n)=n$ or $n-1$ according to whether n is odd or even.

It has long been conjectured (see, e.g., [9]) that the constant $c(n)$ occurring in (6) can be replaced by an absolute constant independent of n. This conjecture was recently proved by Dennis Phillips [13]. Our next two results are improvements of this result of Phillips.

THEOREM 2. *Let A and B be any two $n\times n$ matrices. Then their eigenvalues can be enumerated as $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_n$ in such a way that*

$$\max_i|\lambda_i-\mu_i|\leqslant 4\times 2^{-1/n}(\|A\|_2+\|B\|_2)^{1-1/n}\|A-B\|_2^{1/n}. \tag{7}$$

THEOREM 3. *Let A and B be any two $n\times n$ matrices. Let $\|\cdot\|$ be any operator norm on the space of matrices. Then the eigenvalues of A and B can be enumerated as $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_n$ in such a way that*

$$\max_i|\lambda_i-\mu_i|\leqslant 4\times 2^{-1/n}\times n^{1/n}(2M)^{1-1/n}\|A-B\|^{1/n}, \tag{8}$$

where $M=\max(\|A\|,\|B\|)$.

(We should warn the reader here that in the monograph [1] the symbol $\|\cdot\|$ is consistently used for what has been called $\|\cdot\|_2$ above.)

In Section 2 we give proofs of these results. In Section 3 we discuss how sharp these estimates are and how they could possibly be improved. Section 4 contains some related remarks.

There are two crucial ingredients in our proofs. One is the use of a homotopy method which has been repeatedly employed (see [4–6, 12]). The other is an ingenious use of Chebyshev polynomials made by Phillips in his recent paper [13]. Using this, Phillips obtains the inequality (7) above with the constant 8 instead of 4 and a similar result for other operator norms. While our proofs of Theorems 2 and 3 are simpler and our results stronger than the corresponding ones in [13], the essential ideas used in our proof are the same as used there by Phillips.

2. PROOFS OF THE MAIN RESULTS

A common ingredient in all the proofs will be the following:

LEMMA 1. *Let Γ be a continuous curve in the complex plane with end points a and b. Let $\lambda_1, \dots, \lambda_n$ be any given points in the plane. Then there exists a point λ on Γ such that*

$$\prod_{i=1}^{n} |\lambda - \lambda_i| \geqslant \frac{|b-a|^n}{2^{2n-1}}. \tag{9}$$

Proof. Let L denote the straight line through a and b, and let S be the segment

$$S = \{z : z = a + t(b-a), 0 \leqslant t \leqslant 1\}.$$

Given any point z in the plane, we denote by z' its orthogonal projection onto L. Then $|z-w| \geqslant |z'-w'|$ for all z and w. For the given points λ_i let their projections λ'_i be parametrized as $\lambda'_i = a + t_i(b-a)$ for some $t_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. Let z be any point on L given by $z = a + t(b-a)$ for some $t \in \mathbb{R}$. Then

$$\prod_{i=1}^{n} |z - \lambda_i'| = \prod_{i=1}^{n} |(t-t_i)(b-a)| = |b-a|^n \prod_{i=1}^{n} |t - t_i|. \tag{10}$$

By a classical result of Chebyshev, if p is any monic polynomial of degree n, then

$$\max_{0 \leqslant t \leqslant 1} |p(t)| \geqslant \frac{1}{2^{2n-1}}. \tag{11}$$

See, e.g., [10, p. 194] or [14, p. 31]. So from (10) we can conclude that there exists a point z on the line segment S such that

$$\prod_{i=1}^{n} |z-\lambda_i'| \geqslant \frac{|b-a|^n}{2^{2n-1}}. \tag{12}$$

Since Γ is a continuous curve joining a and b, there exists a λ on Γ such that $\lambda' = z$. Since $|\lambda - \lambda_i| \geqslant |z - \lambda_i'|$ for all $i = 1, 2, \ldots, n$, the inequality (9) follows from (12). ■

Proof of Theorem 1. Let $t \in [0,1]$, and let λ be a root of $(1-t)f + tg$. It is known that $|\lambda| \leqslant \gamma$, where γ is defined by (3). See [12, p. 277]. Since $(1-t)f(\lambda) + tg(\lambda) = 0$, we have

$$\begin{aligned} |f(\lambda)| &= |t[f(\lambda) - g(\lambda)]| \leqslant |f(\lambda) - g(\lambda)| \\ &= \left| \sum_{k=1}^{n} (a_k - b_k)\lambda^{n-k} \right| \leqslant \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k}. \end{aligned} \tag{13}$$

