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Eigenvalue Neutrality in Block Triangular Matrices

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Let $A \in M_n$, $B \in M_m$, and $\lambda \in C$ be given. We say that $X \in M_{n,m}$ is λ-neutral for A and B if the Jordan structure of

$$C = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}.$$

associated with λ is the same as that of

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

We characterize λ-neutrality in terms of the bilinear form of X evaluated at elements of left Jordan chains of A and elements of right Jordan chains of B. This may be applied to convergence of powers of C. Other related matters are also discussed.

1. INTRODUCTION

(a) *Problem Statement* We are primarily concerned with the following general problem. Suppose that $A \in M_n$, $B \in M_m$, and $\lambda \in C$ are given. If the basic Jordan blocks associated with λ in the Jordan canonical form of A are of sizes $n_1, n_2, \ldots, n_k$ and the basic Jordan blocks associated with λ in the Jordan canonical form of B are of sizes $m_1, \ldots, m_p$, what are necessary and sufficient conditions on a matrix $X \in M_{n,m}$ such that the block matrix

$$C = C(A, B; X) = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \tag{1}$$

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has basic Jordan blocks associated with λ in its Jordan canonical form of sizes $n_1, \ldots, n_k, m_1, \ldots, m_p$? In this event, we say that X is *λ-neutral* for A and B. Of course $0 \in M_{n,m}$ is λ-neutral for any A and B, but for given A and B, not scalar matrices, there will always be many nonzero λ-neutral matrices X. Note that we allow the possibility that λ occurs as an eigenvalue of neither or just one of A and B, in which event the solution to our problem is trivial, as any $X \in M_{n,m}$ is λ-neutral. We also note the emphasis here on a single eigenvalue λ (in the most interesting case occurring in both A and B). There are, of course, connections with Roth's theorem [7] (see discussion below), which, in essence, addresses the situation in which X is λ-neutral with respect to *every* eigenvalue λ. In addition to this formal variation from Roth's theorem, it is our purpose (motivated by an application mentioned in part (b)) to present a solution in terms rather different from Roth's theorem. The key will be the bilinear form of X evaluated at elements of left Jordan chains of A and right Jordan chains of B.

(*b*) *Motivation* We were led to study the problem described in part (a) by two rather different considerations.

The first deals with a recent observation arising in the context of computational analysis of homogeneous Markov systems [8]. We call a matrix $Q \in M_n$ *convergent* if $\lim_{t \to \infty} Q^t$ exists. (It should be noted that this definition, which is convenient, is at variance with a common definition of "convergent" that requires further that the limit actually be 0.) It is a known fact, and a simple exercise, that a matrix in M_n is convergent if and only if each of its eigenvalues is either less than 1 in absolute value or is equal to 1 and all the Jordan blocks associated with the eigenvalue 1 are 1-by-1. In [8] the authors were concerned with a block matrix of the form $C = C(A, B; X)$ in which each of the matrices A and B is irreducible row stochastic. In this event, of course, A and B are convergent. Their principal result is that if X is a real matrix with row sums all 0, then the block matrix C is convergent also. Since the eigenvalues of C are just those of A together with those of B, counting multiplicities, it is clear that C is convergent, given that A and B are, if and only if X is 1-neutral. The proof given in [8] is a rather lengthy and painstaking analytic argument (which does not benefit from the observations of the previous sentence). It seemed to us that an algebraic proof would be more natural and efficient. In providing same, we were naturally led to the notion of λ-neutrality and to substantial simplification of the proof and generalization of the result of [8].

The second motivation for our problem is that it is a first step in a slightly different view of a long standing fundamental problem. Often referred to as the "Carlson problem" [1], this is the question of which Jordan structures are possible for the block matrix C in (1), given the Jordan structures of A and B, as X runs through all complex matrices. Of course, this question may be viewed "eigenvalue by eigenvalue", and much is known. In fact, there is a recent "solution" via equivalence to the problem of eigenvalues of an Hermitian sum, a solution of which has been announced by Lidskii [5]. It is not our intent to discuss the Carlson problem itself in any detail, but the connection with our problem is as follows. An "inverse" approach to the Carlson problem might be to consider each proposed Jordan form for C and to characterize those matrices X that achieve this Jordan form. Those Jordan forms that yield a nonvoid set of X's then constitute a solution to the Carlson problem. Of course, a λ-neutral Jordan form for each λ is obviously feasible and provides a first, and perhaps instructive, step in this program. In addition, the problem of partitioning the set of all X's by determining the solution set for each feasible Jordan form is a natural and intriguing problem in the context of Carlson's problem.

