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IDENTIFYING FRACTIONS ON NUMBER LINES

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This study investigated the ways students represented fractions on number lines and the effects of instruction on those representations. Two clinical teaching experiments and one large-group teaching experiment were conducted with fourth and fifth graders (N = 5, 8, and 30). The instruction primarily concerned representing fractions and ordering fractions on number lines. Tests and videotaped interviews indicated that unpartitioning, in particular, is difficult for students, although the instruction seemed to help. Associating symbols with representations also seems difficult and may depend on an understanding of the unpartitioning process.

This study investigated the ways students accurately and inaccurately represent fractions on number lines and the influence of instruction on those representations. The number line model in the guise of "number segments" was chosen for study in large part because of its pervasive use in school mathematics instruction.

As a model for representing fractions, the number line differs from other models (e.g., sets, regions) in several important ways. First, a length represents the unit, and the number line model suggests not only iteration of the unit but also simultaneous subdivisions of all iterated units. That is, the number line can be treated as a ruler. Second, on a number line there is no visual separation between consecutive units. That is, the model is totally continuous. Both sets and regions as models possess visual discreteness. When regions are used, for example, space is typically left between copies of the unit.

Third, the number line *requires* the use of symbols to convey part of the intended meaning. For example, Point A in *a* of Figure 1, when *a* is taken as a number line, has no numerical meaning until at least two reference points are labeled. Two possible number line meanings are given in *b* and *c* of Figure 1. Parts *d* and *e*, however, do convey meaning without any accompanying symbols, though their interpretation requires some standard conventions about the nature of a unit. The significant issue is that the number line requires an integration of two forms of information, visual and symbolic; this integration does not seem essential with other models.

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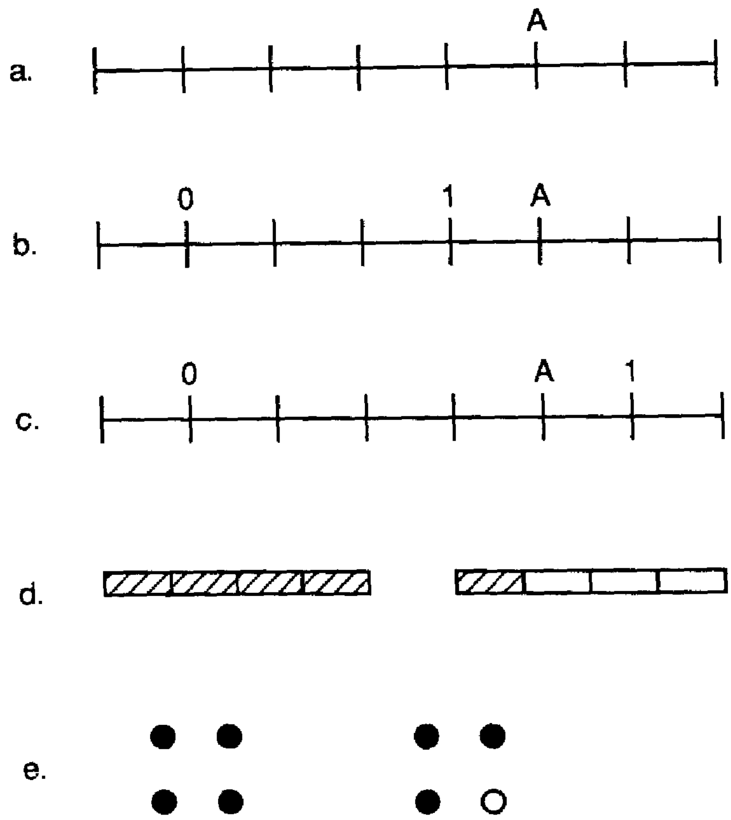


Figure 1. Representations of fractions.

The use of symbols to label points on a number line may focus a student's attention on those symbols rather than on the pictorial embodiment of the fractions. That focusing may in turn cue symbolic processes as the predominant mode of manipulating information. Too, the necessary but not directly used marks on a number line may act as perceptual distractors (Behr, Lesh, Post, & Silver, 1982).

Three experiments were conducted to explore students' understanding of fractions. This paper deals primarily with representations of fractions on number lines. Data come from tests and videotaped interviews. The research goal was to attempt to identify the links between students' understandings and the representations of fractions on number lines.

EXPERIMENTS

An 18-week clinical teaching experiment, a 30-week clinical teaching experiment, and a 30-week large-group teaching experiment were conducted in which lessons related to the number line were embedded within a framework of instruction related to fractions. The instruction in the second and third experiments was modified to attempt to overcome the apparent deficiencies in students' performance during the first experiment.

Clinical Teaching Experiment 1

Subjects. The subjects were five fourth graders (three boys and two girls) in an elementary school in northern Illinois. They were selected through teacher evaluations to have a range of facility with arithmetic concepts. They were subjects in an 18-week teaching experiment reported previously (Behr, Wachsmuth, Post, & Lesh, 1984).

Instruction. The instruction was a 4-day lesson concerning the association of fractions with points, comparisons of fractions represented on a number line, and transformations of fraction representations on a number line. The specific objectives were to (a) associate whole numbers, fractions, and mixed numbers with points on a number line, (b) use number lines to help connect improper fraction names to mixed number names, (c) use number lines to determine which of two fractions is less or whether they are equivalent, and (d) use number lines to generate equivalent fractions. The lesson on number line representations was presented near the end of the teaching experiment, which was conducted during 1981–82.

