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A LIMIT THEOREM ON THE CORE

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The upper hemicontinuity in distribution of the core correspondence even in the case of economies with atoms is proved in a direct way without using prices. The distribution of agents' characteristics together with a specification of the atoms give a sufficient description of an economy as far as analysis of the core for large economies is concerned. Let $\mathscr{D}C(\mathscr{E})$ be the set of distributions of core allocations for the exchange economy $\mathscr{E}$. It is shown that for every distribution μ of agents' characteristics there is a standard representation $\mathscr{E}^{\mu}$ among the economies $\mathscr{E}$ having this distribution μ such that $\mathscr{D}C(\mathscr{E}^{\mu})$ is closed and contains $\mathscr{D}C(\mathscr{E})$.

1. Introduction

The well-known Limit Theorem on the core by Debreu and Scarf (1963) has been generalized recently by several authors: Hildenbrand (1970), Kannai (1970), Arrow and Hahn (1971), Grodal (1971), Nishino (1971), Brown and Robinson (1972), Khan (1973a, b, c), Bewley (1974), Grodal–Hildenbrand (1974) and Hildenbrand (1974). Grodal and Hildenbrand (1974), Bewley (1974) and Hildenbrand (1974) use a Theorem of Vind (1965) on existence of prices to prove the 'approximate' decentralization of core allocations in large but finite economies.

Arrow and Hahn (1971) and Nishino (1971) prove very similar results under different assumptions and in different setups.

Kannai applies the Theorem of Aumann (1964) on the equivalence between the core and Walras equilibria to obtain a continuity result for the core. Like Kannai also Grodal proves her continuity result on the core in the setups of continuous economies. But she does not use prices in her proof.

A quite different approach is the one of Brown and Robinson (1972) and Khan (1973a, b, c). They work with methods of non-standard analysis and prove the 'approximate' decentralization of core-allocations in large economies by applying a 'non-standard version' of the Equivalence Theorem.

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In the present paper, which is an outgrowth of Hildenbrand (1970), we define *competitive sequences* of finite economies, which are more general than the *purely competitive sequences*.

To such a competitive sequence $(\mathscr{E}_n)$ we associate a limit economy $\mathscr{E}^\mu$ and show that for every neighborhood U of $\mathscr{D}C(\mathscr{E}^\mu)$ and n large enough we have: $\mathscr{D}C(\mathscr{E}_n) \subset U$. $\mathscr{D}C(\mathscr{E})$ denotes the set of distributions of core allocations for the economy $\mathscr{E}$.

It is a main object of this paper to give a direct proof for this limit theorem in the atomless case without using prices. In this case one can then apply the Equivalence Theorem to approximately decentralize the core allocations for large economies. But we emphasize that our approach makes it possible to deal also with situations where per capita endowments of some agents do not tend to zero. Therefore the Limit Theorem provides a better insight into the nature of atoms in economies as treated in Gabszewicz–Drèze (1971), Gabszewicz–Mertens (1971) and Shitovitz (1974).

Some further results give a justification for the description of an economy by the distribution of agents' characteristics in the analysis of the core.

2. Notation and definitions

2.1. Notation

(2.1) $\mathbf{R}^l$ denotes the l-dimensional Euclidean space. For $x = (x_1, \ldots, x_l)$, $y = (y_1, \ldots, y_l) \in \mathbf{R}^l$, $x \geqq y$ means $x_i \geqq y_i$ for all i, $x > y$ means $x \geqq y$ and $x \neq y$, $x \gg y$ means $x_i > y_i$ for all i. For $x \in \mathbf{R}^l$ we will choose the norm defined by $\|x\| := \sum_{j=1}^{l} |x_j|$. Clearly, any other norm in $\mathbf{R}^l$ would work as well. $\mathbf{R}^l_+ := \{x \in \mathbf{R}^l \mid x \geqq 0\}$ is the positive orthant of $\mathbf{R}^l$. The closure of a set $M \subset \mathbf{R}^l$ is denoted by $\overline{M}$. For any set $M \subset \mathbf{R}^l_+$ and for any $\alpha \in \mathbf{R}_+ \setminus \{0\}$, $\alpha M := \{\alpha x \mid x \in M\}$.

2.2. Definitions

A *preference relation* is a continuous, irreflexive, transitive binary relation $\succ$ on $\mathbf{R}^l_+$. $\succ$ is *monotone* if $x > y$ implies $x \succ y$. $\mathscr{P}$ and $\mathscr{P}_{\mathrm{mo}}$ denote the sets of preferences and of monotone preferences, respectively. A consumer a is characterized by an element $(\succ(a), e(a)) \in \mathscr{P}_{mo} \times \mathbf{R}^l_+$, where the first component is his preference relation, the second one his bundle of initial endowment.

An *exchange economy* $\mathscr{E}$ is a measurable mapping from a probability space $(A, \mathscr{A}, \nu)$ into $\mathscr{P}_{\mathrm{mo}} \times \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ with finite mean endowment $\int \boldsymbol{e} \circ \mathscr{E} \, \mathrm{d}\nu$. The mapping $\boldsymbol{e}$ denotes the projection of $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ onto $\mathbf{R}^l_+$. With $\mathscr{E} = (\succ, e)$ we have therefore $e = \boldsymbol{e} \circ \mathscr{E}$.

An *allocation* f for the economy $\mathscr{E}$ is an integrable function from A into $\mathbf{R}^l_+$. It is *attainable* if $\int f \, \mathrm{d}\nu = \int e \, \mathrm{d}\nu$.

An exchange economy is called:

finite, if A is a finite set, every element of which has positive measure;

simple, if it is finite and ν is the equal distribution;

atomless, if the space $(A, \mathscr{A}, \nu)$ is atomless;

competitive with m atoms, if A consists of an atomless part T_0 with $\nu(T_0) > 0$ and m atoms t_i, $i = 1, \ldots, m$.

A set $S \in \mathscr{A}$ is a *coalition*; it is *effective* for the allocation f if

$$\int_S f \, \mathrm{d}\nu = \int_S e \, \mathrm{d}\nu.$$

The coalition S *can improve upon* an allocation f for the economy $\mathscr{E}$, if $\nu(S) > 0$ and if there exists an allocation g for $\mathscr{E}$ such that S is effective for g and $g(a) \succ_a f(a)$, ν – a.e. in S. The set of all attainable allocations for $\mathscr{E}$ which cannot be improved upon by any coalition in $\mathscr{A}$ is called the *core* of the economy $\mathscr{E}$ and is denoted $C(\mathscr{E})$.