2. λ-NEUTRALITY IN THE INDEX 1 CASE

The index of an eigenvalue λ is the size of the largest Jordan block corresponding to λ. Thus λ is an eigenvalue of *index* 1 when the geometric multiplicity of λ equals the algebraic multiplicity. In this section we consider the special case in which the eigenvalue λ is of index 1 for A and for B.

For $A \in M_n$, call a nonzero vector f a *left eigenvector* of A corresponding to the eigenvalue λ if $f^T A = \lambda f^T$.

THEOREM 1 *Let $A \in M_n$, $B \in M_m$ and $X \in M_{n,m}$. Assume that λ is an eigenvalue of index 1 for A and for B. Then X is λ-neutral for A and B if and only if for any left eigenvector f of A associated with λ and for any right eigenvector e of B associated with λ we have*

$$f^T X e = 0. \tag{2}$$

Proof Let

$$C = C(A, B; X) = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}.$$

For the eigenvalue λ, let $f_1, f_2, \ldots, f_k$ be a basis of the left eigenspace for A, $g_1, g_2, \ldots, g_k$ a basis for the right eigenspace of A, and $e_1, e_2, \ldots, e_p$ a basis of the right eigenspace of B. If $\begin{bmatrix} h \\ w \end{bmatrix}$ is an eigenvector of C associated with λ, then $Ah + Xw = \lambda h$ and $Bw = \lambda w$. The vectors $\begin{bmatrix} g_1 \\ 0 \end{bmatrix}, \begin{bmatrix} g_2 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} g_k \\ 0 \end{bmatrix}$ are k linearly independent eigenvectors of C associated with λ. Moreover, in any linearly independent collection $\begin{bmatrix} h_1 \\ w_1 \end{bmatrix}, \ldots, \begin{bmatrix} h_r \\ w_r \end{bmatrix}$ of eigenvectors of C associated with λ, at most k of the w_j can be the zero vector. Since $\begin{bmatrix} h \\ w \end{bmatrix}$ an eigenvector of C, with $w \neq 0$, implies that w is an eigenvector of B associated with λ, it is sufficient to consider the case $w = e_j$.

A vector $\begin{bmatrix} h_j \\ e_j \end{bmatrix}$ is an eigenvector for C associated with λ if and only if $(A - \lambda)h_j = -Xe_j$. We claim that this is equivalent to $f_i^T X e_j = 0$, $i = 1, 2, \ldots, k$. To show this, complete a basis of $\mathbf{C}^n$ by adjoining the vectors $v_1, v_2, \ldots, v_{n-k}$ to $f_1, \ldots, f_k$. Let $V = [v_1, \ldots, v_{n-k}]^T$, $F = [f_1, \ldots, f_k]^T$ and $M = \begin{bmatrix} V \\ F \end{bmatrix}$. Then M is nonsingular, $F(A - \lambda I) = 0$, and the rank of $V(A - \lambda I)$ equals $n - k$. The claim is established from the equivalence of the following statements.

(i) $\begin{bmatrix} h_j \\ e_j \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ associated with λ.

(ii) $(A-\lambda I)h = -Xe_j$ has a solution $h = h_j$.

(iii) $M(A-\lambda I)h = -MXe_j$ has a solution $h = h_j$.

(iv) $\begin{bmatrix} V(A-\lambda I) \\ F(A-\lambda I) \end{bmatrix} h = -\begin{bmatrix} VXe_j \\ FXe_j \end{bmatrix}$ has a solution $h = h_j$.

(v) $\begin{bmatrix} V(A-\lambda I) \\ 0 \end{bmatrix} h = -\begin{bmatrix} VXe_j \\ FXe_j \end{bmatrix}$ has a solution $h = h_j$.