The instruction included a variety of activities. The notion of subdivision of the unit was introduced and reinforced by use of centimeter rods to develop the analogy with the set-subset fraction concept and by repeated focusing on the 0 (left-hand) endpoint of the first unit. Considerable attention was paid to the equivalence of improper fractions and mixed numbers and to ordering fractions using number lines. For example, $2\frac{2}{6}$ and $2\frac{1}{3}$ were compared directly on number lines, rather than by emphasizing symbolic processes for reducing $\frac{2}{6}$ to $\frac{1}{3}$.

Test. The fraction test of Larson (1980) was given immediately before and immediately after the number line lessons. Sample items are given in Figure 2. This 16-item multiple-choice test can be partitioned into pairs of 8-item subscales in several ways: (a) fraction given with representation to be chosen versus representation given with fraction to be chosen, (b) number line showing 0 to 1 versus number line showing 0 to 2, and (c) representation on number line showing unreduced fraction versus representation showing reduced fraction. Thus, there are six nonindependent subscales. For each item there were five choices, one of which was "None of the above"; this choice was never the correct choice. In all cases, the fraction symbol in the correct fraction-representation pair was reduced even if the representation was for an unreduced equivalent fraction.

Results. For five of the subscales, all scores increased or remained constant from pretest to posttest. The sole exception was when the representation was unreduced and the fraction symbol was reduced. As a follow-up, the scores in this subscale were separated according to the other categories of items. With one exception, the students were unable to choose a reduced fraction name when an unreduced equivalent form was represented on a number line.

To help determine what processes the students might be using, we examined the incorrect responses on the unreduced representation subscale.

On the pretest, 10 of the 31 incorrect responses were "None of the above"; two were blanks. On the posttest, however, 28 of the 30 incorrect responses were "None of the above"; none were blanks. On the pretest, the students generally behaved as if they knew what fraction the representation was associated with. On the posttest, the students perhaps knew that the representation was not the simplest or most obvious one for any of the fractions listed, but they were perhaps not able to reinterpret the representation to match the correct fraction.

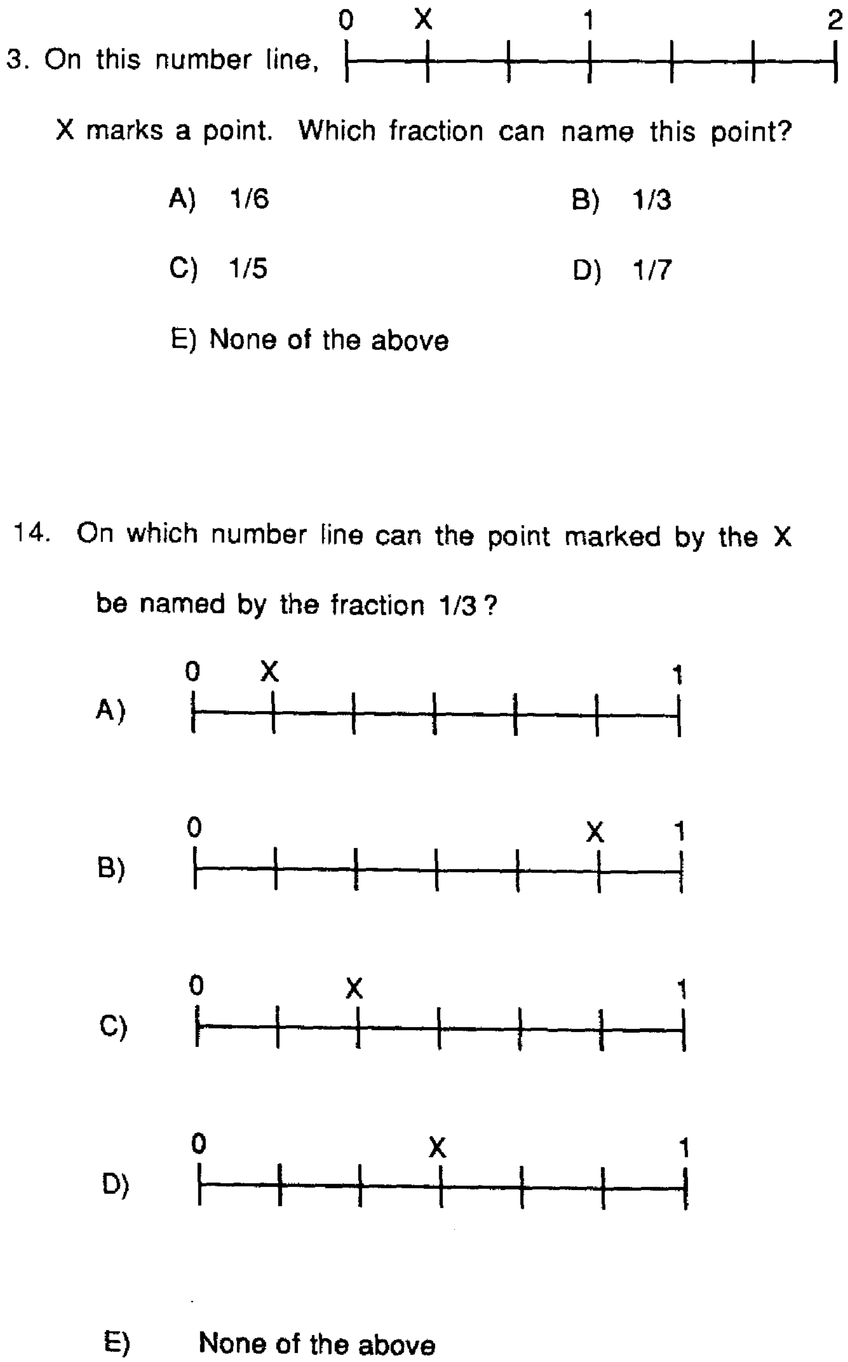


Figure 2. Sample items from *Locating Fractions on the Number Line,* developed by C. N. Larson. Reprinted by permission of the author.