Since for every agent $a \in A$ holds $\succ(a) \subset \mathbf{R}^l_+ \times \mathbf{R}^l_+$ and $\{e(a)\} \subset \mathbf{R}^l_+$, we can identify $\mathscr{E}(a)$ with the subset $\succ(a) \times e(a)\}$ of $(\mathbf{R}^l_+ \times \mathbf{R}^l_+) \times \mathbf{R}^l_+$. According to (2.1) the sets $g(a) \cdot \mathscr{E}(a)$ are well defined for any real-valued function g on A.

For any measurable g the function $g \cdot \mathscr{E}$ therefore is an economy if $g \cdot e$ is integrable.

For any economy $\mathscr{E} : (A, \mathscr{A}, \nu) \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ we define its *normalized version* by

$$\mathscr{E}^* : (A, \mathscr{A}, \nu^*) \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+,$$

with

$$\mathscr{E}^*(\cdot) := \|e(\cdot)\|^{-1} \cdot \mathscr{E}(\cdot),$$

and

$$\nu^*(S) := \left(\int_A \|e\| \, \mathrm{d}\nu\right)^{-1} \int_S \|e(\cdot)\| \, \mathrm{d}\nu,$$

for any $S \in \mathscr{A}$.

We will conclude this section with some measure theoretical notions which are used later.

If $(A, \mathscr{A}, \nu)$ is a measure space, the *norm* $|Q|$ of the measurable partition $Q = \{E_1, \ldots, E_k\}$ of A is the number $\max_{i=1,\ldots,k} \nu(E_i)$.

A sequence $(Q_n)_{n \in N}$ of partitions is *dense* – in $(A, \mathscr{A}, \nu)$ – if, for all $E \in \mathscr{A}$ and for all $\varepsilon > 0$, there exist $n \in N$ and $F_n \in \mathscr{A}$ such that

(1) F_n is a union of sets in Q_n;

(2) $\nu(E \triangle F_n) < \varepsilon$.

A correspondence $\varphi : (A, \mathscr{A}, \nu) \to \mathbf{R}^l_+$ is *measurable* if the graph of φ is an element of the product σ-algebra $\mathscr{A}_\nu \times \mathscr{B}[\mathbf{R}^l_+]$; ($\mathscr{A}_\nu$ denotes the completion of $\mathscr{A}$ relative to ν).

For a correspondence $\varphi : (A, \mathscr{A}, \nu) \to \mathbf{R}^l_+$, $\mathscr{L}_\varphi := \{f : A \to \mathbf{R}^l_+ \mid f \text{ integrable}, f(a) \in \varphi(a)\ \nu - \text{a.e.}\}$ is the set of measurable selections of φ.

A family M of measures on a metric space S is called *tight* if, for every $\varepsilon > 0$, there exists a compact set $K \subset S$ such that $\mu(K) > 1 - \varepsilon$ for every $\mu \in M$.

For a probability μ on a topological space T the smallest closed subset of T with measure one is called the support of μ and denoted supp(μ).

3. Competitive sequences

The atomless measure space of consumers as introduced by Aumann (1964) represents one way to describe perfect competition. Another one, the purely competitive sequence, has been defined by Hildenbrand (1974). We recall the definition:

Definition. A sequence $(\mathscr{E}_n)_{n \in N}$ of simple economies is called purely competitive if

(1) $A_n \to \infty$;

(3.1) (2) *the sequence $(\mu_n)_{n \in N}$ of distributions $\mu_n := \nu_n \circ \mathscr{E}_n^{-1}$ of economies $\mathscr{E}_n$ converges weakly in $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ to a distribution μ*;

(3) $\lim_{n \in N} \int e \, \mathrm{d}\mu_n = \int e \, \mathrm{d}\mu \gg 0$.

For bounded initial endowments, Condition (3) is implied by Condition (2), which simply describes convergence in distribution. Condition (3) asserts that the sequence (e_n) is uniformly integrable. But this is equivalent to

(3.2) $\nu_n(S_n) \to 0 \Rightarrow \int_{S_n} e_n \, \mathrm{d}\nu_n \to 0$.

Thus Condition (3) guarantees that for a group, whose measure tends to zero, also the mean endowment does so. Therefore for a group with a positive fraction of total endowments it is impossible to be negligible in the limit.

The notion of a purely competitive sequence was born out of the idea to describe perfect competition as an asymptotic property of a sequence of economies. The result was a theorem of Hildenbrand (1974) asserting that:

(3.3) *To every purely competitive sequence $(\mathscr{E}_n)$ one can associate an atomless limit economy $\mathscr{E}^\mu$ such that for every neighborhood U of $W(\mathscr{E})$ and for n large enough $C(\mathscr{E}_n) \subset U$.*

Combining this theorem with Aumann's Equivalence Theorem one gets upper hemicontinuity in distribution for the core in the case of an atomless limit economy.

Now if we replace atomless measure spaces by measure spaces with atoms and purely competitive sequences by sequences of economies which are not purely competitive, can we find a limit theorem connecting these concepts? Can we prove the above formulated limit theorem in situations where the Equivalence Theorem does not hold? From Gabszewicz–Mertens (1971) and Shitovitz (1974) we know that there exist measure spaces with atoms for which the Equivalence Theorem holds. But these situations seem to be exceptional. Moreover, even in these cases the concept of a purely competitive sequence does not work to get a limit theorem.

Before we try to answer the above questions let us look at an example of a sequence that is not purely competitive.

Example (Gabszewicz). Let $(\mathscr{E}_n)$ be a sequence of simple economies defined as follows:

$$\mathscr{E}_n : A_n \to \mathscr{P}_{\text{mo}} \times \mathbf{R}^l_+ ; \qquad A_n := \{1, \ldots, n+1\} ;$$

(3.4) $\nu_n(i) = 1/\# A_n = 1/(n+1)$ for all $i \in A_n$ by definition since the sequence is simple.

$$\succ_n(i) := \{(x, y) \in \mathbf{R}^2_+ \times \mathbf{R}^2_+ \mid \sqrt{x_1} + \sqrt{x_2} > \sqrt{y_1} + \sqrt{y_2}\},$$

for all $i \in A_n$; $e_n(1) := (0, 4n)$, $e_n(i) := (4, 0)$ for all $i \in A_n \backslash \{1\}$; $n \in N$.

