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On the Uniqueness of Individual Demand at Almost Every Price System

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A simple new proof, based on Fubini's theorem, is given for the uniqueness of individual demand at almost every price system, even if preferences are nonconvex. *Journal of Economic Literature* Classification Number: 022.

1. INTRODUCTION

The fact that even in case of a nonconvex preference relation the demand set is a singleton at almost every budget situation has been proved by Mas-Colell [2] and by Mas-Colell and Neuefeind [3]. Mas-Colell's proof made use of the theory of Hausdorff measures on lower dimensional subspaces of a Euclidian space. The proof due to Mas-Colell and Neuefeind relies on an application of the projection theorem for analytic sets due to Marczewski and Ryll-Nardzewski [1] together with Fubini's theorem. In the present note I shall give a new proof which relies only on the disintegration theorem (cf. Parthasarathy, [4, Theorem 8.1, p. 147]), i.e., a general version of Fubini's theorem.

2. RESULT

Consider $l \geqslant 2$ perfectly divisible commodities. An agent is described by his *consumption set P*, the positive orthant of the commodity space $\mathbb{R}^l$, his *preference relation* $\succcurlyeq$ a reflexive, transitive, continuous binary relation on P, and his wealth $w \in L \equiv (0, \infty)$. The preference relation $\succcurlyeq$ is moreover

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assumed to be *weakly monotone*, i.e., $(\forall i \in \{1,\ldots,l\}: x_i > y_i) \Rightarrow (x \succ y)$. The space of normalized *price systems* p is L^{l-1}. Denote by λ^{l-1} and λ^1 the Lebesgue measures on $(L^{l-1}, \mathscr{B}(L^{l-1}))$ and on $(L, \mathscr{B}(L))$, respectively, and by $\#M$ the cardinality of the set M. The demand set of an agent, described by $(\succcurlyeq, w)$ at the price system p is

$$\varphi(\succcurlyeq, w, p) = \{x \in P \mid (y \succ x) \Rightarrow (py > w)\},$$

i.e., the set of $\succcurlyeq$-maximal commodity bundles in his budget set.

PROPOSITION. *Let $\succcurlyeq$ be a weakly monotone continuous preference relation on P. Then*

$$\lambda^{l-1} \times \lambda^1(\{(p, w) \in L^l \mid \#\varphi(\succcurlyeq, w, p) > 1\}) = 0.$$

Proof. By Fubini's theorem any measurable set of line segments in an $(l-1)$-dimensional cube is an λ^{l-1}-null set. Although only this is needed, in case $l = 2$ the even stronger statement that a line can contain at most countably many disjoint segments is well known. To simplify notation we choose in the following $l = 2$ without loss of generality.

Let μ be a probability on $(L^2, \mathscr{B}(L^2))$ which is equivalent to λ^2, i.e., which has the same null sets as λ^2. Let the utility function u represent $\succcurlyeq$ and let v be the indirect utility function defined by

$$v: L^2 \to L: (p_1, w) \mapsto u(\varphi(\succcurlyeq, w, p)).$$

The map v is a continuous, hence measurable map onto its image. Therefore, since L^2 and image(v) are Polish spaces, μ has a disintegration

$$\mu = \int_L \zeta_t \mu \circ v^{-1}(dt),$$

where the probabilities ζ_t on L^2 live on the fibres $v^{-1}(t)$, $t \in$ image(v). One can easily derive that ζ_t is equivalent to λ^2, hence atomless for $\mu \circ v^{-1}$—almost every $t \in L$. This follows from the translation invariance of the Lebesgue measure λ^2.

Denote by N the measurable set of pairs $(p_1, w) \in L^2$ with $\#\varphi(\succcurlyeq, w, p) > 1$. For any $t \in$ image(v) the set $N \cap v^{-1}(t)$ can be at most countable, since an indifference curve can have at most countably many disjoint line segments. Therefore we get

$$\lambda^2(N) = \mu(N) = \int_L \zeta_t(N)\, \mu \circ v^{-1}(dt) = \int_L \zeta_t(N \cap v^{-1}(t))\, \mu \circ v^{-1}(dt) = 0. \quad \blacksquare$$

References

1. E. Marczewski and C. Ryll-Nardzewski, Projections in abstract sets, *Fund. Math.* **40** (1953), 160–164.
2. A. Mas-Colell, A remark on a smoothness property of convex, complete preorders, *J. Math. Econom.* **3** (1976), 103–105.
3. A. Mas-Colell and W. Neuefeind, Some generic properties of aggregate excess demand and an application, *Econometrica* **45** (1977), 591–599.
4. K. R. Parthasarathy, "Probability Measures on Metric Spaces," Academic Press, New York, 1967.