Let

$$\Omega = \{z : z \text{ is a root of } (1-t)f + tg \text{ for some } t,\ 0 \leqslant t \leqslant 1\}.$$

Let Ω' be a connected component of the set Ω in the complex plane. Then by a familiar homotopy argument [1, 12] Ω' contains as many roots of f as of g. So, if d is an upper bound for the diameter of Ω', then the roots of f and g lying in Ω' can be enumerated as $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$, respectively, in such a way that $\max_{1 \leqslant i \leqslant m} |\alpha_i - \beta_i| \leqslant d$.

Let a and b be any two points in Ω'. By Lemma 1 we can find a point λ in Ω' such that

$$|f(\lambda)| = \prod_{i=1}^{n} |\lambda - \alpha_i| \geqslant \frac{|a-b|^n}{2^{2n-1}}.$$

So from (13) we get

$$|a-b|^n \leqslant 2^{2n-1} \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k},$$

and hence

$$|a-b| \leqslant 4 \times 2^{-1/n} \left(\sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k} \right)^{1/n}. \tag{14}$$

Since a, b were any two points in Ω', and Ω' was any connected component of Ω, the right hand side of (14) is an upper bound for the diameter of each connected component of Ω. So, by our remark above, we get the theorem. ■

Proof of Theorem 2. Let Ω' be any connected component at the set Ω defined as

$$\Omega = \{z : z \text{ is an eigenvalue of } (1-t)A + tB \text{ for some } t, 0 \leqslant t \leqslant 1\}.$$

Then by the same argument as used in the proof of Theorem 1, Ω' contains as many eigenvalues of A as of B. So we only need to show that if a and b are any two points in Ω', then $|a-b|$ is bounded by the right hand side of (7).

Assume $\|A\|_2 \leqslant \|B\|_2$ without any loss of generality. By Lemma 1, we can find a point λ in Ω' such that

$$|\det(A - \lambda I)| = \prod_{i=1}^{n} |\lambda - \lambda_i| \geqslant \frac{|b-a|^n}{2^{2n-1}}. \tag{15}$$

Recall from [7] that if X and Y are two $n \times n$ matrices and if λ is an eigenvalue of Y, then

$$|\det(X - \lambda I)| \leqslant \|X - Y\|_2 (\|X\|_2 + \|Y\|_2)^{n-1}.$$

Apply this to (15) with $X = A$ and $Y = (1-t)A + tB$, where t is a point in $[0,1]$ such that λ is an eigenvalue of Y. This gives

$$|b-a|^n \leqslant 2^{2n-1} \|A - B\|_2 (\|A\|_2 + \|B\|_2)^{n-1}.$$

Taking nth roots, we see that $|b-a|$ is bounded by the right hand side of (7). ■

Proof of Theorem 3. Let Ω, Ω' be as in the proof of Theorem 2, and let λ be as in (15) above. Let t be a point in $[0,1]$ such that λ is an eigenvalue of $Y=(1-t)A+tB$.

If $\|\cdot\|$ is any operator norm on $\mathscr{B}(\mathbb{C}^n)$, we have from [9] for any two operators S and T

$$|\det S - \det T| \leqslant n\|S-T\|[\max(\|S\|,\|T\|)]^{n-1}. \tag{16}$$

Let $S = A - \lambda I$, $T = Y - \lambda I$. Then note that

$$\det T = 0, \tag{17}$$

$$\|S-T\| \leqslant \|A-B\|, \tag{18}$$

$$\max(\|S\|,\|T\|) \leqslant 2\max(\|A\|,\|B\|) = 2M. \tag{19}$$

The relations (15) to (19) together show that the right hand side of (8) is an upper bound for the diameter of Ω'. So the theorem follows as before. ■

3. SHARPNESS OF THE BOUNDS

The bounds derived above can all be slightly improved by an argument mentioned in [13]. A modification of the usual proof of (11) shows that if p is a monic polynomial of degree n vanishing at 0, then

$$\max_{0\leqslant t\leqslant 1} |p(t)| \geqslant 2^{1-2n}\left(\cos\frac{\pi}{4n}\right)^{-2n}. \tag{20}$$

Now, in the proof of Theorem 1 we could choose the point a to be a root of the polynomial f. This, then, leads to an improvement of the inequality (5) by a factor $\cos^2(\pi/4n)$.