Let us see now why this sequence is not purely competitive. Clearly Condition (1) of Definition (3.1) is fulfilled. For Condition (2) only the marginal distributions on $\mathbf{R}^2_+$ are interesting since $\succ_n(i)$ is constant for all $i \in A_n$ and all $n \in N$. If we denote them again by μ_n and μ and define μ by $\operatorname{supp}(\mu) := \{(4, 0)\} \subset \mathbf{R}^2_+$, also (2) is satisfied. But since $\int e \, d\mu_n = (0, 4n) \cdot 1/(n+1) + (4, 0) \cdot n/(n+1) = (4n/(n+1), 4n/(n+1))$ tends to (4,4), whereas $\int e \, d\mu = (4, 0) \not\gg 0$, our example violates Condition (3). Obviously the mean endowment in the limit is too small, because a significant part of the total endowment, i.e. all of commodity 2, is concentrated on a null set. Therefore it cannot be taken into account by the integral. Formally we describe this situation by saying that the sequence (e_n) of initial endowments is not uniformly integrable. Looking at the sequence we can observe the following phenomenon. As long as we are not in the limit the increase in resources is balanced by the decrease in measure for the agents. What really counts seems to be the product of resources and measures. But we cannot expect to get a suitable description of the limit behavior of the sequence unless we guarantee that this product does not degenerate in the limit. Therefore we have to look for a new representation of the economic situation described in the example. In this new representation both factors of the product, the measure and the initial endowment of every agent, have to be bounded away from zero and infinity. Clearly the new measure must be different from the counting measure. One way of getting such a new measure is the following one.

Let us consider an economy $\mathscr{E}$. We normalize every agent's endowment in the given economy $\mathscr{E}$, i.e., we replace $e(a)$ by $e^*(a) := e(a)/\|e(a)\|$. To keep the product of resources and measures constant when changing from the old ones to the new ones we must define the new measure for a by $\nu(a)\|e(a)\|$. But since we want ν^* to be a probability measure we have to normalize the measure $\|e(\cdot)\|\,\nu$. So we get $\nu^* : \mathscr{A} \to [0, 1]$ with $\nu^*(S) = (\int \|e\| \,\mathrm{d}\nu)^{-1} \int_S \|e(\cdot)\| \,\mathrm{d}\nu$ for any $S \in \mathscr{A}$, i.e., $\nu^* = (\int \|e(\cdot)\| \,\mathrm{d}\nu)^{-1}\|e(\cdot)\|\,\nu$. Replacing now the preferences $\succ(a)$ by $\succ^*(a) := 1/\|e(a)\| \succ (a)$, we get the normalized version $\mathscr{E}^*$ of $\mathscr{E}$. If we now replace all allocations f for $\mathscr{E}$ by the allocations $f^*(\cdot) := \|e(\cdot)\|^{-1}f(\cdot)$ we have a complete new description of the same economic situation.

To make this clear, we have to prove that all notions which are important for analyzing the core are invariant under the transition from $\mathscr{E}$ to $\mathscr{E}^*$. These notions are 'preferred to', 'monotone', 'attainable', 'effective'. We will prove now this invariance:

'*preferred to*'

$$(f(a), g(a)) \in\, \succ (a) \Leftrightarrow (\|e(a)\|^{-1}f(a),\, \|e(a)\|^{-1}g(a)) \in \|e(a)\|^{-1} \succ (a).$$

'*monotone*'

$$\begin{array}{ccc} f(a) > g(a) & \Rightarrow & (f(a), g(a)) \in\, \succ (a) \\ \Updownarrow & & \Updownarrow \\ \|e(a)\|^{-1}f(a) > \|e(a)\|^{-1}g(a) & \Rightarrow & (\|e(a)\|^{-1}f(a),\, \|e(a)\|^{-1}g(a)) \\ & & \in \|e(a)\|^{-1} \succ (a). \end{array}$$

As the diagram shows any of the two implications implies the other one.

'*effective*'

$$\int_S e^* \,\mathrm{d}\nu^* = \int_S \frac{e}{\|e\|}\, \|e\| \left(\int \|e\| \,\mathrm{d}\nu\right)^{-1} \mathrm{d}\nu = \left(\int \|e\| \,\mathrm{d}\nu\right)^{-1} \int_S e \,\mathrm{d}\nu.$$

In the same way we get

$$\int_S f^* \,\mathrm{d}\nu^* = (\int \|e\| \,\mathrm{d}\nu)^{-1} \int_S f \,\mathrm{d}\nu.$$

Therefore S is effective for f in $\mathscr{E}$ if and only if it is effective for f^* in $\mathscr{E}^*$. Putting $S = A$ we get that also '*attainable*' is invariant.

We will replace now the sequence of economies in our example by the sequence of their normalized versions.

We get:

$$\mathscr{E}_n^*: A_n \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^2\,; \qquad A_n := \{1, \ldots, n+1\}\,;$$

$$e_n^*(1) = \frac{(0, 4n)}{\|(0, 4n)\|} = (0, 1)\,; \qquad e_n^*(i) = \frac{(4, 0)}{\|(4, 0)\|} = (1, 0),$$

for all $i \in A_n \setminus \{1\}$;

(3.5) $\succ_n^*(i) = \succ_n(i)$, for all $i \in A_n \setminus \{1\}$;

$$v_n^*(1) = \left(4n \cdot \frac{1}{n+1} + n \cdot 4 \cdot \frac{1}{n+1}\right)^{-1} \cdot 4n \cdot \frac{1}{n+1} = \frac{1}{2}\,;$$

$$v_n^*(i) = \left(4n \cdot \frac{1}{n+1} + n \cdot 4 \cdot \frac{1}{n+1}\right)^{-1} \cdot 4 \cdot \frac{1}{n+1} = \frac{1}{2n},$$

for all $i \in A_n \setminus \{1\}$.