(vi) $f_i^T X e_j = 0$, $i = 1, 2, \ldots, k$.

Thus, there is a set $\begin{bmatrix} g_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} g_k \\ 0 \end{bmatrix}, \begin{bmatrix} h_1 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} h_p \\ e_p \end{bmatrix}$ of $k+p$ linearly independent eigenvectors of C if and only if $f_i^T X e_j = 0$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$. As the f_i's and e_j's form bases of the respective eigenspaces, this is equivalent to (2). ■

If A and B are convergent matrices, the convergence of $C = C(A, B; X)$ depends entirely on the eigenvalue $\lambda = 1$ being of index 1 in C.

COROLLARY 2 *Let $A \in M_n$ and $B \in M_m$ be convergent matrices, $X \in M_{n,m}$. Then $C = C(A, B; X)$ is convergent if and only if X is 1-neutral for A and B.*

COROLLARY 3 [8, Theorem 2.1] *Let $A \in M_n$ and $B \in M_m$ be irreducible row stochastic matrices, $X \in M_{n,m}$. If all of the row sums of X are zero then $C(A, B; X)$ is convergent.*

Proof The eigenvalue $\lambda = 1$ of B is of index 1 and multiplicity 1. The vector $e = [1, 1, \ldots, 1]^T$ is a basis of the eigenspace of B associated with $\lambda = 1$. The condition that all row sums of X are zero is equivalent to $Xe = 0$. Thus X is 1-neutral for A and B by condition (2). ■

For $f_1, f_2, \ldots, f_k$ and $e_1, e_2, \ldots, e_p$ defined as in the proof of Theorem 1, $F = [f_1, \ldots, f_k]^T$ and $E = [e_1, \ldots, e_p]$, we have established that for λ of index 1 for A and B, the matrix X is λ-neutral for A and B if and only if

$$FXE = 0.$$

It is of interest to know the status of λ for $C = C(A, B; X)$ when λ is of index 1 in each of A and B, but $FXE \neq 0$. In order to do this we first consider a decomposition under similarity of C. This decomposition, which does not require that λ be of index 1, will also be used in the solution of the general case.

Let S and T be nonsingular matrices producing the Jordan forms of A and B. That is, $S^{-1}AS = \begin{bmatrix} \tilde{J}_A & 0 \\ 0 & J_A(\lambda) \end{bmatrix}$ and $T^{-1}BT = \begin{bmatrix} J_B(\lambda) & 0 \\ 0 & \tilde{J}_B \end{bmatrix}$, in which $J_A(\lambda)$ denotes the $\lambda 1$-portion and $\tilde{J}_A$ represents the remaining blocks of the Jordan form for A, with corresponding notation for B. Then

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} S^{-1}AS & S^{-1}XT \\ 0 & T^{-1}BT \end{bmatrix} = \begin{bmatrix} \tilde{J}_A & 0 & V_1 & V_2 \\ 0 & J_A(\lambda) & Y & V_3 \\ 0 & 0 & J_B(\lambda) & 0 \\ 0 & 0 & 0 & \tilde{J}_B \end{bmatrix}.$$

Since λ is not on the diagonal of $\tilde{J}_A$ or $\tilde{J}_B$, this last matrix is similar to

$$\begin{bmatrix} \tilde{J}_A & 0 & 0 & V \\ 0 & J_A(\lambda) & Y & 0 \\ 0 & 0 & J_B(\lambda) & 0 \\ 0 & 0 & 0 & \tilde{J}_B \end{bmatrix} \qquad \text{(see [4, Theorem 2.4.8])}.$$

By means of a permutation similarity on this matrix we obtain the fact that

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \text{ is similar to } \begin{bmatrix} J_A(\lambda) & Y & 0 & 0 \\ 0 & J_B(\lambda) & 0 & 0 \\ 0 & 0 & \tilde{J}_A & V \\ 0 & 0 & 0 & \tilde{J}_B \end{bmatrix},$$

and note that λ is not an eigenvalue of either $\tilde{J}_A$ or $\tilde{J}_B$. Thus, in examining the status of λ in $C(A, B; X)$ it is sufficient to consider the upper block $C(J_A(\lambda), J_B(\lambda); Y)$.