Additional information was available from videotaped interviews. In three interview tasks, the students were to find equivalent fractions, $5/3 = ?/12$, $8/6 = ?/3$, and $8/6 = ?/12$. A variety of strategies were employed to solve these problems. In every case, however, the students had considerable difficulty coordinating their symbolic work with number line representations.

One student solved the problems symbolically and used the number line only to plot the fractions involved. His plots were accurate and verified the equality of the fractions, but the number line seemed to play no role in his thinking.

One student solved all three problems correctly through use of the number line. He plotted the given fraction in each case and then subdivided or collapsed markings to create a number line appropriate for the new fraction. He then counted appropriately.

Another student solved the first and third problems correctly by plotting the given fractions and then adding marks to the number line to be able to represent the larger denominators. For the second problem, however, she plotted $8/6$ and then reasoned that $3 \times 2 = 6$ so $8 \times 2 = 16$. She counted 16 sixths as 16 thirds, and marked $16/3$. This appears to be an instance of a well-known but not very frequent mistake in dealing with fractions (see Bright & Harvey, 1982).

One student solved only the last two problems correctly. For the second problem, he reasoned that $6 - 3 = 3$ so $8 - 4 = 4$ and then marked fractions appropriately. For the third problem, he plotted $8/6$ and then created markings for twelfths. He reasoned that $6 + 6 = 12$ so $8 + 8 = 16$. For the first problem, however, he said $4/12$ as the answer, perhaps thinking that $1/3 = 4/12$.

The fifth student solved only the last problem correctly. For the first problem, she plotted $5/3$, but then labeled $7/3$ as $7/12$. For the second problem, she labeled $8/6$ correctly but then also labeled it as $5/3$, perhaps because it was 2 marks past $3/3$, which she knew to be 1. She then labeled $7/6$ as $4/3$, again consistent with its being 1 mark past $3/3$. She reasoned that $6 - 3 = 3$ so $8 - 4 = 4$ and claimed that $8/6 = 4/3$, though her markings did not verify this. For the third problem, she plotted $8/6$ and then created markings for twelfths and counted $16/12$.

The tasks requiring adding marks to the number lines were clearly easier to solve. The 2:1 ratio of denominators in two of the problems may also have made those problems easier than the other problem. There was little evidence, however, that use of the number lines played an important role in the thinking processes of the students. Plotting fractions seemed independent of the thinking about equivalence.

Clinical Teaching Experiment 2

Subjects. The subjects were eight students during the end of their fourth-grade year and the beginning of their fifth-grade year (four boys and four girls) in the same elementary school as in Clinical Teaching Experiment 1. They were selected to have a range of arithmetic facility and were also subjects in an extended teaching experiment conducted during the academic years 1981–82 and 1982–83 (Behr, Post, & Lesh, 1981).

Instruction. The instruction on the use of number lines lasted 8 days, 14–24 September 1982. The instruction in Clinical Teaching Experiment 1 was extended by including more activities on equivalence, on translations between the number line and area models, and on using equivalent fractions to name a single point on a number line. Experience was also given in translating between area and discrete model representations and number line representations.

Tests. The Larson test was given on 13 September and 27 September. A separate Number Line Test was given on the same dates. Sample items are shown in Figure 3.