After having replaced the counting measure v by the measure v^* we may ask what is measured by v^*. To answer this we look at our example. For $i \in A_n$, we get

$$\begin{aligned} v^*(i) &= \left(\int \|e\| \,\mathrm{d}v\right)^{-1} \int_{\{i\}} \|e(i)\| \,\mathrm{d}v \\ &= \frac{\|e(i)\| \cdot (1/\# A_n)}{\sum_{j \in A_n} \|e(j)\| \cdot 1/(\# A_n)} \qquad (3.6) \\ &= \frac{\|e(i)\|}{\sum_{j \in A_n} \|e(j)\|}. \end{aligned}$$

$v^*(i)$ expresses the fraction of total endowments possessed by agent i. Using the norm we have chosen means that for evaluating an agent's fraction one simply adds up butter and tea.

While in the economies of the original sequence all agents have general common units for the l commodities, now in the normalized versions, every agent has his own, possibly different, scale for counting commodities, which depends on the size of his own fraction of the total endowments. Whereas the old measures describe coalitions' sizes, the new ones describe the relative sizes of coalitions' endowments.

Let us check now whether the new sequence $\mathscr{E}_n^*$ is purely competitive.Defining $\mu^* := \mu_n^*$, we get

$$\int e \,\mathrm{d}\mu_n^* = \int e \,\mathrm{d}\mu^* = (0, 1) \cdot \tfrac{1}{2} + (1, 0) \cdot \tfrac{1}{2} = (\tfrac{1}{2}, \tfrac{1}{2}) \gg 0.$$

Obviously all conditions of definition (3.1) are satisfied. But the economies $\mathscr{E}_n^*$ are not simple. Therefore we are going to define a more general class of sequences:

Definition. A sequence $(\mathscr{E}_n)$ *of finite economies* $\mathscr{E}_n : A_n \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ *with* $A_n = \{r_1^n, \ldots, t_m^n, a_1^n, \ldots, a^n_{\# A_n - m}\}$ *is called competitive with m atoms, if*

(3.7)

(1) $\# A_n \to \infty$;

(2) (μ_n) *converges weakly on* $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ *to* μ;

(3) $\lim \int e \, \mathrm{d}\mu_n = \int e \, \mathrm{d}\mu \gg 0$;

(4) $\sup_{i=1,\ldots,\# A_n - m} v_n(a_i^n) \to_{n\to\infty} 0$;

(5) *for all* $i \in \{1, \ldots, m\}$: (a) $(v(t_i^n))_{n\in N}$ *is a constant sequence,* and (b) $(\mathscr{E}_n(t_i^n))_{n\in N}$ *converges in* $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$.

Apparently our sequence $(\mathscr{E}_n^*)$ is competitive with one atom:

$$t_n^1 \equiv 1, \qquad \text{for all} \quad n \in N,$$

$$a_i^n \equiv i+1, \qquad \text{for all} \quad i \in \{1, \ldots, A_n - 1\} = \#\{1, \ldots, n\}.$$

This is only true since the limit preferences are monotone. In the special case of our example the old preferences and the new ones are identical with their monotone limit preference since they are homothetic, i.e., $\succ = \lambda \succ$ for $\lambda > 0$. But in general any sequence of preferences with $\lim \succ_n^* \in \mathscr{P}_{\mathrm{mo}}$ is possible.

We are going now to discuss Conditions (4) and (5) of the above definition.

Since the economies of a competitive sequence are not necessarily simple, we need a condition that guarantees that all the agents, who don't have a significant portion of the total endowment are asymptotically negligible. That is just the meaning of Condition (4).

It seems to be perhaps more difficult to justify Condition (5). It relates to those agents who are not negligible. While the a_i^n for growing n lose more and more their identity and, as we will see, eventually disappear in the ocean of a limit economy, the t_i^n will remain observable and distinguishable even in the limit. The t_i^n and a_i^n represent the discrete and continuous part of an hypothetical distribution in the nth sample of size $\# A_n$. Since the economies of the sequence for growing n should describe approximately the same economic situation, the characteristics of the significant economic agents should be approximately the same. Therefore the convergence of the two terms in Condition (5) is a reasonable condition. That we postulate the first sequence even to be constant has only technical reasons.[1]

[1]If in our example $e_n(t_1^n) = (0, 4n)$ is replaced by $e_n(t_1^n) = (0, 4n-(1/n))$, we get

$$v_n^*(t_1^n) < v^*_{n+1}(t_1^{n+1}) < \ldots < \lim v_n^*(t_1^n) = v(t_1) = \tfrac{1}{2}.$$

For $\mathscr{E}_n$ the only Walras allocation is f_n with

$$(3.8) \qquad f_n(1) = (2n, 2n), \qquad f_n(i) = (2, 2), \qquad \text{for all} \quad i \in A_n \backslash \{1\},$$

with prices $p = (\frac{1}{2}, \frac{1}{2})$.

But the core contains the set of all allocations g_n^λ with $g_n^\lambda(1) = (\lambda n, \lambda n)$, $g_n^\lambda(i) = (4-\lambda, 4-\lambda)$ for all $\lambda \in [2, 3]$ and for all $i \in A_n \backslash \{1\}$. For f_n^*, $g_n^{*\lambda}$ we get

$$f_n^*(1) = \frac{(2n, 2n)}{4n} = (\tfrac{1}{2}, \tfrac{1}{2}),$$

$$f_n^*(i) = \frac{(2, 2)}{4} = (\tfrac{1}{2}, \tfrac{1}{2}),$$

$$(3.9) \quad g_n^{*\lambda}(1) = \frac{(\lambda n, \lambda n)}{4n} = \left(\frac{\lambda}{4}, \frac{\lambda}{4}\right), \qquad \lambda \in [2, 3],$$

$$g_n^{*\lambda}(i) = \frac{(4-\lambda, 4-\lambda)}{4} = \left(\frac{4-\lambda}{4}, \frac{4-\lambda}{4}\right), \qquad \lambda \in [2, 3],$$

for all $\quad i \in A_n \backslash \{1\}$.

For the normalized versions we get

$$f^*(1) = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad g^{*\lambda}(1) = \left(\frac{\lambda}{4}, \frac{\lambda}{4}\right),$$

$$(3.10) \quad \lim_n f_n^*(i_n) = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad \lim_n g_n^{*\lambda}(i_n) = \left(\frac{4-\lambda}{4}, \frac{4-\lambda}{4}\right),$$

$\lambda \in [2, 3]$ for any sequence (i_n) with $i_n \in A_n \backslash \{1\}$.