By the same argument the inequalities (7) and (8) can also be improved by the factor above.

We believe that a more substantial sharpening of (5) might be possible. However, we show now that the factor 4 occurring in this inequality cannot be replaced by anything smaller than 2.

For polynomials f and g defined by (1) and (2) let

$$v(f,g) = \min_{\sigma} \max_{1\leqslant i\leqslant n} |\alpha_i - \beta_{\sigma(i)}|. \tag{21}$$

where σ varies over all permutations on n symbols. Let

$$\Theta(f,g)=\left\{\sum_{k=1}^{n}|a_k-b_k|\gamma^{n-k}\right\}^{1/n}, \tag{22}$$

where γ is defined as in (3). Let

$$c(n)=\sup\left\{\frac{v(f,g)}{\Theta(f,g)}:f,g \text{ monic polynomials of degree } n\right\}. \tag{23}$$

We have shown above that

$$c(n)\leqslant 4\times 2^{-1/n}\cos^2\frac{\pi}{4n}. \tag{24}$$

Let

$$c=\sup_n c(n). \tag{25}$$

We will show that $c\geqslant 2$. So the bound (24) is off by a factor not larger than 2 from the best possible bound for $c(n)$.

We remark that Ostrowski [12, p. 280] showed that $c(3)\geqslant 2^{1/3}$. The construction of our example below is inspired by Ostrowski's example.

We shall construct, for each positive integer n, monic polynomials f and g of degree n with the following properties:

(i) f has 1 as a root with the multiplicity $[(n+1)/2]$, and its remaining roots lie in the disk $\{z:|z|<1\}$;

(ii) g has 0 as a root with multiplicity $[(n+2)/2]$, and its remaining roots lie in the disk $\{z:|z-1|<1\}$;

(iii) if f and g are expressed as in (1) and (2), then $a_k=b_k$ for $k=1,2,\dots,n-1$,

$$a_n=(-1)^{[n/2]}\binom{n-1}{[n/2]}^{-1},$$

and $b_n=0$ [this is a consequence of (ii) above].

Here, as usual, $[x]$ denotes the integer part of x, and $\binom{n}{k}$ denotes the binomial coefficient.

Notice that the above conditions imply

$$c(f,g)=1, \tag{26}$$

$$\Theta(f,g)=\binom{n-1}{[n/2]}^{-1/n}. \tag{27}$$

Now, using Stirling's formula for large factorials:

$$m!\approx\sqrt{2\pi m}\cdot m^m e^{-m},$$

we get from (27) that $\Theta(f,g)$ approaches $\frac{1}{2}$ as n becomes large. So once we establish the existence of f and g satisfying (i), (ii), and (iii) above, we shall also have shown that $c\geqslant 2$.

Let s and k be positive integers. Using the binomial theorem

$$(1-z)^{-s}=\sum_{r=0}^{\infty}\binom{-s}{r}(-z)^r=\sum_{r=0}^{\infty}\binom{s+r-1}{r}z^r,$$

we see that the polynomial

$$(1-z)^s\sum_{r=0}^{k}\binom{s+r-1}{r}z^r-1$$

has a root of order $k+1$ at 0. Hence

$$(1-z)^s\sum_{r=0}^{k}\binom{s+r-1}{r}z^r=1+z^{k+1}\tilde{p}_{s-1}(z), \tag{28}$$

where $\tilde{p}_{s-1}$ is a polynomial of degree $s-1$. Multiply both sides by the constant $\varphi_{k,s}$ defined by

$$\varphi_{k,s}=(-1)^s\binom{s+k-1}{k}^{-1}, \tag{29}$$

and write the resulting equation as

$$(z-1)^s p_{k,s}(z)=\varphi_{k,s}+z^{k+1}q_{s-1,k+1}(z), \tag{30}$$

where $p_{k,s}$ is a monic polynomial of degree k given by

$$p_{k,s}(z) = \binom{s+k-1}{k}^{-1} \sum_{r=0}^{k} \binom{s+r-1}{r} z^r, \tag{31}$$

and $q_{s-1,k+1}$ is a monic polynomial of degree $s-1$. This polynomial can be explicitly found by making the substitution $w = 1-z$ in (30). This leads to the identity

$$q_{s-1,k+1}(z) = (-1)^{s-1} p_{s-1,k+1}(1-z),$$

and so we can rewrite (30) as

$$(z-1)^s p_{k,s}(z) = \varphi_{k,s} + (-1)^{s-1} z^{k+1} p_{s-1,k+1}(1-z). \tag{32}$$