Now let λ be of index 1 for A and for B, and let F and E be defined as above. It is possible to find matrices W and U such that $S^{-1} = \begin{bmatrix} W \\ F \end{bmatrix}$ and $T = [E\ U]$. In this case $Y = FXE$. Let $N = C(J_A(\lambda), J_B(\lambda); Y) - \lambda I = C(0, 0; FXE)$. Then $N^2 = 0$ and if $FXE \neq 0$, λ will be of index 2 in $C = C(A, B; X)$ and there will be $r = \text{rank}(FXE)$ blocks of the form $J_2(\lambda)$ in the Jordan form of C. We use $J_k(\lambda)$ to represent a basic Jordan block, a k-by-k upper triangular matrix with λ's on the main diagonal, 1's on the superdiagonal and all other entries zero (see [4, definition 3.1.1]). In general, the matrices $J_A(\lambda)$ and $J_B(\lambda)$ will be direct sums of basic Jordan blocks of various sizes.

THEOREM 4 *Let λ be an eigenvalue of index* 1 *for $A \in M_n$ and for $B \in M_m$, and let $X \in M_{n,m}$. Let $f_1, \ldots, f_k$ be a basis of the left eigenspace for A associated with λ and $e_1, \ldots, e_p$ a basis of the right eigenspace for B associated with λ, and let $F = [f_1, \ldots, f_k]^T$ and $E = [e_1, \ldots, e_p]$. Then for the matrix $C = C(A, B; X)$, λ will have index less than or equal to* 2 *and there will be $r = \text{rank}(FXE)$ 2-by-2 basic Jordan blocks associated with λ in the Jordan canonical form of C.*

3. λ-NEUTRALITY IN THE GENERAL CASE

We now consider the general situation in which A has basic Jordan blocks of order $n_1, n_2, \ldots, n_k$ corresponding to the eigenvalue λ and B has basic Jordan blocks of order $m_1, m_2, \ldots, m_p$ for λ. The discussion regarding decomposition presented after

Corollary 3 shows that to examine the Jordan structure of $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ corresponding to λ it is sufficient to consider the matrix $\begin{bmatrix} J_A(\lambda) & Y \\ 0 & J_B(\lambda) \end{bmatrix}$. The key tool in our investigation will be the theorem of Roth [7] which states that $\begin{bmatrix} A & Y \\ 0 & B \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ if and only if the matrix equation $AW - WB = Y$ has a solution W. We use an immediate block version of this result.

LEMMA 5 *Let* $A_1, A_2, \dots, A_k, B_1, \dots, B_p$ be square matrices. The matrix

$$\begin{bmatrix} A_1 & & & 0 & Y_{11} & \cdots & Y_{1p} \\ & A_2 & & & \vdots & \cdots & \vdots \\ & & \ddots & & \vdots & \cdots & \vdots \\ 0 & & & A_k & Y_{k1} & \cdots & Y_{kp} \\ & & & & B_1 & & 0 \\ & 0 & & & & \ddots & \\ & & & & 0 & & B_p \end{bmatrix}$$

is similar to $\mathrm{Diag}(A_1, A_2, \dots, A_k, B_1, \dots, B_p)$ if and only if for each $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, p$ the matrix equation $A_i W_{ij} - W_{ij} B_j = Y_{ij}$ has a solution W_{ij}.

In view of this fact, it is sufficient to investigate a matrix of the form $\begin{bmatrix} J_r(\lambda) & Y \\ 0 & J_t(\lambda) \end{bmatrix}$ in which $Y \in M_{r,t}$. We shall be interested in the sums along certain diagonals in the lower left corner of Y. We use the term *lower complete diagonal* to indicate that the first entry is in the first column and the last entry is in the last row, i.e., $y_{i1}, y_{i+1,2}, \dots, y_{r,r-i+1}$ are the entries of a typical lower complete diagonal of Y. The matrix Y has $s = \min\{r, t\}$ lower complete diagonals. The sum of the entries in such a diagonal will be called a *complete lower diagonal sum*. For example, $Y \in M_{6,3}$ has exactly three complete lower diagonal sums, $y_{61}, y_{51} + y_{62}, y_{41} + y_{52} + y_{63}$.