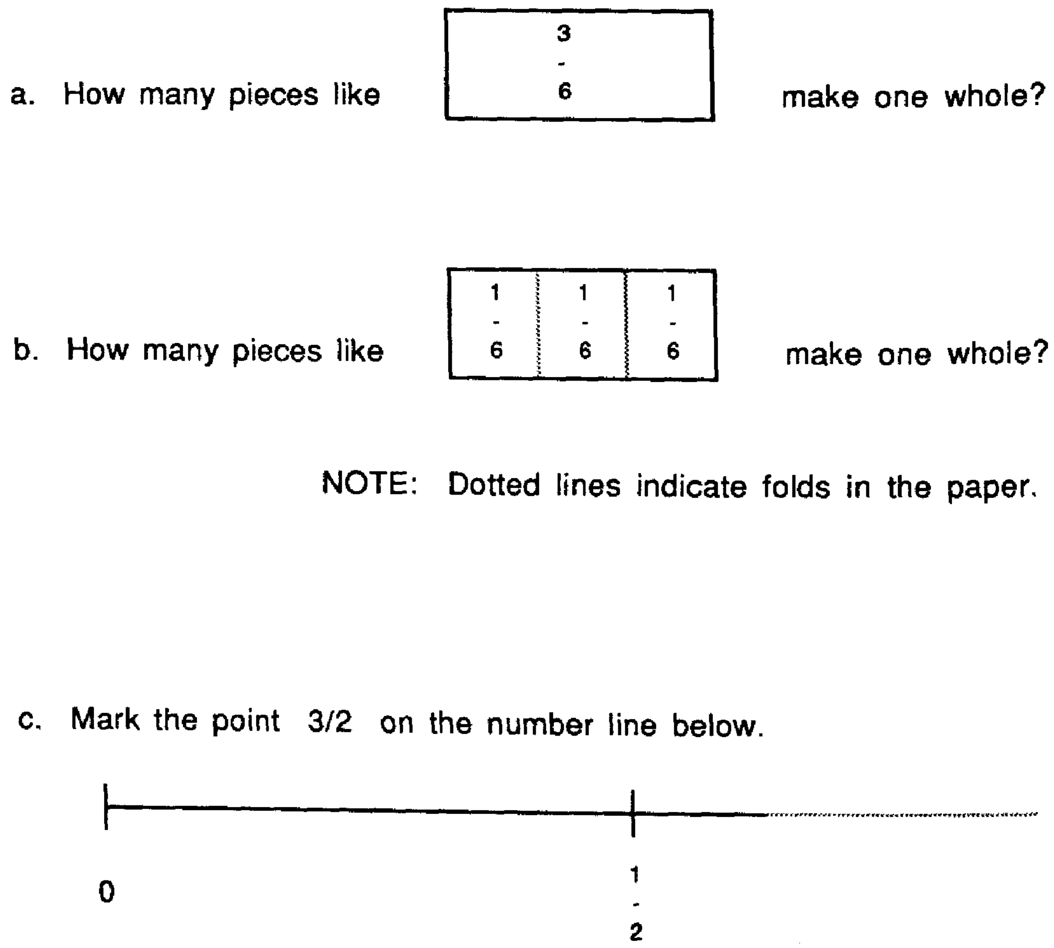


Figure 3. Sample items from Number Line Test.

Results. The performance on two subscales of the Larson test showed considerable improvement on the reduced representation subscale ($p < .05$)

but not on the unreduced representation subscale. Further refinement of the scores on the latter subscale along with analysis of the errors revealed shifts in the students' error patterns. On the pretest, 25 of the 47 incorrect responses (53%) were "None of the above" and 3 more (6%) were consistent with use of the interval from 0 to 2, incorrectly, as the unit. On the posttest, these results were 30 (79%) and 7 (18%) of the 38 incorrect responses. This shift to perhaps more restrictive interpretations of what a number line actually represents is consistent with the data of Teaching Experiment 1.

The performance on the Number Line Test improved dramatically (the mean score increased from 0.75 to 7.75), but since the items were closely related to the instruction, this increase may reflect only a practice effect and not a transfer effect. An error analysis on these tests is given in Table 1. The avoidance of the use of counting marks instead of intervals on the posttest was expected because the instruction explicitly dealt with the number line from a measurement rather than a counting interpretation.

Table 1
Errors on Number Line Test

	Pretest		Posttest	
Error	No.	%	No.	%
Use wrong unit	38	42	11	32
Count marks instead of intervals	14	16	0	0
Represent inverse of fraction	12	13	4	12
I don't know	14	16	8	24
Others	12	13	11	32

The same interview questions were given as in Clinical Teaching Experiment 1. The two most common strategies for obtaining answers in the interviews were multiplication (or division) and skip counting; these strategies were applied only to the symbolic statements of the problems. When asked to solve the problems on the number line, most students had great difficulty.

For the first problem, presented as $5/3 = ?/12$, along with a number line 0 through 4, partitioned into thirds, five of the eight students first solved the problem symbolically and only tried to use the number line when asked to do so by the interviewer. Not all students who successfully gave a symbolic solution were able to do so with the number line. The following protocol is one example.

Interviewer: Use the number line to solve the problem [writes it].

Student: Five thirds [counts 5 intervals on the number line partitioned into thirds, places a dot at the end of the fifth interval, writes $5/3$ over the dot] . . . five thirds.

I: Explain.

S: There is the five thirds, so three times four is twelve [gestures from $5/3$ to $?/12$], and five times four is twenty.

I: You didn't really use the number line, you multiplied. Could you also use the number line to obtain that?

S: Sure, twenty twelfths, make that into twelfths. [Starts at 0 and indicates with finger motions to partition each third into three parts] . . . Oh, it's hard to get 'em into twelfths [continues to "partition" each of the 0-1 and 1-2 units into 12 parts but makes no partition marks] . . . here is twenty-four [pointing to the end of the 1-2 unit] as twenty would be around here [makes dot a little to the right of his dot for $5/3$] . . . I just took a guess. [Discussion verifies student's intention that there be two dots to represent $5/3$ and $20/12$.]

S: The two fractions are equal, though.

I: But the points are different.

S: So they're not [equal], they have to be the same to be equal.

I: Why?

S: Well, because if the fraction is equal, it would always be the same dot.

The most common mistake was to subdivide the given thirds into three parts to make ninths, a problem evident in the protocol above. Two of the other three students explained their work through a multiplication or skip-counting process, which can be observed in the following protocol.

I: Use the number line to solve this problem [writes $5/3 = ?/12$].

S: I don't know.

I: Can you solve it without the number line?

S: Uh-huh [writes 20].

I: Explain.

S: Well, twelve; three, six, nine, twelve, and five, five, ten, fifteen, twenty.

I: Do you have any idea how to solve it on the number line?

S: No.

Only one student appeared to actually solve the problem using the number line.

S: [Writes $5/3$ over the correct tick mark, then marks 4 subunits in each third between 0 and 1, counts to make sure he now has 12, then does the same for the unit 1-2.] Let's see [points with his left hand to successive 12ths up to 20, then writes 20 in the box]. Twenty twelfths.

I: How would you solve that without a number line?

