Journal for Research in Mathematics Education
1985, Vol. 16, No. 1, 18–36

ORDER AND EQUIVALENCE OF RATIONAL NUMBERS: A COGNITIVE ANALYSIS

THOMAS R. POST, *University of Minnesota*
IPKE WACHSMUTH, *Northern Illinois University*
RICHARD LESH, *Northwestern University*
MERLYN J. BEHR, *Northern Illinois University*

Fourth-grade students' understanding of the order and equivalence of rational numbers was investigated in an 18-week teaching experiment. Data from observations of, and interviews with, two children were employed to identify patterns over time in the strategies used in performing tasks. Three related characteristics of thinking are hypothesized to be related to the successful performance of tasks on order and equivalence: (a) thought flexibility in coordinating between-mode translations, (b) thought flexibility for within-mode transformations, and (c) reasoning that becomes increasingly independent of specific concrete embodiments.

In an earlier report on children's thinking when dealing with tasks concerned with the order or equivalence of rational numbers (Behr, Wachsmuth, Post, & Lesh, 1984), we looked across children at the various strategies they appeared to use in performing the tasks. In this report, by taking a look at how two children approached the tasks at various times, we attempt to provide a deeper analysis of their thinking and how their strategies changed or remained the same.

An 18-week teaching experiment was conducted in schools located in St. Paul, Minnesota, and DeKalb, Illinois. The subjects were 12 fourth graders, 6 at each site. The instructional program consisted of 13 lessons, from 3 to 8 days each. Working individually and in a group, the children were introduced to the part-whole interpretation of rational number by means of circular and rectangular pieces of laminated colored paper as well as other manipulative aids. For details on the instructional sequence, see Behr et al., 1984. Each child was interviewed individually on 11 separate occasions at intervals of approximately 8 days. The analysis in this report is based on transcriptions of recordings of the interviews with two of the children and on notes taken during the lessons by a participant observer.

We are indebted to the following people who assisted in the research: Nik Azis Bin Nik Pa, Kathleen Cramer, Mary Patricia Roberts, Robert Rycek, Constance Sherman, and Juanita Squire. An earlier version of this paper was presented at the annual meeting of the American Educational Research Association, New York, March 1982. The research was supported in part by the National Science Foundation under Grants No. SED 79-20591 and No. SED 81-12643. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

THEORETICAL ISSUES

Our analysis indicates that the development of children's rational number understanding appears to be related to three characteristics of thinking: (a) flexibility of thought in coordinating translations between modes of representing rational numbers, (b) flexibility of thought for transformations within a given mode of representation, and (c) reasoning that becomes increasingly free from a reliance on concrete embodiments of rational numbers.

Thought Flexibility in Coordinating Translations

Children's initial understanding of a fraction—symbolized by a mathematical symbol of the form m/n, where m and n are natural numbers — is not derived from the natural numbers m and n. Nor is their initial comprehension of a mathematical symbol of the form m/n itself derived from the natural numbers m and n. Instead, children's understanding of both the fraction symbolized by m/n and their comprehension of the symbol itself are derived from embodiments — which we denote as $(m)/n$ — of the fraction, such as a picture of an object partitioned into n equal pieces with m of them shaded, or a set of n white poker chips with m of them covered with red chips.

For children to derive meaning from embodiments of fractions, they need information about existing agreements for how fractions are embodied with pictures and manipulative objects. To comprehend a mathematical symbol of the form m/n, they need information about existing agreements for identifying the symbol associated with given pictorial and manipulative displays. Taken together, information about these agreements is adequate to make translations in either direction (see Figure 1) between the mathematical symbol representation and the embodiment representation. The extent to which children can indeed make these directional translations is indicative of their *personal* understanding of the fractions involved.

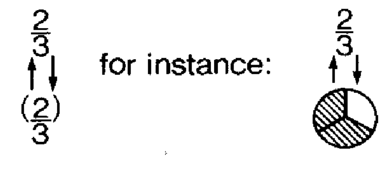

Figure 1. Translation between mathematical symbol and embodiment.