LEMMA 6 *Let* $Y \in M_{r,t}$. *The following are equivalent.*

(i) $\begin{bmatrix} J_r(\lambda) & Y \\ 0 & J_t(\lambda) \end{bmatrix}$ *is similar to* $\begin{bmatrix} J_r(\lambda) & 0 \\ 0 & J_t(\lambda) \end{bmatrix}$.

(ii) $\begin{bmatrix} J_r(0) & Y \\ 0 & J_t(0) \end{bmatrix}$ *is similar to* $\begin{bmatrix} J_r(0) & 0 \\ 0 & J_t(0) \end{bmatrix}$.

(iii) *The complete lower diagonal sums of* Y *all equal zero. That is,*

$$\sum_{j=1}^{h} y_{r-h+j,j} = 0, \qquad h = 1, 2, \dots, s; \quad s = \min\{r, t\}. \tag{3}$$

Proof The equivalence of (i) and (ii) is immediate. To show the equivalence of

(ii) and (iii) let $N = J_r(0)$ and $M = J_t(0)$. Then (ii) is equivalent to the existence of a matrix W which is a solution of $NW - WM = Y$. For $W \in M_{r,t}$,

$$NW - WM = \begin{bmatrix} w_{21} & w_{22} - w_{11} & w_{23} - w_{12} & \cdots \\ w_{31} & w_{32} - w_{21} & w_{33} - w_{22} & \cdots \\ \vdots & \vdots & & \\ w_{r1} & w_{r2} - w_{r-1,1} & w_{r3} - w_{r-1,2} & \cdots \\ 0 & -w_{r1} & -w_{r2} & \cdots \end{bmatrix}.$$

The element in position (i, j), $i \neq r$, $j \neq 1$, is $w_{i+1,j} - w_{i,j-1}$. The complete lower diagonal sums of $NW - WM$, beginning in the lower left corner, are 0, $w_{r1} + (-w_{r1}) = 0$, $w_{r-1,1} + (w_{r2} - w_{r-1,1}) + (-w_{r2}) = 0, \ldots$. The complete lower diagonal sum beginning in position $(i, 1)$ is

$$w_{i+1,1} + \sum_{j=2}^{r-i} (w_{i+j,j} - w_{i+j-1,j-1}) + (-w_{r,r-i}) = 0.$$

Due to telescoping, in calculating these $s = \min\{r, t\}$ complete lower diagonal sums of $NW - WM$, while there are $\frac{s(s+1)}{2}$ entries of $NW - WM$ involved, only $\frac{s(s-1)}{2}$ elements w_{ij} of W are included. Moreover, the arrangement of the w_{ij} in the remaining diagonals allows for the systematic evaluation of the remaining w_{ij}. Thus, there will be a solution W to $NW - WM = Y$ if and only if the complete lower diagonal sums of Y all equal zero. By Roth's Theorem, (ii) and (iii) are equivalent. ∎

An ordered set $f_1, f_2, \ldots, f_r$ of nonzero vectors satisfying $f_r^T A = \lambda f_r^T$ and $f_i^T A = \lambda f_i^T + f_{i+1}^T$, $i = 1, 2, \ldots, r-1$ is called a *left Jordan chain* of A corresponding to λ. Note that f_r is a left eigenvector of A associated with λ. In like manner, an ordered set $e_1, e_2, \ldots, e_t$ of nonzero vectors satisfying $Be_1 = \lambda e_1$ and $Be_j = \lambda e_j + e_{j-1}$, $j = 2, 3, \ldots, t$ is called a *(right) Jordan chain* of B corresponding to λ. In this chain, e_1 is the eigenvector. In going from $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ to $\begin{bmatrix} S^{-1}AS & S^{-1}XT \\ 0 & T^{-1}BT \end{bmatrix}$ to the principal submatrix $\begin{bmatrix} J_r(\lambda) & Y \\ 0 & J_t(\lambda) \end{bmatrix}$ to be considered, the matrix Y is of the form $Y = FXE$, with $F = [f_1, \ldots, f_r]^T$ and $E = [e_1, \ldots, e_t]$ where $f_1, \ldots, f_r$ is a left Jordan chain of A corresponding to λ and $e_1, \ldots, e_t$ is a right Jordan chain of B corresponding to λ. In particular, $y_{ij} = f_i^T X e_j$. Therefore, using Lemmas 5 and 6, our result is obtained.