S: Well, there's, three times four equals twelve, five times four equals twenty [marks as he talks:

$$\frac{5}{3} \xrightarrow{\times 4} \frac{20}{12} \quad (5 \xrightarrow{\times 4} 20,\ 3 \xrightarrow{\times 4} 12)$$

]

Two other students indicated that the units (between 0 and 1 or between 1 and 2) could be divided in 12ths but showed no desire or ability to do so.

For the second problem, presented as $\frac{8}{6} = \frac{?}{3}$ along with the same number line, most students first approached the problem through a number line representation, apparently because they expected that the interviewer demanded it. An eloquent solution is shown in the next protocol.

S: [Marks off into sixths the segment presented as marked into thirds by putting a mark between each of the existing marks, first for the unit 0-1, then for 1-2; pauses, counts 8 sixths, and writes $\frac{8}{6}$ above the tick mark.]

I: Explain.

S: Well, you divide one third into two parts, there's two, four, six, then you divide the same thing here [points to the 1-2 unit], there's eight parts up to here [i.e., up to $\frac{8}{6}$], there's one, two, three, four [pointing to alternate tick marks that denote thirds].

Three students first represented $\frac{4}{3}$; that is, they seemed to have solved the problem in their heads and then tried to represent the problem solution on the number line. In this situation, they split each third into two pieces to create the sixths. The following protocol is an example.

S: [Writes 4 in the box, puts dot at $1\frac{1}{3}$, partitions each third in half to make sixths up to $1\frac{4}{6}$, counts sixths by pointing at each mark] one, two, three . . . eight [points to the dot for $\frac{4}{3}$] and eight [sixths] would be right [there].

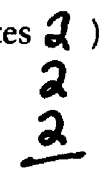
I: Eight what?

S: Eight sixths.

One student gave the answer $\frac{16}{3}$, which reflects a known syntactic error. This student's difficulty with the number line was extensive:

S: Do I have to use the number line?

I: Yes.

S: Eight times two (writes ). Okay [writes 16 in the box]. Umm, okay now, one, two, three, four, . . . eight [counts thirds up to 8, places dot at $2\frac{2}{3}$], there's eight sixteenths.

I: What is that?

S: Eight sixteenths, or I mean eight thirds.

I: How did you find sixteen thirds [refers to the 16 the student wrote in $\frac{?}{3}$].

S: One, two, three, four, five, six, seven . . . [points to successive one-third tick marks and trails off with a gesture to suggest counting could continue].

In all instances, there was a period of struggling with the representation on the number line. The students clearly did not have automatic responses to the problem of representing the fractions. They all reflected on their work, though none demonstrated a clear understanding of whether they were on the right track, except through trial and error. One student who was not successful with $5/3 = ?/12$ demonstrated this struggling:

I: I want you to show me [on the number line] that that [$5/3$] is $20/12$.

S: [Long pause.]

I: Any ideas?

S: No.

I: How do you make $1/3$ into 12ths?

S: [Divides each third into 3 more parts, whispers 3, 6, 9, 12, 15.] Oh.

I: You want twelfths.

S: Yeah, no.

I: What did you make it into?

S: Threes.

I: So this unit is how many parts now?

S: Fifteen [that is, the 5 thirds are subdivided into 15 parts].

A second student who was successful demonstrated the difficulties in thinking ahead for using the number line:

S: Then I'd be . . . [makes motion to cut each third in half and counts silently 2, 4, 6; stops at 1 unit; starts over and motions to cut each third into three pieces, marks them and counts 9 parts in 1 unit] one, two, three, . . . , nine. [Goes back and marks 4 in each third and counts up to 12 in 1 unit.] One, two, . . . , twelve.

Large-Group Teaching Experiment

Subjects. Thirty-four children (20 boys and 14 girls) in a single classroom in an elementary school in St. Paul, Minnesota, were the subjects. As in Clinical Teaching Experiment 2, the study extended across the end of the fourth grade and the beginning of the fifth grade. The data for four students were incomplete.

Instruction. The instruction on use of number lines lasted 8 days during September 1983. The instruction was identical to that of Clinical Teaching Experiment 2 and was intended to illustrate the translation of the small-group instruction to a whole-class setting.

Tests and analysis. The tests of Clinical Teaching Experiment 2 were used with these students. Unfortunately, one item on the Larson (1980) test was misdrawn, with the result that "None of the above" was the correct answer. A set of 28 items was selected for analysis, 16 from the Larson test and 12 from the other tests. These data were analyzed by a two-way analysis of variance on test administration time *(T)* and on the characteristics of the items: (a) *L*, 0-to-1 number line versus 0-to-2 number line (Larson items), (b) *G*, fraction given versus representation given (Larson items), (c) *P*, repartition required (i.e., correct response required dealing with unreduced representation) versus not required (Larson items), and (d) *I*, complete and precise information given (e.g., a task requiring fourths using a number line divided into fourths) versus extraneous marks included on number line (e.g., a task requiring fourths using a number line divided into eighths) versus incomplete information given (e.g., a task requiring eighths using a number line divided into fourths) versus perceptual distractors included on the number line (e.g., a task requiring fourths using a number line divided into thirds).