To use such embodiment-based knowledge in making a judgment about the order relation between two fractions such as 2/5 and 2/3, a child has to recognize that a smaller part of a unit (e.g., a circular region) is covered in (2)/5 than in (2)/3. This judgment about fraction embodiments is translated into a judgment about the fractions 2/5 and 2/3. The translation process is more complex than a cursory glance might suggest. It appears that, at a minimum, the child must (a) perform bidirectional translations between 2/5 and (2)/5 and store this information in short-term memory (STM), (b) perform bidirectional translations between 2/3 and (2)/3 and store this information in STM, (c) make a judgment about (2)/5 and (2)/3 and store this information in STM, and (d) coordinate the information from (a), (b), and (c) to infer a judgment about the order relation on 2/5 and 2/3 (Figure 2). In the context of our instruction and interviews, a good answer that reflected this coordination would likely be accompanied by an explanation such as "Two fifths is less than two thirds because there are two pieces in each, but the pieces in two fifths are smaller, so a smaller amount of the unit is covered for two fifths."

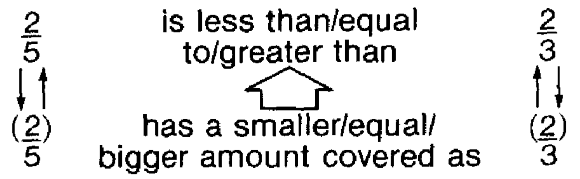

Figure 2. Use of embodiment-based knowledge in judging fraction order.

It is this coordination of information that we refer to as *coordinating translations* between the representational modes of mathematical symbols and fraction embodiments. The comparison process also involves translating a relational judgment on given fraction embodiments to (or from) a relational judgment on the fractions embodied. Further, the process often involves relating the physical transformation among, and within, fraction embodiments to a corresponding symbolic arithmetic operation.

Thought Flexibility for Transformations Within a Mode of Representation

To operate with mathematical symbols for fractions, one must be adept at making transformations within the symbolic mode; similarly, operations with embodiments for fractions require transformations within the system of embodiments. This notion can be illustrated with tasks involving the recognition or the construction of (a) a mathematical symbol for a fraction, and (b) an embodiment that represents a fraction equivalent to the one given. For example, solving the open sentence 4/6 = □ /3 within the representational system of mathematical symbols can be done by means of a transformation algorithm, such as dividing 4 and 6 each by 2. Similarly, recognition that 4/6 = 2/3 can be accomplished with the same transformation. To solve the open sentence within the representational system of fraction embodiments would require the ability to transform (4)/6 to (2)/3. This transformation involves either a physical or a mental (i.e., imagined) repartitioning (Figure 3). It is the ability to make these transformations within a representational system that we refer to as *thought flexibility for transformations.* Flexibility is used in this context to refer to the adeptness, insight, or understanding that a child displays in making these transformations.

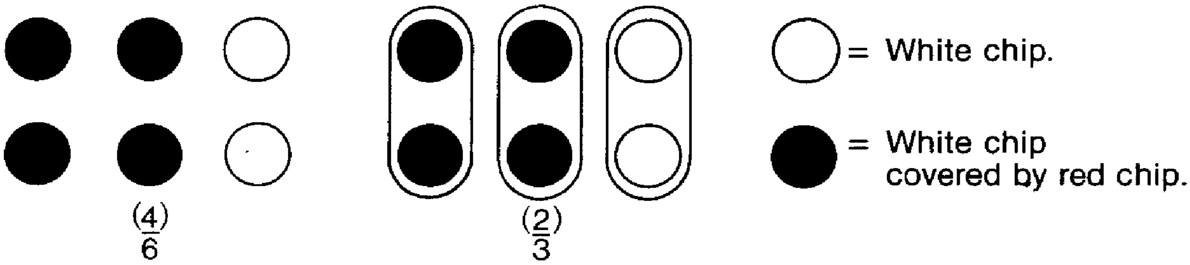

Figure 3. Poker chip embodiment used to solve the open sentence 4/6 = □/3.

We view thought flexibility for transformations within representational modes as related to thought flexibility in coordinating translations between representational modes. For example, the transformation of a chip display for 4/6 to a chip display for 2/3 should facilitate the understanding that 4/6 = 2/3 in the symbolic mode. Such understanding requires the coordination of three sources of information — namely, (a) the observation that (4)/6 and (2)/3 have equal amounts covered, (b) the translation of (4)/6 to 4/6, and (c) the translation of (2)/3 to 2/3 — and it provides for the ability to make the subsequent inference that 4/6 = 2/3. The ease with which children can accomplish these transformations, translations, and inferences is related to their ability to understand at the symbolic level issues of the order and equivalence of fractions. Limited flexibility of thought in accomplishing any of (a), (b), or (c) above places extra demands on the child's information-processing capacity and inhibits the acquisition of flexibility of thought in making transformations within the representational mode of mathematical symbols.