Since $(\mathscr{E}_n^*)$ is competitive with one atom we may apply the limit theorem of the next section. Therefore the limits of f_n^* and $g_n^{*\lambda}$ can be described by core allocation of a competitive economy with one atom. We get

$$e(t_1) := \lim e_n^*(1) = (0, 1)$$

If by a continuous representation, as defined in the next section, the $t_1{}^n$ and t_1 are imbedded in an abstract measure space, we would get $t_1{}^n \subset t_1{}^{n+1} \subset \ldots \subset t_1$. But this implies by $0 < \nu(t_1{}^n) = \nu_n(t_1{}^n) < \nu(t_1)$ that t_1 cannot be an atom. In this case we can normalize in a slightly modified way and get a sequence of economies where Condition (5) is fulfilled. For details see Trockel (1974).

$$v(t_1) := \lim v_n^*(1) = \tfrac{1}{2}$$

$$f(t_1) := f^*(1) = (\tfrac{1}{2}, \tfrac{1}{2}), \qquad g^\lambda(t_1) := g^{*\lambda}(1) = \left(\frac{\lambda}{4}, \frac{\lambda}{4}\right).$$

(3.11) $$e(a) = \lim e_n^*(i_n) = (1, 0)$$

$$f(a) = \lim f_n^*(i_n) = (\tfrac{1}{2}, \tfrac{1}{2})$$

$$g^\lambda(a) = \lim g_n^{*\lambda}(i_n) = \left(\frac{4-\lambda}{4}, \frac{4-\lambda}{4}\right),$$

for all $\lambda \in [2, 3]$, for any sequence (i_n) with $i_n \in A_n \backslash \{1\}$, and for every $a \in T_0$ with $v(a) = 0$ and $v(T_0) = \frac{1}{2}$.

In our original example in all economies of the sequence $\mathscr{E}_n$ the set of Walras allocations is a proper subset of the core. We would have supposed that also in the limit there is no perfect competitition. But as the example was formulated it was impossible to make this conjecture precise, because there was no limit economy. By transition to the normalized versions we made it possible to associate a limit economy to our sequence, to make the conjecture precise and, applying the limit theorem, even to prove it. Thus we have produced an effective connection between competitive sequences with m atoms and competitive economies with m atoms. Moreover every such economy is the limit of a suitably chosen sequence of this kind. This is particularly true for the economies with atoms as defined in Gabszewicz–Mertens (1971) and Shitovitz (1974) as long as the number of atoms is finite. We have intentionally restricted ourselves to the case of finitely many atoms, since otherwise a sequence of economies could not describe approximately the economic situation of the limit economy because no economy of the sequence contains all significant agents, i.e., atoms.

To point out the generality of our method we emphasize that the 'rich' agent in a sequence of economies needs not to be a monopolist in some commodity. This fact in our example is unessential. We also can treat situations where the mean endowment of a group whose measure tends to zero, does not tend to zero. As long as the endowment of every member of the group tends to zero, we get an atomless limit economy. Also this situation cannot be analyzed in the context of purely competitive sequences, because one needs measures which are not simple. Let us try now to interpret the atoms in a competitive economy.

The 'rich' individuals can be identified in each member economy of the sequence and even in the limit, where they become proper atoms. This possibility of tracing back the atoms of the limit economy to the finite economies means that we may interpret the atoms in the same way we did their predecessors in the finite economies of the sequences. However, we want to point out that the t_i^n in the sequence $(t_i^n)_{n \in N}$ need not be identical. But the fact that all t_i^n, $n \in N$, for fixed i, have the same measures and, for large n, have approximately the same characteristics, thus characterizing the atom t_i of the limit economy, justifies the notion 'competitive sequence with m atoms'. Therefore an atom must be interpreted as an agent who owns and consumes much more than an average agent of the economy. His standard of evaluating commodities and exchanging is totally different from the average one. Clearly, this allows us to interpret a 'rich' man as a manager of a syndicate of identical traders, however questionable this concept may be. On the other hand this seems to be the only way to explain that an agent really consumes much more than the average agent.

4. The limit theorem

The following definition provides an important tool for the proof of the limit theorem:

Definition. A continuous representation for a competitive sequence $(\mathscr{E}_n)_{n\in N}$ *is a triple* $[(A, \mathscr{A}, \nu), \mathscr{E}, (\alpha_n)_{n\in N}]$ *with the following properties:*

(4.1)

(1) $(A, \mathscr{A}, \nu)$ *is a probability space;*

(2) $\mathscr{E}: A \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ *and* $\alpha_n : A \to A_n$, $n \in N$, *are measurable mappings;*

(3) $\nu \circ \alpha_n^{-1} = \nu_n$, $n \in N$;

(4) $\lim_{n\in N} \mathscr{E}_n(\alpha_n(a)) = \mathscr{E}(a)$, ν-a.e. *in* A.

Lemma 1 (Halmos). *Every sequence* $(Q_n)_{n\in N}$ *of Borel partitions of* $[0, 1]$ *with* $\lim_n |Q_n| = 0$ *is dense in* $([0, 1], \mathscr{B}[0, 1], \lambda_{[0,1]})$.

For a proof see Halmos (1950, p. 172).

Lemma 2. $\mathscr{P}_{\mathrm{mo}}$ *is a* G_δ *set in the set* $\mathscr{P}$ *of preferences.*

This lemma has been proved independently by Grodal (1974) and Trockel (1974).

Lemma 3. For every competitive sequence $(\mathscr{E}_n)_{n\in N}$ *with* m *atoms there exists a continuous representation* $[(A, \mathscr{A}, \nu), \mathscr{E}, (\alpha_n)_{n\in N}]$ *such that* $A = [0, 1] \cup \bigcup_{i=1}^m \{i+1\}$,

$$\mathscr{A} = \sigma\left(\mathscr{B}[0, 1] \cup \bigcup_{i=1}^{m} \{i+1\}\right), \qquad \nu(\{i+1\} = \nu_n(t_i^n),$$

$$\nu|_{[0,1]} = \left[1 - \sum_{i=1}^{m} \nu(\{i+1\})\right] \cdot \lambda_{[0,1]}.$$

Proof. The proof using Skorokhod's Theorem is essentially the same as the proof of Proposition 2 in Hildenbrand (1974, p. 139). The extension to the more general case is straight forward. The atomless part can be chosen to be [0, 1] since by Lemma 2 $\mathscr{P}_{\mathrm{mo}}$ is topologically complete. Therefore we define for the rest of the paper $A := [0, 1] \cup \bigcup_{i=1}^{m} \{i+1\}$.