Results. Descriptive data and results of the ANOVAs are presented in Tables 2 and 3. Several observations seem notable. First, as would be expected, the posttest scores are significantly higher than the pretest scores. The instruction seemed to be effective. Second, the 0-to-1 number lines were easier than the 0-to-2 number lines. This result was also expected because in these settings the students did not have to consciously identify the unit. Third, fraction-given items were easier than representation-given items. The instruction did not seem particularly biased toward either type of item. Fourth, no repartition required was easier than repartition required. Fifth, the relative difficulty of the four item types shifted from pretest to posttest. For the posttest, from easiest to hardest, these are as follows:

1. complete information given (COM)
2. incomplete information given (INC)
3. perceptual distraction included on number line (DIST)
4. extraneous marks included on number line (EXT)

Prior to instruction, the orders were COM, EXT, INC, and DIST. EXT dropped from second to fourth position.

DISCUSSION

In many ways the instruction seems to have been effective. The shifts in error patterns, in the clinical teaching experiments in particular, suggest that the instruction at least sensitized students to the need to attend to some characteristics of the number line model. For example, in Clinical Teaching Experiment 1, the increase in "None of the above" responses may have resulted from learning to look for a representation with a unit that is subdivided as indicated by the denominator of the fraction. Failure to recognize unreduced representations, however, may indicate either an inability to *unpartition* (Behr et al., 1982), a lack of skill at reducing fractions, or an inflexibility in translating between modes of representation. In Clinical Teaching Experiment 2, the shift in errors on the Number Line Test also seems to support the assessment of effectiveness of the instruction. The decrease in very inappropriate responses (e.g., counting marks instead of intervals and representing the inverse of the given fraction) and the concurrent rise in the percent of "I don't know" responses suggest that these students at least learned the major characteristics of the model that needed to be attended to.

Table 2
Descriptive Statistics for Large-Group Experiment (N = 30)

Variable	*M*	*SD*
Pretest		
Number line 0–1	4.9	2.3
Number line 0–2	2.0	2.0
Fraction given	3.9	1.9
Representation given	3.0	1.5
Repartition required	2.7	1.9
No repartition needed	4.2	1.8
Complete information given	.49	.24
Extraneous marks on number line	.34	.24
Incomplete information given	.13	.18
Perceptual distractors on number line	.05	.09
Posttest		
Number line 0–1	6.8	1.2
Number line 0–2	6.4	1.5
Fraction given	6.9	1.4
Representation given	6.3	1.3
Repartition required	5.6	2.3
No repartition needed	7.6	0.6
Complete information given	.96	.08
Extraneous marks on number line	.70	.29
Incomplete information given	.79	.26
Perceptual distractors on number line	.73	.32

Note. The first six variables from each task have a maximum score of 8; the last four are proportion correct.

Table 3
ANOVA Summary for Large-Group Experiment (N = 30)

Source	*F*	*df*	*p*
Number Line (L)	30.8	1,29	.00
Time (T)	81.0	1,29	.00
L × T	17.8	1,29	.00
Given (G)	18.7	1,29	.00
Time (T)	81.0	1,29	.00
G × T	0.7	1,29	.40
Partition (P)	30.6	1,29	.00
Time (T)	81.0	1,29	.00
P × T	1.0	1,29	.33
Item Type (I)	27.0	3,87	.00
Time (T)	319.9	3,87	.00
I × T	7.9	3,87	.00

The instruction of the second clinical teaching experiment and the large-group teaching experiment also seems to have been marginally more successful at helping students deal with unreduced representations. This greater success may have been due to the added attention given to translations between part-whole displays and number lines, to finding units on number lines, or to greater emphasis on the measure construct. In Clinical Teaching Experiment 1, only one student spontaneously used symbolic algorithms for finding equivalent fractions in interview tasks. In Clinical Teaching Experiment 2, however, six of the eight students were able in interview tasks to use symbolic algorithms for generating equivalent fractions. Only four of the eight, however, seemed to have any success at coordinating the symbols with the number line model. Clear and easy access to these symbolic algorithms may indicate a well-developed concept of fractions.

The data from Clinical Teaching Experiment 2 also indicate that unpartitioning of a given representation is possible; that is, if a reduced fraction is given and a correct representation is to be chosen, students can sometimes identify the proper representation, even when it is of an unreduced equivalent fraction. However, when the representation is given in unreduced form, students have difficulty choosing the correct reduced symbolic fraction. The data support this observation; the subscale of items requiring repartitioning had the lowest posttest mean. Perhaps students do not look back at the given representation and try to make each symbolic fraction choice fit that number line. Perhaps the symbol takes on an identity of its own once it is generated from a given representation and the connection to its perceptual base is lost.

IMPLICATIONS

Number line instruction is difficult. During the instruction, the students seemed to be able to perform adequately; the improved performance on the Number Line Test supports this observation. However, transfer of knowledge to slightly different situations (e.g., the Larson test) was not particularly successful, especially when the representations of the fractions were in unreduced form.

One primary feature of the number line model may help explain this observation. Since the model consists of pictorial information with accompanying symbols, there may be a difficulty in connecting the information contained in the two modes of representation. With items in which there was one number to represent a fractional area or segment, the students could generally identify the proper unit. However, with items in which there were several numbers, the students did not do as well, possibly because there are more symbols to coordinate in the representation of the item information. There may be an overload on the capacity of the students to coordinate the two modes of information. A hypothesis arising out of this analysis is that *the need to coordinate symbolic and pictorial information with the number line model poses difficulty in matching fraction names with number line representations.* This need is not unique to the number line model, but it is relevant to that model because of the large role the number line plays in elementary school mathematics instruction.

