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ON APPROXIMATE CORES OF NON-CONVEX ECONOMIES

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In this note we investigate, for non-convex finite economies, the relationship between the existence of approximate core allocations and the size of an economy.

1. Introduction

The existence of approximate equilibria for exchange economies whose agents have non-convex preferences was established by Starr (1969). For a stronger concept of approximate equilibria the existence result was obtained by Hildenbrand, Schmeidler and Zamir (1973) (HSZ hereafter). Moreover, it was shown in HSZ, that the approximation can be made as good as one likes by choosing the number of agents in the economy large enough.

Under assumption of completeness of preferences, the analogous existence result for approximate cores was also obtained in HSZ. In a consequent paper Grodal (1976) has extended this result to the non-complete case.

In a recent contribution Anderson (1982) proved a 'rate-of-convergence' theorem for approximate equilibria in the framework of quite general sequences of finite exchange economies, including those sequences considered in HSZ. A special case of Anderson's result was proved subsequently by Weber (1980) for the sequences of HSZ by modifying arguments given already there.

In this note we establish a relationship between the existence of approximate core allocations and the size of a non-convex economy. The result is directly related to the previous papers by HSZ and Grodal (1976).

To obtain our proposition the only additional assumption we need is a cone-monotonicity of preference relations which is defined below.

2. Model

We follow HSZ very closely, and all notations, assumptions and definitions, except of C-monotonicity, are reproduced from there.

Let T denote an infinite set of potential traders. For every $t \in T$ there are defined:

(i) a preference relation $\succ_t$ on the positive orthant R^l_+ of the l-dimensional Euclidean space $R^l (l \geqslant 2)$,
(ii) a vector of the initial endowment w_t in $R^l_+ \setminus \{0\}$.

We assume that:

A.1. For all $t \in T$, $\succ_t$ is irreflexive, transitive, open [the set $\{(x, y) \in R^l_+ \times R^l_+ | x \succ_t y\}$ is open in the relative topology of $R^l_+ \times R^l_+$], and strongly monotone ($x \in R^l_+$, $u \in R^l_+ \setminus \{0\}$ implies $x + u \succ_t x$).

A.2. Uniform boundedness of initial endowments. There is a positive real number M such that $w_t \leqslant Me$ for all t in T, where e denotes the unit vector $(1, \ldots, 1) \in R^l$.

A.3. Compactness of preferences. The set $\{\succ_t)_{t \in T}$ is compact. [For a precise definition of the topology on the set of preferences see HSZ (p. 1164). It is shown there, for example, that the set of all irreflexive, transitive, open, and monotone preferences is compact with respect to this topology.]

Now, let $C \subset R^l$ be a proper cone such that int $C \supset R^l_+ \setminus \{0\}$. [Recall that C is a proper cone if (i) $C + C \subset C$, (ii) $\lambda C \subset C$ for all $\lambda > 0$, (iii) $C \cap (-C) = \{0\}$. Cf. Shaefer (1966).]

Definition 1. A preference relation on R^l_+ is *C-monotone at* $x \in R^l_+$ if $[u \in C \setminus \{0\},\ x + u \in R^l_+]$ implies $[x + u \succ x]$. A preference relation is *C-monotone on a set* $A \subset R^l_+$ if it is C-monotone at x for every $x \in A$. A preference relation is *C-monotone* if it is C-monotone on R^l_+.

Now denote for any $\lambda \in (0, 1)$ and $i \in \{1,2,\ldots,l\}$ the vector

$$\Big(-\lambda,\ldots,-\lambda,\underset{\substack{\uparrow \\ i\text{th place}}}{1},-\lambda,\ldots,-\lambda\Big)$$

by e^i_λ. Then C_λ denotes the convex cone generated by the set $\{0, e^1_\lambda, e^2_\lambda,\ldots,e^l_\lambda\}$. Note that there is $\lambda \in (0, 1)$ such that $C \supset C_\lambda$.

We denote by S the open simplex

$$\left\{ p = (p^1,\ldots,p^l) \in R^l_+ \,\middle|\, \sum_{i=1}^{l} p^i = 1 \quad \text{and} \quad p^i > 0 \quad \text{for all} \quad i \right\}$$

in R^l_+. The demand correspondence ψ: $T \times S \to R^l_+$ is defined by

$$\psi(t, p) = \{ x \in \Omega \mid px \leqslant pw_t \quad \text{and} \quad y \succ_t x \quad \text{implies} \quad py > pw_t \}.$$

It is shown in Schmeidler (1969) that ψ is well defined and, by the monotonicity assumption, $px = pw_t$ holds for all x in $\psi(t, p)$. An *exchange economy* is by definition a finite subset, say E, of T. An *allocation* for the economy E is a collection $\{x_t\}_{t \in E}$ of elements of R^l_+ satisfying $\sum_{t \in E}(x_t - w_t) = 0$.