Lemma 4 (Hildenbrand). *For every competitive sequence $(\mathscr{E}_n)_{n\in N}$ of economies every sequence $(f_n)_{n\in N}, f_n \in C(\mathscr{E}_n)$ is uniformly integrable.*

Proof. The proof given in Hildenbrand (1974, p. 182) for purely competitive sequences works as well in our general case.

The mappings α_n in the next lemma are the same as in Definition (4.1). They have the property that

$$\mathscr{E}_n : A_n \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+,$$

and

$$\tilde{\mathscr{E}}_n := \mathscr{E}_n \circ \alpha_n : A \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+, \qquad n \in N,$$

have the same distributions.

Lemma 5. For every integrable function $g : A \to \mathbf{R}^l$ the sequence $(\tilde{g}_n)_{n\in N}$ with $\tilde{g}_n := E(g|\alpha_n^{-1}(PA_n))$[2] *converges to g ν – a.e.*

Proof. Assume to the contrary that $\nu\{a \in A \mid \tilde{g}_n(a) \to g(a)\} < 1$. Then there exists an $\varepsilon > 0$ such that for infinitely many $m \in N$ the measure of the set

$$(4.2) \quad C_m^\varepsilon := \left\{ a \in A \,\middle|\, \sup_{n \geqq m} \left| \tilde{g}_n(a) - g(a) \right| \geqq \varepsilon \right\}$$

is larger than $\rho > 0$. Since $C_m^\varepsilon \supset C_{m+1}^\varepsilon$ for all $m \in N$ we have $\nu(C_m^\varepsilon) > \rho$. Therefore $\lim_m \nu(C_m^\varepsilon) \geqq \rho$ exists. By continuity of the measure ν this implies

$$\lim_m \nu(C_m^\varepsilon) = \nu\left(\bigcap_m C_m^\varepsilon \right) \geqq \delta.$$

Then without loss of generality for some $h \in \{1, \ldots, l\}$ the set

$$(4.3) \quad C^\varepsilon := \left\{ a \in A \,\middle|\, \inf_m \sup_{n \geqq m} (\tilde{g}_n^h(a) - g^h(a)) \geqq \varepsilon \right\} \in \mathscr{A}$$

[2] PA_n denotes the set of subsets of A_n, $E(g \mid \alpha_n^{-1}(PA_n))$ denotes the conditional expectation of g given the σ-algebra $\alpha_n^{-1}(PA_n)$.

has measure $\delta_\varepsilon > 0$. The sequence $(\alpha_n^{-1}(\boldsymbol{P}A_n) \cap [0,1])_{n\in N}$ is by Lemma 1 and by (4) of (3.7) dense in $([0,1], \mathscr{B}[0,1], \nu|_{[0,1]})$. Therefore there exist $Q' \subset Q$ and for all $n \in Q'$ a $C_n \in \alpha_n^{-1}(\boldsymbol{P}A_n)$ such that $\nu(C^\varepsilon \triangle C_n)_{n\in Q'} \to 0$, since by definition the restrictions of $\tilde{g}_n$ and g_n to the atoms $\{i+1\} = \alpha_n^{-1}(t_i)$, $i \in \{1, \ldots, m\}$, are the same. By definition of the $\tilde{g}_n$, $n \in N$, that implies:

$$
\begin{aligned}
0 &= \textstyle\int_{C_n} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu \\
&= \textstyle\int_{C_n \cap C^\varepsilon} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu + \int_{C_n \setminus C^\varepsilon} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu \\
&= \textstyle\int_{C^\varepsilon} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu - \int_{C^\varepsilon \setminus C_n} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu \\
&\quad + \textstyle\int_{C_n \setminus C^\varepsilon} (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu \\
&> \textstyle\int_{C^\varepsilon} \varepsilon\,\mathrm{d}\nu - \eta_n,
\end{aligned} \tag{4.4}
$$

with

$$\eta_n := \textstyle\int (\mathbf{1}_{C_n \setminus C^\varepsilon} - \mathbf{1}_{C^\varepsilon \setminus C_n}) \cdot (g^h - \tilde{g}_n^h)\,\mathrm{d}\nu. \tag{4.5}$$

Since by Neveu (1965, p. 124) $(\tilde{g}_n)_{n\in N}$ is uniformly integrable by the integrability of g also $(g^h - \tilde{g}_n^h)_{n\in N}$ is so. Therefore η_n in (4.5) converges with $n \in Q'$ to zero. For n large enough we therefore have with

$$0 > \textstyle\int_{C^\varepsilon} \varepsilon\,\mathrm{d}\nu - \eta_n = \delta_\varepsilon \cdot \varepsilon - \eta_n > 0, \tag{4.6}$$

a contradiction. Q.E.D.

Before formulating and proving the Limit Theorem we will give a useful alternative definition of the core. For an attainable allocation of the economy $\mathscr{E}$ we define the correspondence $\psi_{\mathscr{E},f}$ from A into $\mathbf{R}^l$ by

$$\psi_{\mathscr{E},f}(a) := \{x \in \mathbf{R}^l \mid f(a) \prec_a x + e(a)\} \cup \{0\}.$$

Proposition 1. Let f be an attainable allocation for the economy $\mathscr{E}$. Then the following three assertions are equivalent:

$$
\begin{aligned}
&\text{(a)}\ \ f \in C(\mathscr{E}); \\
&\text{(b)}\ \ \textit{for all}\ \ h \in \mathscr{L}_{\psi_{\mathscr{E},f}}: \ \textstyle\int h\,\mathrm{d}\nu \nless 0; \\
&\text{(c)}\ \ \textit{for all}\ \ h \in \mathscr{L}_{\psi_{\mathscr{E},f}}: \ \textstyle\int h\,\mathrm{d}\nu \nless 0.
\end{aligned} \tag{4.7}
$$

Proof. To prove ((a) ⇒ (c)) and ((b) ⇒ (a)) one needs only standard arguments. For ((a) ⇒ (c)) see also Hildenbrand (1974, p. 134). (c) ⇒ (b): Let

$$\bar{h} \in \mathscr{L}_{\psi_{\mathscr{E},f}}, \quad \textstyle\int \bar{h}\,\mathrm{d}\nu < 0, \quad S := \{a \in A \mid \bar{h}(a) \neq 0\}.$$