A similar situation was observed by Gerace and Mestre (1982). In their study, Hispanic high school students in a beginning algebra class were initially very rigid in labeling number lines; that is, the first tick to the right of 0 was always supposed to be labeled with a 1. Later, the number lines frequently seemed to be labeled with incorrect labels. For example, in *a* of Figure 4, the first ⅓ denotes the first third, whereas the second ⅓ denotes the third closest to 1. In *b* of Figure 4, the labeling illustrates a common mistake the students made in ordering common fractions. In more concrete situations (e.g., in problems in which the number line represented distances or in which there was use of east-west designations), the students were noticeably more successful.

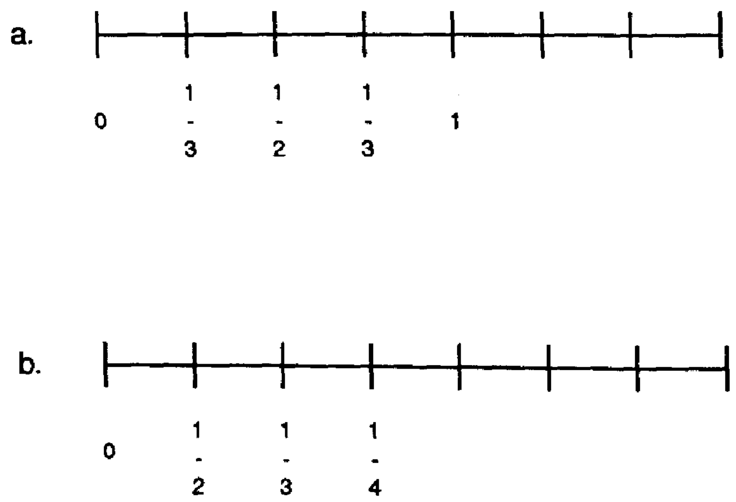


Figure 4. Mislabeling of number line.

From a slightly different perspective, the data of this study and other studies indicate that students' difficulties in marking partitioning points to generate higher-term fractions or in mentally removing partitioning points to generate lower-term fractions is not unique to the number line model (Behr et al., 1982; Payne, 1976). Moreover, the greater difficulty children have generating lower-term fractions by unpartitioning pervades work with both continuous and discrete models and with symbolic equivalent-fractions tasks. Generating higher-term fractions symbolically seems to be easier than generating lower-term fractions. At the symbolic level, this difference in difficulty may be due to children's greater facility with multiplication than division.

With manipulative tasks children seem to rely heavily on the visual representation of a fraction. Flexibility in the perception of equivalent fractions, independent of the given representations, has not yet been achieved. Children not only seem distracted by extra points of partition but also seem to question the rules of the game. That is, some children have been observed to add partitioning points, but when faced with the removal of points, these same children hesitate and may query the teacher or interviewer about whether it is all right to take out points. Other children have been found totally unable to perceive lower-term fractions in the presence of extra points. More generally, the partitioning-unpartitioning phenomenon seems to pervade many children's work with most models for rational numbers. A hypothesis arising out of this analysis is that *as long as partitioning and unpartitioning are difficult for children, number line representations of fractions may not be easily taught.*

The instruction of the second year seemed more effective at helping the students deal with representations on number lines from 0 to 1 than on number lines from 0 to 2. In Clinical Teaching Experiment 1, the instruction seemed to be ineffective at helping students deal with representations on both kinds of number lines. Perhaps the increased emphasis on identifying the unit during the instruction phase of Clinical Teaching Experiment 2 was responsible for the different pattern of effectiveness. If so, then increased emphasis on identifying the points on a number line that represent the first unit, second unit, third unit, and so on, might help students further generalize their concepts of fraction representations.

A major hypothesis of the research project of which this study is one part is that translations between and within modes of representation facilitate learning (Behr et al., 1981). As noted earlier, the instruction provided models of translations of three types: (a) symbol → number line, (b) symbol → number line → different number line, and (c) number line → symbol → different symbol. More work on translations such as symbol → number line → different number line → different symbol might have helped children make the symbol → different symbol translations in generating equivalent fractions (see Figure 5). Chaffe-Stengel and Noddings (1982) also called attention to the use of multiple number lines in the process of moving from one symbol to a different symbol. The symbol → different symbol translation should be viewed as a condensed version of the longer string of translations. Until students are able to collapse this sequence, it may be helpful to provide settings in which all parts of the sequence are explicit. (See Bernard & Bright, 1984, for further discussion of the notion of collapsing of processes and operations.) Flexibility with the use of manipulatives may be one way to foster the collapsing process.

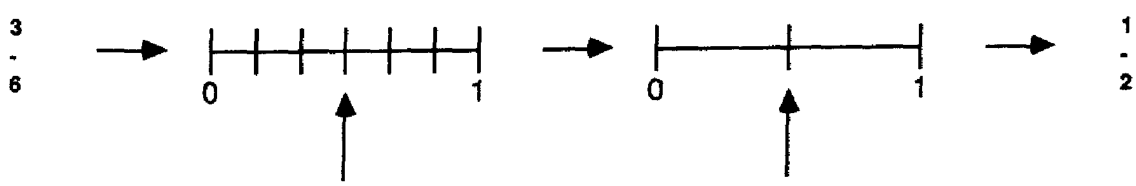


Figure 5. **Multiple representations and translations.**

More translations between different kinds of models might also have been helpful. The inclusion in the second two studies of translations between part-whole models and number lines may have been partially responsible for the apparent improvement in performance. If so, more instruction of this type might have enhanced the improvement.