Put $k = s-1$ in the above equation to get

$$(z-1)^s p_{s-1,s}(z) = \varphi_{s-1,s} + (-1)^{s-1} z^s p_{s-1,s}(1-z), \tag{33}$$

where

$$\varphi_{s-1,s} = (-1)^{s-1} \binom{2s-2}{s-1}^{-1}. \tag{34}$$

Put $k = s$ in (32) to get

$$(z-1)^s p_{s,s}(z) = \varphi_{s,s} + (-1)^{s-1} z^{s+1} p_{s-1,s+1}(1-z), \tag{35}$$

where

$$\varphi_{s,s} = (-1)^s \binom{2s-1}{s}^{-1}. \tag{36}$$

Now let n be any odd integer $n = 2s-1$. Let $f(z) = (z-1)^s p_{s-1,s}(z)$. Then f is a polynomial of degree n having 1 as a root with multiplicity s. The rest of its roots are the roots of $p_{s-1,s}$. If we write this polynomial as

$$p_{s-1,s}(z) = \delta_0 z^{s-1} + \delta_1 z^{s-2} + \cdots + \delta_{s-1}, \tag{37}$$

then using (31), we find that the coefficients δ_j in (37) satisfy

$$\delta_j < \delta_{j-1}, \qquad j = 1, 2, \ldots, s-1.$$

Hence, by the Eneström-Kakeya theorem [12, p. 99] all roots of $p_{s-1,s}$ have modulus smaller than 1. Thus the polynomial f has the required property (i).

Let $g(z) = (-1)^{s-1} z^s p_{s-1,s}(1-z)$. The same reasoning shows that the polynomial g has the required property (ii). From (33)

$$f - g = \varphi_{s-1,s}, \tag{38}$$

and property (iii) now follows from (34) and (38).

If n is an even integer, $n = 2s$, then using (35) and (36) in place of (33) and (34), we obtain f and g satisfying (i), (ii), and (iii) by the above construction. As remarked earlier, this shows that $c \geqslant 2$.

4. SOME REMARKS

Remark 1. We could derive bounds for the distance between roots of the polynomials f and g by going over to their companion matrices and then appealing to Theorem 2 or 3. The latter seems more suited to such an application. Let

$$C_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix} \tag{39}$$

be the companion matrix of the polynomial f given by (1), and let C_g be defined analogously for g. The roots of f and g are the eigenvalues of C_f and C_g respectively. Let $|C_f|$ denote the matrix obtained from C_f by taking the absolute values of each of its entries. Let $\mu_f = \rho(|C_f|)$, where ρ denotes the spectral radius of a matrix. Then μ_f is also the unique positive root of the polynomial equation

$$z^n - |a_1| z^{n-1} - \cdots - |a_n| = 0. \tag{40}$$

The polynomial in (40) is called the *comparison polynomial* for f. Let

$$\mu = \max(\mu_f, \mu_g). \tag{41}$$

Define a norm $\|\cdot\|_{(\mu)}$ on the space of $n \times n$ matrices by setting

$$\|A\|_{(\mu)} = \max_{1 \leqslant i \leqslant n} \sum_{k=1}^{n} |a_{ik}| \mu^{k-i}, \tag{42}$$

where a_{ik} are the entries of the matrix A. Let $\|\cdot\|_p$ be the operator norm on $\mathbb{M}(n)$ associated with the l_p vector norm on $\mathbb{C}^n$. Then in particular

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{k=1}^{n} |a_{ik}|.$$

If X is the diagonal matrix with entries $1, \mu, \dots, \mu^{n-1}$ down its diagonal, then one can see that

$$\|A\|_{(\mu)} = \|X^{-1}AX\|_\infty, \tag{43}$$

and hence it indeed defines a norm. In fact, it is an operator norm corresponding to the vector norm

$$\|x\|_{(\mu)} = \max_{1 \leqslant i \leqslant n} |x_i \mu^{-i+1}| = \|X^{-1}x\|_\infty$$

defined on n-vectors $x = (x_1, \dots, x_n)$. From the definition of μ given by (41) one can check that

$$\|C_f\|_{(\mu)} = \mu, \qquad \|C_g\|_{(\mu)} = \mu. \tag{44}$$

Now apply Theorem 3 for $A = C_f$, $B = C_g$, and the norm $\|\cdot\|_{(\mu)}$. Using (44), we get

$$\max_i |\alpha_i - \beta_i| \leqslant 4 \times 2^{-1/n} n^{1/n} (2\mu)^{1-1/n} \left\{ \sum_{k=1}^{n} |a_k - b_k| \mu^{1-k} \right\}^{1/n}$$

$$= 4 \times 2^{1-2/n} n^{1/n} \left\{ \sum_{k=1}^{n} |a_k - b_k| \mu^{n-k} \right\}^{1/n}$$