Then we have $\bar{h}(a) = \lim_n z_n(a)$ for a suitable sequence $(z_n(a))_{n\in N}$ with

$z_n(a)+e(a) \succ_a f(a)$ for all $n \in N$ or $\bar{h}(a) = 0$ ν–a.e. in A. We define h by

$$(4.8) \quad h(a) := \begin{cases} \bar{h}(a) - \dfrac{1}{2\nu(S)} \displaystyle\int \bar{h}\,\mathrm{d}\nu, & \text{for } a \in S, \\ \bar{h}(a) = 0, & \text{for } a \notin S. \end{cases}$$

Since $z_n(a) \to_n \bar{h}(a)$ ν–a.e. in S, by continuity of preferences there exists ν–a.e. in S a $n_a \in N$ with $h(a)+e(a) \succ_a z_{n_a}+e(a) \succ_a f(a)$. By transitivity we therefore have ν – a.e. in S

$$h(a)+e(a) \succ_a f(a),$$

and, since $h(a) = 0$ for $a \notin S$, $h \in \mathscr{L}_{\psi_{\mathscr{E},f}}$. Finally, we have also

$$(4.9) \quad \int h\,\mathrm{d}\nu = \int \bar{h}\,\mathrm{d}\nu - \frac{1}{2\nu(S)}\int \bar{h}\,\mathrm{d}\nu < 0.$$

Q.E.D.

Theorem 1 (Limit Theorem). For every competitive sequence $(\mathscr{E}_n)_{n\in N}$ and for every sequence $(f_n)_{n\in N}, f_n \in C(\mathscr{E}_n)$, there exist subsequences $(\mathscr{E}_n)_{n\in Q}, (f_n)_{n\in Q}$, $Q \subset N$, an economy $\mathscr{E}$ and an allocation $f \in C(\mathscr{E})$ such that the joint distributions τ_n of $\mathscr{E}_n$ and f_n converge weakly to the joint distribution τ of $\mathscr{E}$ and f.

The theorem is essentially a generalization of Theorem 5 in Grodal (1971). We will only sketch the proof. For details see Trockel (1974).

From Lemma 4 we get uniform integrability of the sequence (f_n) and thus convergence in distribution. Using Skorokhod's Theorem[3] and some standard arguments we get a continuous representation $[(A, \mathscr{A}, \nu), \mathscr{E}, (\alpha_n)]$ with $f_n := f_n \circ \alpha_n$ converging a.e. to an attainable allocation f.

It remains to show that f cannot be improved upon. The proof makes use of the characterization of the core given in Proposition 1. We assume $f \notin C(\mathscr{E})$ and can choose therefore an $h \in \mathscr{L}_{\psi_{\mathscr{E},f}}$ with $\int h\,\mathrm{d}\nu < 0$. Then we define a sequence (h_n) by slightly modifying the $\bar{h}_n := E(h \mid \alpha_n^{-1}(PA_n))$. By Lemma 5 we get the convergence a.e. of this sequence to h. Using a quite general form of Egorov's Theorem and Lemma 1 we get by some standard arguments $h_n \in \mathscr{L}_{\psi_{\tilde{\mathscr{E}}_n,\tilde{f}_n}}$ and $\int h_n\,\mathrm{d}\nu < 0$. But this by Proposition 1 is equivalent to $f_n \notin C(\mathscr{E}_n)$.

[3] *Skorokhod's Theorem: Let $(\mu_n)_{n\in N}$ be a weakly convergent sequence of probabilities on a separable metric space T with limit μ. Then there exist a probability space $(A, \mathscr{A}, \nu)$ and measurable mappings f and f_n, $n \in N$, from A into T such that μ and μ_n, $n \in N$, are their respective distributions,* i.e., $\mu = \nu \circ f^{-1}$, $\mu_n = \nu \circ f_n^{-1}$, $n \in N$, *and* $\lim_n f_n = f$ ν–a.e. *in A. If T is complete one chooses $(A, \mathscr{A}, \nu)$ to be* $([0, 1], \mathscr{B}[0, 1], \lambda_{[0,1]})$. For a proof see Skorokhod (1965).

Remark. It is Lemma 5 which allows us to get rid of the assumption that $\alpha_n^{-1}(\boldsymbol{P}A_n) \subset \alpha_n^{-1}(\boldsymbol{P}A_{n+1})$ for all n. Since this lemma is true only in the case that the atomless part of A in the standard representation is the unit interval, we need Lemma 2 which allows us to choose A in that way. Since we know from Grodal (1974) that all interesting sets of preferences are Borelian, too, one could ask why we restrict ourselves to the case of monotone preferences in our Theorem. But monotony is essential for Lemma 4 which is needed in the proof of our Limit Theorem.

5. Standard representations

Let us define now the correspondence γ by

$$(5.1) \quad \gamma : \mathscr{P}_{\text{mo}} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+ \twoheadrightarrow \mathbf{R}^l_+$$

$$(r, y) \mapsto \{x \in \mathbf{R}^l \mid y \prec_r x + e_r\} \cup \{0\},$$

with $r = (\succ_r, e_r)$. Then we have

$$(5.2) \quad \gamma \circ (\mathscr{E}, f)(a) = \gamma(\mathscr{E}(a), f(a)) = \psi_{\mathscr{E},f}(a)$$

$$= \{x \in \mathbf{R}^l \mid f(a) \prec_a x + e(a)\} \cup \{0\}.$$

By Theorem 1 we get for an attainable allocation f for $\mathscr{E}$

$$(5.3) \quad f \in \boldsymbol{C}(\mathscr{E}) \Leftrightarrow \int \bar{\gamma} \circ (\mathscr{E}, f) \,\mathrm{d}\nu \cap \mathbf{R}^l_- = \{0\}.$$

(5.4) *Definition. Let $\mathscr{E}_j$, $j = 1, 2$, be two competitive economies with m atoms such that $\mathscr{E}_1(t_i^1) = \mathscr{E}_2(t_i^2)$ and $\nu_1(\{t_i^1\}) = \nu_2(\{t_i^2\})$ for $i \in \{1, \ldots, m\}$ and $\nu_1 \circ \mathscr{E}_1^{-1} = \nu_2 \circ \mathscr{E}_2^{-1}$. Then $\mathscr{E}_1$ and $\mathscr{E}_2$ are called m-equivalent.*

Proposition 2. Let $\mathscr{E}_j$, $j = 1, 2$, be m-equivalent. Let f_j be allocations for $\mathscr{E}_j$, $j = 1, 2$, with $f_1(t_i^1) = f_2(t_i^2)$, $i \in \{1, \ldots, m\}$ and $\tau := \nu_1 \circ (\mathscr{E}_1, f_1)^{-1} = \nu_2 \circ (\mathscr{E}_2, f_2)^{-1}$. Then $f_1 \in \boldsymbol{C}(\mathscr{E}_1) \Leftrightarrow f_2 \in \boldsymbol{C}(\mathscr{E}_2)$.