THEOREM 7 *Let $A \in M_n$ and $B \in M_m$. Then $X \in M_{n,m}$ is λ-neutral for A and B if and only if for any left Jordan chain $f_1, f_2, \ldots, f_r$ of A corresponding to λ and for any right Jordan chain $e_1, e_2, \ldots, e_t$ of B corresponding to λ, we have*

$$\sum_{j=1}^{h} f_{r+h-j}^T X e_j = 0, \qquad h = 1, 2, \ldots, s; \quad s = \min\{r, t\}. \tag{4}$$

There is a local version of this result (Theorem 7). It can be shown that if $f_1, \ldots, f_r$ is a left Jordan chain of A corresponding to λ such that the equations (4) hold for all right Jordan chains of B corresponding to λ, then the basic Jordan block $J_r(\lambda)$ of A associated with that left Jordan chain will be a basic Jordan block of $C(A, B; X)$. The corresponding result for a fixed right Jordan chain of B for which the equations (4) hold for all appropriate left Jordan chains of A also holds.

Describing the Jordan structure of $C(A, B; X)$ for λ when X fails to be λ-neutral is far more complicated in the general situation than it was in the index 1 case. We consider only the case in which each matrix has exactly one Jordan block for λ. Let $C = C(J_r(0), J_t(0); Y)$ and let $e = [0, 0, \dots, 0, 1]^T$. If $q = \max\{r, t\}$, then $w = C^q e = [\dots, y_{r-1,1} + y_{r2}, y_{r1}, 0, \dots, 0]^T$ where the first $s = \min\{r, t\}$ coordinate entries of w are precisely the s complete lower diagonal sums of Y. In the product Cw, the only nontrivial action on w is caused by the leading principal block $J_r(0)$ of C, which has the effect of moving each entry up one position. When $C^k w$ is a nonzero multiple of the eigenvector $[1, 0, 0, \dots, 0]^T$, the sequence $C^k w, \dots, Cw, w = C^q e, \dots, Ce, e$ will be a right Jordan chain of C. If h $(h \leqslant s)$ is the least positive integer such that $v = \sum_{j=1}^{h} y_{r-h+j,j} \neq 0$, then $C^k w = [v, 0, 0, \dots, 0]^T$. In this case, C has Jordan blocks of size $r + t - (h-1)$ and $h - 1$.

THEOREM 8 *Let $A \in M_n$ and $B \in M_m$ each have precisely one basic Jordan block associated with λ. Let $f_1, f_2, \dots, f_r$ be a left Jordan chain of A corresponding to λ and $e_1, e_2, \dots, e_t$ a right Jordan chain of B corresponding to λ.*

(i) *If $f_r^T X e_1 \neq 0$, then the Jordan canonical form of $C = C(A, B; X)$ has exactly one basic Jordan block of order $r + t$ associated with λ.*

(ii) *If h is the least positive integer, $1 < h \leqslant \min\{r, t\}$, such that $\sum_{j=1}^{h} f_{r-h+j}^T X e_j \neq 0$, then the Jordan canonical form of $C = C(A, B; X)$ has two basic Jordan blocks associated with λ of order $h - 1$ and $r + t - (h - 1)$.*

We close with two brief comments. First, it follows from Theorem 7 that for fixed A, B and λ the set of λ-neutral matrices is a vector space. Second, for corresponding analysis of block triangular matrices of the form $\begin{bmatrix} A & 0 \\ X & B \end{bmatrix}$ one may use complete upper diagonals and exchange the adjectives right and left in our results.

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