Too, interconnecting the symbolic generation of equivalent fractions to the number line models needs more attention. Some children were quite successful at symbolic tasks but did not spontaneously use their symbolic skills in number line situations. Perhaps they did not see the connection between these two kinds of tasks (or modes of presentation of information).

Two possible teaching techniques seem to arise from these considerations. First, multiple number line representations of a single fraction might be presented. At most, one of these representations would be the reduced representation of the fraction, whereas all others would be unreduced representations of the fraction. Some of these unreduced representations would be labeled with the unreduced fraction name, and others would be labeled with the reduced fraction name. Illustrations of appropriate number lines are given in Figure 6.

Second, number lines with different subdivisions might be matched and then labeled. Symbolic expressions of the equivalence of the fraction represented in the several ways could then be presented. Illustrations of sample number lines are given in Figure 7.

A knowledge of equivalent fractions seems to be important to the full utilization of number line representations. Knowledge that is developed only through symbolic algorithms may be isolated and not called upon in the context of manipulative tasks. Work with the number line model during instruction on equivalent fractions would then probably be needed. For example, partitioning units of a number line first into halves, then fourths, and so on, would illustrate the notion that to every point on the number line there are associated many equivalent fractions. Before students use the number line for more complex tasks (e.g., to model addition and subtraction, especially of unlike fractions), more skill with equivalent fractions in the context of the number line seems essential. The automatic generation of equivalent fraction representations, through further partitioning or unpartitioning of the number line, "in the mind's eye," could facilitate flexibility in perception. Such flexibility might significantly enhance students' performance.

Too, further investigation of the ways students translate between different representations of knowledge is needed. Experts (e.g., teachers) seem to make these translations easily, and frequently they seem not to be consciously aware that translations are used. In some sense, experts seem to view all modes of presentation of information as equivalent. Novices (e.g., students), on the other hand, need explicit help in learning how to make these translations. Much more needs to be known about processes that students use in translating before instruction can be effectively modified to help students learn to make translations between the modes of representation.

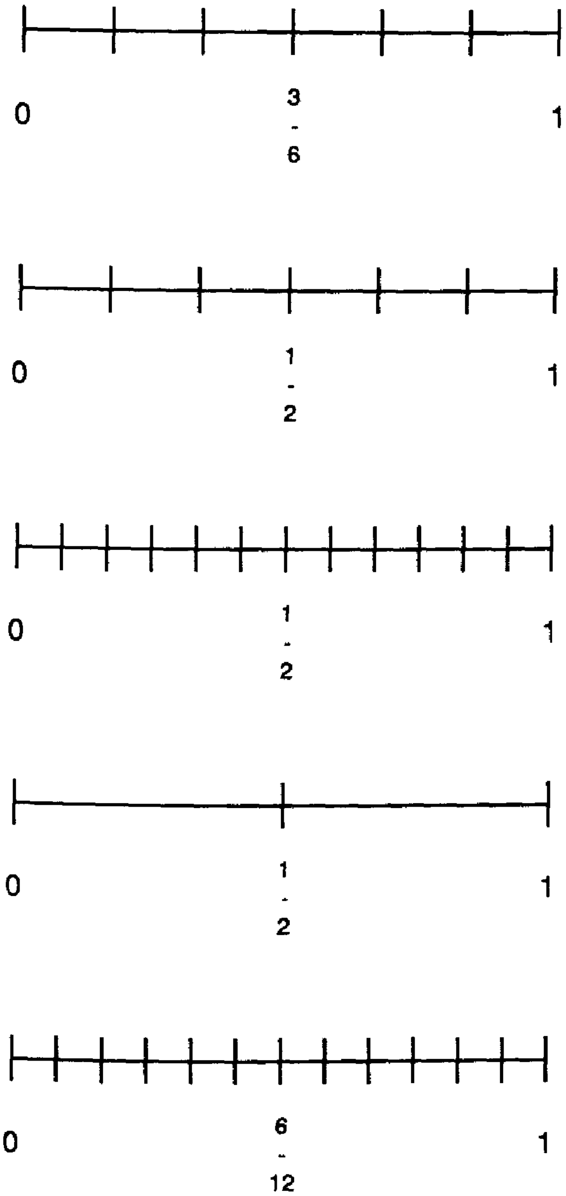


Figure 6. Some representations of one half.

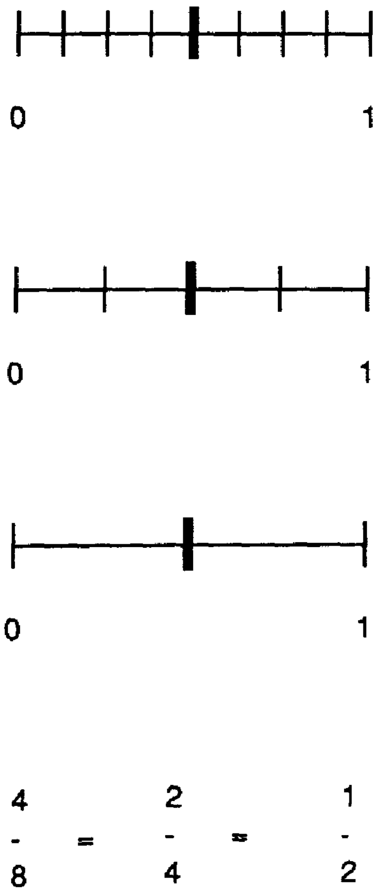


Figure 7. Equivalent pictorial and symbolic representations.

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