Proof. By the Theorems 4 and 5 in Hildenbrand (1974, pp. 64, 67) we get

$$\int \bar{\gamma} \circ (\mathscr{E}_1, f_1)\, \mathrm{d}\nu_1$$

$$= \int_{T_0{}^1} \bar{\gamma} \circ (\mathscr{E}_1, f_1)\, \mathrm{d}\nu_1 + \sum_{i=1}^{m} \bar{\gamma}((\mathscr{E}_1, f_1)(t_i^1))\, \nu_1(\{t_i^1\})$$

$$(5.5) \quad = \int \bar{\gamma}\, \mathrm{d}\left(\nu_1 \circ (\mathscr{E}_1, f_1) \Big|_{T_0{}^1}^{-1} \right) + \sum_{i=1}^{m} \bar{\gamma}((\mathscr{E}_1, f_1)(t_i^1))\, \nu_1(\{t_i^1\})$$

$$= \int \bar{\gamma}\, \mathrm{d}\left(\nu_2 \circ (\mathscr{E}_2, f_2) \Big|_{T_0{}^2}^{-1} \right) + \sum_{i=1}^{m} \bar{\gamma}((\mathscr{E}_2, f_2)(t_i^2))\, \nu_2(\{t_i^2\})$$

$$= \int \bar{\gamma} \circ (\mathscr{E}_2, f_2)\, \mathrm{d}\nu_2 .$$

Q.E.D.

Now we will answer the question whether there is a natural representation $\mathscr{E}$ of a distribution μ on $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l$. Clearly, it is possible to find two competitive economies with different numbers of atoms having the same distribution μ on $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l$. Thus, since μ does not give us enough information to find a standard economy, we need the additional specification of the number of atoms and their measures and characteristics for a full description of the economy. The notion of a standard representation of an economy in the atomless case has been introduced by Hart et al. (1974). We extend this definition for the case of competitive economies.

Definition. Let μ *be a probability on* $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l$, $c_i \in (0, 1)$, $r_i \in \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l$, $i = 1, \ldots, m$, *with* $\mu(r_i) \geqq c_i, \sum c_i < 1$. *A standard representation for* $(\mu, (c_i, r_i)_{i=1,\ldots,m})$ *is a mapping*

$$(5.6) \quad \mathscr{E}^{\mu}_{c_i, r_i, m} : \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l \times [0, 1] \to \mathscr{P}_{\mathrm{mo}} \times \mathbf{R}_+^l$$

$$(\succ, e, \xi) \mapsto (\succ, e)$$

such that for $\nu = \mu \otimes \lambda_{[0,1]}$: $\nu(\{(\succ_j, e_j)\} \times [\alpha_j, \beta_j]) = c_j$ *and* $\mathscr{E}^{\mu}_{c_i, r_i, m}(\{(\succ_j, e_j)\} \times [\alpha_j, \beta_j] = r_j$ *for some* $\alpha_j, \beta_j \in [0, 1], j \in \{1, \ldots, m\}$.

Remark. Every standard representation represents an m-equivalence class. If the meaning is clear from the context we will write $\mathscr{E}^{\mu}_{m}$ instead of $\mathscr{E}^{\mu}_{c_i, r_i, m}$.

For an economy $\mathscr{E}$ we denote the set of all distributions of core allocations by $\mathscr{D}C(\mathscr{E})$, i.e.,

(5.7) $\mathscr{D}C(\mathscr{E}) := \{\mathscr{D}f \mid f \in C(\mathscr{E}), \mathscr{D}f := \nu \circ f^{-1}\}$.

The next two theorems will give some information on standard representations.

Proposition 3. For a standard representation $\mathscr{E}_m$ the set $\mathscr{D}C(\mathscr{E}^\mu_m)$ is weakly closed.

The proof of Proposition 3 consists of two parts. One part, the proof for the atomless case has been given in Hart et al. (1974, p. 164). The second part is a lengthy construction of a modified version of the measure space provided by Skorokhod's Theorem. Its result is that we can restrict the proof to the atomless case. For details see Trockel (1974).

Proposition 4. For every competitive economy $\mathscr{E}'$ which is m-equivalent to $\mathscr{E}^\mu_m$ holds

$$\overline{\mathscr{D}C(\mathscr{E}')} \subset \mathscr{D}C(\mathscr{E}^\mu_m).$$

Proof. Let $\delta \in \mathscr{D}C(\mathscr{E}')$. Then there is an $f' \in C(\mathscr{E}')$ with $\delta = \nu' \circ f'^{-1} = \mathscr{D}f'$. We define $\tau := \nu' \circ (\mathscr{E}', f')^{-1}$ and get a mapping from $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+ \times [0, 1]$ into $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+$ with distribution τ such that $(r, \xi) \mapsto (r, f(r, \xi))$. This works in the same way as in Proposition 3 and the corresponding Theorem 2 of Hart et al. (1974, p. 164). By Proposition 2 then, we have $f \in C(\mathscr{E}^\mu_m)$ and therefore $\nu \circ f^{-1} = \delta \in \mathscr{D}C(\mathscr{E}^\mu_m)$. Since $\mathscr{D}C(\mathscr{E}^\mu_m)$ is closed we are finished. Q.E.D.

Remark. The inclusion in Proposition 4 can be sharpened to equality. This is an immediate consequence of the Equivalence Theorem and Theorem 1 of Hart et al. (1974, p. 163).

Basic for the formulation of our Limit Theorem has been the view that the distributions of consumption characteristics essentially determine an exchange economy and its core. This interpretation is justified by the Propositions 3 and 4. Clearly, one could not expect, that the distributions alone could also describe the competitive structure of an economy. The formulation of Propositions 3 and 4 shows that this additional necessary information is given by the exact specification of the atoms.

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