In order to state our proposition we need the following:

Definition 2. Let $\epsilon > 0$ and an economy E be given. Then the *ϵ-core of E* is defined to be the set of all allocations $\{x_t\}_{t \in E}$, such that there do not exist a non-empty subset S of E and an allocation $\{y_t\}_{t \in E}$ for E satisfying

(i) $y_t \succ_t x_t$ for every $t \in S$,
(ii) $\sum_{t \in S} y_t \leqslant (\sum_{t \in S} w_t)\theta |S| \epsilon e$.

($|A|$ denotes the cardinality of a set A, and for any $x, y \in R^l_+$ the vector whose jth coordinate is $\max\{0, x^j - y^j\}$ for all $1 \leqslant j \leqslant l$ is denoted $x\theta y$.)

Proposition. Let A.1–A.3 be satisfied and $\delta > 0$. Suppose that for any $n \in N$ there is a $\lambda_n > 0$ so that every $\succ_t$ is C_{λ_n}-monotone on $[0,n]^l$. Then for *any sequence of economies in T with* $\sum_{t\in E_n} w_t > |E_n|\delta e$ *for each n, there are sequences* $\{\{x_t^n\}_{t\in E_n}\}_{n\in N}$ *of allocations and* $\{\epsilon_n\}_{n\in N}$ *of positive numbers with* $\epsilon_n = 0(1/|E_n|)$, *such that* $\{x_t^n\}_{t\in E_n} \in \epsilon_n$*-core of* E_n *for every* $n \in N$.

3. Preliminary statements and proof of the proposition

Let us state the following lemma, which is useful for the proof of the proposition:

Lemma. *Let* $\delta > 0$. *Under assumptions A.1–A.3, for any sequence* $\{E_n\}_{n\in N}$ *of economies in T with* $\sum_{t\in E_n} w_t > |E_n|\delta e$ *for each n, there are sequences* $(\{x_t^n\}_{t\in E_n}, p^n)_{n\in N}$ *of allocations and of prices in S, such that for each* $n \in N$ *and each* $t \in E_n$ *one has* $p^n x_t^n = p^n w_t$ *and* $\rho(x_t^n, \psi(t, p^n)) = 0(1/|E_n|)$. *(For* $x \in R^l$ *and* $A \subset R^l$, $\rho(x, A)$ *denotes* $\inf_{y\in A} \| x - y \|$).

This Lemma due to Weber (1980) is a corollary of Theorem 2 in Anderson (1982).

Consider the following two statements concerning an irreflexive preference relation $\succ$:

(i) for every $x,y,x',y' \in R^l_+ : y \geqslant x, y \overset{+}{\neq} x$, $\| y - y' \| \leqslant \| y - x \| / K$ and $\| x - x' \| \leqslant \| y - x \| / K \Rightarrow y' \succ x'$.
(ii) $\succ$ is C_λ-monotone.

Then we can state:

Claim 1. (i) implies that (ii) holds for every $\lambda \leqslant 1/(l-1)(\sqrt{l}K + 1)$.

Claim 2. (ii) implies that (i) holds for $K \geqslant 4\sqrt{l}/\lambda$.

Proof of Claim 1. Assume (i) is satisfied and consider $x \in R^l_+$, $u \in C_\lambda \setminus \{0\}$, where $\lambda \leqslant 1/(l-1)(\sqrt{l}K + 1)$. Then, $u = \sum_{i=1}^l \delta_i e^i_\lambda$ for some $\delta_i \geqslant 0$. Let $\delta = \max_{i=1,\dots,l} \delta_i > 0$. Now define $y' = x + u$ and assume that $y' \in R^l_+$. We have to prove that $y' \succ x$. Therefore define $v \in R^l_+$ by $v_i = \max(0, \delta_i - \lambda\sum_{h\neq i}\delta_h)$ for $i = 1,\dots,l$ and let $y = v + x$. Then, $\| y - x \| = \| v \| \geqslant \delta - \lambda(l-1)\delta$. Moreover

$$\| y' - y \| = \| u - v \| \leqslant \left\| \left(\sum_{h\neq i} \lambda\delta_h \right)_{i=1}^{l} \right\| \leqslant \left((l-1)^2\delta^2\lambda^2 l \right)^{\frac{1}{2}} = (l-1)\delta\lambda\sqrt{l}\,.$$

Since $\lambda \leqslant 1/(l-1)(K\sqrt{l}+1)$, we have that

$$(l-1)\delta\lambda\sqrt{l} \leqslant \frac{\delta - \lambda(l-1)\delta}{K} \leqslant \frac{\|y-x\|}{K}.$$

Consequently, as $y \geqslant x$, we have by (i) that $x+u \succ x$. Q.E.D.

