A Note on the Hoffman-Wielandt Theorem

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ABSTRACT

An elementary proof of a recent generalization of the Hoffman-Wielandt theorem to commuting m-tuples of normal matrices by Bhatia and Bhattacharyya is given. In the case $m = 1$ this provides a derivation, which, though using the same tools, avoids some unnecessary calculations and is thus shorter than the original one and that to be found in the standard literature.

Let $A = (A^{(1)}, \ldots, A^{(m)})$ be an m-tuple of commuting normal n-by-n matrices. There exists a unitary matrix U such that

$$U^H A^{(j)} U = \Lambda_j = \operatorname{diag}\left(\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}\right), \qquad j = 1, \ldots, m_F. \tag{1}$$

The vector $\alpha_k = (\alpha_k^{(1)}, \ldots, \alpha_k^{(m)}) \in \mathbb{C}^m$, $1 \leqslant k \leqslant n$, is called a joint eigenvalue of A. In [1], Bhatia and Bhattacharyya prove:

THEOREM. *Let A, B be two m-tuples of commuting normal operators with joint eigenvalues α_k, β_k, where $\beta_k = (\beta_k^{(1)}, \ldots, \beta_k^{(m)})$, $1 \leqslant k \leqslant n$. There exist permutations $\sigma, \tilde{\sigma}$ of $\{1, \ldots, n\}$ such that*

$$\sum_{k=1}^{n} \|\alpha_k - \beta_{\sigma(k)}\|_2^2 = \sum_{k=1}^{n} \sum_{j=1}^{m} |\alpha_k^{(j)} - \beta_{\sigma(k)}^{(j)}|^2 \leqslant \sum_{j=1}^{m} \|A^{(j)} - B^{(j)}\|_F^2$$

$$\leqslant \sum_{k=1}^{n} \|\alpha_k - \beta_{\tilde{\sigma}(k)}\|_2^2 = \sum_{k=1}^{n} \sum_{j=1}^{m} |\alpha_k^{(j)} - \beta_{\tilde{\sigma}(k)}^{(j)}|^2. \tag{2}$$

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Here $\| \|_2$ is the Euclidean vector norm and $\| \|_F$ the Frobenius matrix norm.

We give an elementary proof based only on

(i) the Birkhoff-König theorem stating that the set of doubly stochastic matrices is the convex hull of the permutation matrices,

(ii) the fact that a linear functional defined on a convex polyhedron achieves its optimal value on a set which includes a vertex.

Proof. Using (1) and

$$V^H B^{(j)} V = M_j = \operatorname{diag}(\beta_1^{(j)}, \dots, \beta_n^{(j)}), \qquad j = 1, \dots, m, \tag{3}$$

for a suitable unitary matrix V, we get

$$A^{(j)} - B^{(j)} = V(W\Lambda_j - M_j W)U^H,$$

where $W = V^H U = (w_{rs})_{r,s=1,\dots,n}$ is unitary. As the Frobenius norm $\| \|_F$ is unitarily invariant, we have

$$\sum_{j=1}^{m} \|A^{(j)} - B^{(j)}\|_F^2 = \sum_{j=1}^{m} \|W\Lambda_j - M_j W\|_F^2 = \sum_{j=1}^{m} \sum_{r,s=1}^{n} |w_{rs}(\alpha_s^{(j)} - \beta_r^{(j)})|^2$$

$$= \sum_{r,s=1}^{n} |w_{rs}|^2 h_{rs}, \qquad h_{rs} = \sum_{j=1}^{m} |\alpha_s^{(j)} - \beta_r^{(j)}|^2. \tag{4}$$

Define the linear functional $l(X) = \sum_{r,s=1}^{n} h_{rs} x_{rs}$ on the set $\mathscr{M}$ of doubly stochastic matrices, and let $\hat{W} = (|w_{rs}|^2) \in \mathscr{M}$. By (ii) above, l achieves its maximum and minimum on vertices of $\mathscr{M}$, which by (i) are permutation matrices. Hence there are permutation matrices P_1, P_2 such that

$$l(P_1) \leqslant l(\hat{W}) \leqslant l(P_2),$$

i.e., there are permutations $\sigma, \tilde{\sigma}$ of $\{1, \dots, n\}$ such that

$$\sum_{k=1}^{n} h_{\sigma(k),k} \leqslant l(\hat{W}) = \sum_{j=1}^{m} \|A^{(j)} - B^{(j)}\|_F^2 \leqslant \sum_{k=1}^{n} h_{\tilde{\sigma}(k),k}. \tag{5}$$

The equality here is shown in (4). But (5) is exactly the statement (2). ∎

We remark that for $m = 1$ this is the famous Hoffman-Wielandt theorem; see [2] and many standard books on matrix theory. The proof given above is also slightly simpler than that in [2], where the convexity argument is applied to the expression $\|A - B\|_F^2 - \|A\|_F^2 - \|B\|_F^2$.

REFERENCES

1 R. Bhatia and T. Bhattacharyya, A generalization of the Hoffman-Wielandt theorem, *Linear Algebra Appl.*, to appear.

2 A. J. Hoffman and W. H. Wielandt, The variation of the spectrum of a normal matrix, *Duke Math. J.* 20:37–39 (1953).

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