Now if γ is defined by (3), then $\mu \leqslant \gamma$, as remarked earlier. So we have

$$\max_i |\alpha_i - \beta_i| \leqslant 4 \times 2^{1-2/n} n^{1/n} \left\{ \sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k} \right\}^{1/n}. \tag{45}$$

This estimate is surely weaker than (5). The reason we have derived it above is that if some argument were found to improve the matrix estimate (8), then we would get a corresponding improvement of (45) which could possibly be better than (5).

Remark 2. In [4] and [5] bounds for the distance between the eigenvalues of matrices A and B were derived by using their characteristic polynomials. If f and g given by (1) and (2) are the characteristic polynomials of A and B respectively, then it was shown in [4] that

$$|a_k - b_k| \leqslant k \binom{n}{k} M^{k-1} \|A - B\| \tag{46}$$

for $k = 1, 2, \ldots, n$, where $M = \max(\|A\|, \|B\|)$ and $\|\cdot\|$ denotes the spectral norm $\|\cdot\|_2$. This inequality, however, remains valid for several other operator norms. Let $\|\cdot\|$ be any operator norm such that $\|PAP\| \leqslant \|A\|$ whenever P is a projection operator in $\mathbb{C}^n$ corresponding to a subspace spanned by some of the coordinate axes. Then using (16) and the fact that the coefficients a_k and b_k are the sums of the $\binom{n}{k}$ principal $k \times k$ minors of A and B respectively, we see that (46) holds for all such norms. This is true, in particular, if the underlying vector norm is an absolute norm. All l_p norms, $1 \leqslant p \leqslant \infty$, are examples of such norms. Combining (46) with Theorem 1, we obtain the estimate (8) of Theorem 3 for all such norms.

Remark 3. Motivated by some problems in combinatorics, some authors (see, e.g., [11]) have studied the polynomial

$$\operatorname{per}(zI - A) = z^n + a_1' z^{n-1} + a_2' z^{n-2} + \cdots + a_n', \tag{47}$$

where $\operatorname{per} A$ denotes the permanent of A. It is known that

$$|\operatorname{per} A - \operatorname{per} B| \leqslant n M_p^{n-1} \|A - B\|_p, \qquad p = 1, 2, \infty, \tag{48}$$

where $\|A\|_p$ denotes the operator norm induced by the vector norm $\|x\|_p$

and where $M_p = \max(\|A\|_p, \|B\|_p)$. For $p=2$ this was shown in [3], and for $p = 1,\infty$ in [8].

It is also known [11] that the roots of (47) lie in the union of the Geršgorin disks of A. It follows that these roots are bounded by $\rho(|A|)$, the spectral radius of the matrix $|A|$ obtained by taking the absolute value of each entry of A Hence these roots are bounded by $\|A\|_1$, $\|A\|_\infty$, and $\||A|\|_2$, though not necessarily by $\|A\|_2$. Following the proof of Theorem 3 with "det" replaced by "per" and using (48) instead of (16), we can show that the roots λ'_i, of (47) and the roots μ'_i of $\mathrm{per}(zI - B)$ can be arranged so that we have for $i = 1,2,\dots,n$

$$|\lambda'_i - \mu'_i| \leqslant 4\times 2^{-1/n} n^{1/n}(2M'_p)^{1-1/n}\|A-B\|_p^{1/n}, \tag{49}$$

for $p = 1,2,\infty$. Here $M'_p = M_p = \max(\|A\|_p, \|B\|_p)$ when $p=1$ or ∞, and $M'_2 = \max(\||A|\|_2, \||B|\|_2)$.

REMARK 4. Since this paper was submitted for publication, the inequality (48) has been proved for all values of p, $1 \leqslant p \leqslant \infty$ (R. Bhatia and L. Elsner, On the variation of permanents, *Linear and Multilinear Algebra*, to appear).

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Note added in proof:. The Chebyshev polynomial argument used in our Lemma 1 and by Phillips [13] has been used earlier in a similar context by Schönhage. See A. Schönhage, Quasi-GCD computations, *J. Complexity* 1:118–137 (1985).

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