Proof of Claim 2. Assume (ii) and consider $x, y, x', y' \in R^l_+$ such that $y \neq x$, $y > x$, and $\max(\|y-y'\|, \|x-x'\|) \leqslant \|y-x\|/K$, where $K \geqslant 4\sqrt{l}/\lambda$. It suffices to show that $\underline{y}' - \bar{x}' \in C_\lambda$, where $\underline{y}' := y(-\|y-x\|/K)(1,\ldots,1)$, $\bar{x}' := x + (\|y-x\|/K)(1,1,\ldots,1)$ (clearly $y' \geqslant \underline{y}'$ and $\bar{x}' > x'$). Note that $y \geqslant x$ implies that $y_h \geqslant x_h$ for all $1 \leqslant h \leqslant l$, and there exists i, $1 \leqslant i \leqslant l$, such that $y_i \geqslant x_i + \|y-x\|/\sqrt{l}$. Therefore, we have that $\underline{y}' \geqslant \bar{x}' + (\|y-x\|/\sqrt{l})(0,0,\ldots,1,0,\ldots,0) - 2(\|y-x\|/K)(1,\ldots,1)$. Let

$$u = \frac{\|y-x\|}{2\sqrt{l}}\left(\frac{-4\sqrt{l}}{K},\ldots,\frac{-4\sqrt{l}}{K}, 2\left(1-\frac{2\sqrt{l}}{K}\right), \frac{-4\sqrt{l}}{K},\ldots,\frac{-4\sqrt{l}}{K}\right).$$

Consider $v \in R^l$, where

$$v = \left(\frac{-4\sqrt{l}}{K},\ldots,\frac{-4\sqrt{l}}{K}, 1, \frac{-4\sqrt{l}}{K},\ldots,\frac{-4\sqrt{l}}{K}\right).$$

Since $4\sqrt{l}/K < \lambda < 1$, we conclude that $v \in C_\lambda \setminus \{0\}$. Consequently, since $\underline{y}' \geqslant \bar{x}' + u \geqslant \bar{x}' + (\|y-x\|/2\sqrt{l})v$, it follows that $\underline{y}' - \bar{x}' \in C_\lambda$. As, moreover, $y' \in R^l_+$ and $y' \neq x'$, we obtain that $y' \succ x'$. Q.E.D.

Proof of the proposition. Let $\delta > 0$ and let a sequence of economies $\{E_n\}_{n\in N}$ with $\Sigma_{t\in E_n} w_t > |E_n|\delta e$ be given. Then, by the lemma, there exist an independent constant H and sequences $\{\{x^n_t\}_{t\in E_n}, p^n\}_{n\in N}$ of allocations and prices, so that for each $n \in N$ and each $t \in E_n$ one has $p^n x^n_t = p^n w_t$ and $\rho(x^n_t, \psi(t, p^n)) \leqslant \alpha_n$, where $\alpha_n = H/|E_n|$. Let $\eta = \eta(\delta/2)$ be determined by Lemma 2 of HSZ. Denote by Q a minimal integer, which exceeds $M/2$. Let B be a cube $[0,Q]^l$ and $\lambda_Q > 0$, so that each preference relation $\succ_t$, $t \in T$, is a C_{λ_Q}-monotone on B.

Consider an economy E_n, which belongs to the above sequence of economies. Suppose $x, y, x', y' \in R^l_+$, $y \geqslant x$, $\|y-x\| = (4\sqrt{l}/\lambda_Q)\alpha_n$. Then by Claim 2, $\max(\|x-x'\|, \|y-y'\|) \leqslant \alpha_n$ implies that $y' \succ_t x'$ for any $t \in E_n$. Denote $\beta_n = (4\sqrt{l}/\lambda_Q)\alpha_n$ and $\epsilon_n = 2\beta_n/\eta(\delta/2) = (2/\eta(\delta/2))(4\sqrt{l}/\lambda_Q)(H/|E_n|)$.

Thus, applying the standard arguments used in Grodal (1976), we conclude that an economy E_n has a non-empty ϵ_n-core, and this completes the proof of proposition. Q.E.D.

Remark. Let us note, that although for any $\lambda > 0$ every C_λ-monotone preference relation is strongly monotone, but for given $\bar{\lambda} > 0$, a strongly monotone preference relation cannot, in general, be approximated by a sequence of $C_{\bar{\lambda}}$-monotone preference relations. This can be illustrated by the following:

Example. Let $\bar{\lambda} > 0$. Define $\succ$ on R^l_+ by $x \succ y \Leftrightarrow x - y \in \text{int} C_{\bar{\lambda}/2}$. (Clearly, $\succ$ is strongly monotone.) Consider a sequence $(\succ_n)_{n \in N}$ of $C_{\bar{\lambda}}$-monotone preference relations and for each $n \in N$ denote $Gr(\succ_n) := \{(x, y) | x \succ_n y,\ x \in R^l_+,\ y \in R^l_+\}$. Then $[x, y \in R^l_+, x - y \in C_{\bar{\lambda}},\ x \neq y]$ implies that $[x \succ_n y,\ \forall n \in N]$. Moreover, $Gr(\succ_n) \supset \{(x, y) | x - y \in C_{\bar{\lambda}} \setminus \{0\}, y \in R^l_+,\ x \in R^l_+\}$ for all n and since $R^l_+ \times R^l_+ \setminus Gr(\succ_n)$ is closed we have that $(\succ_n)$ does not converge to $\succ$.

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