Progressive Independence of Thought from Embodiments

A third characteristic of thought that seems fundamental for children's success with tasks on order and equivalence, especially as those tasks become more complex, is the progressive independence of thought from the embodiments used to represent fractions. For an illustration, we consider the problem of ordering 5/6 and 2/3. If a child compares sectors of a given circular unit, the fractions can be ordered by comparing the *absolute size* of the area covered. The ultimate aim of instruction, however, is for children to understand the order relation between 5/6 and 2/3 without making a reference to a physical display. Children should eventually become able to make a judgment based on the relation (ratio) between 5 and 6 and between 2 and 3. This judgment requires that they observe that 5/6 is *relatively* larger than 2/3 regardless of the common unit chosen. The need for children's understanding of rational number concepts to accommodate successive abstraction suggests that thought originally directed at actions on an embodiment must become successively more independent of that embodiment.

Children can be helped toward thought that is independent of embodiments by becoming involved in situations that require a decision in advance (a plan) as to how to construct a manipulative display or to otherwise illustrate a fraction idea. An example is the use of a discrete model such as poker chips to determine the order of 5/6 and 2/3 (Figure 4). With chips, there is no predetermined unit; the child must choose a number of chips to serve as a unit for representing both fractions. A child might choose the unit by mentally manipulating various sets of chips, realizing that a set of 6 (or a multiple of 6) chips could be grouped, and regrouped, into 6 subsets and 3 subsets. That is, the child predetermines by mental manipulation, rather than by trial and error, that a unit of 6 (or a multiple of 6) is necessary. From this point on, the child might use (5)/6 and (2)/3 to determine that 5/6 is greater than 2/3. The important point is that these perceptually facilitating embodiments can be obtained only if information about the fractions is used in *planning* both displays in a coordinated way. A comparison of the embodiments (5)/6 and (2)/3 in terms of sixths is less likely to occur if the child does not anticipate a need for sixths before constructing the embodiments. In this case, the child's understanding of the concepts 5/6 and 2/3 and his or her understanding of fraction equivalence assists in the choice of an appropriate unit. Such understandings and anticipations develop into thought that is independent of the actual use of embodiments for fractions. For children who have progressed to that point, the use of embodiments appears as a *confirmation* of a prejudgment based on a mental manipulation. This use of embodiments is both a more sophisticated and more desirable form of reasoning than relying on embodiments for support in making a judgment.

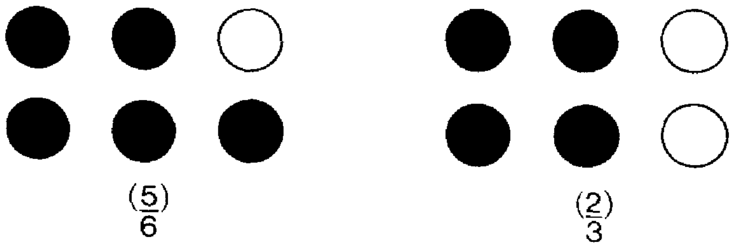

Figure 4. Poker chip embodiment used to order 5/6 and 2/3.

SUPPORTING DATA

We present data from two fourth graders to exemplify the three major characteristics of thinking just discussed. Passages from interview transcriptions and the observer's notes that illustrate the thinking strategies the children employed have been selected. Our major intent is to describe how different thought strategies affect the two children's progress in tackling specific and successively more complex tasks on the order and equivalence of rational numbers. The data have limitations; they do not illustrate why the children exhibited different, more-or-less fully developed strategies, nor do they bear on the questions of how less well-developed strategies (more dependent on concrete materials) might be improved and whether instruction might be delayed for some children. The reader will also observe that even under the conditions of instruction in the teaching experiment—which employed a rich diversity of manipulative aids, placed a heavy emphasis on concept development, and took more time than traditional instruction—many rational number concepts remained exceedingly difficult for these children.

The two children, Bob and Jane (not their real names), were participants in the study at the DeKalb site. (We are grateful to Robert Rycek, who served as the observer and the recorder in collecting these data.) Bob and Jane had different levels of achievement before the teaching experiment began. Bob, although not precocious, was clearly the more able in mathematics and in general school achievement. Jane's ability in mathematics was in the low to middle range. She was by no means at the bottom of the class from which she was chosen; she performed slightly below average in all school subjects. She had no identified learning handicaps.

Thought Flexibility in Coordinating Translations

Differences between Bob's and Jane's ability to make meaningful translations were observed early in the instruction.

1. (Interview following 2 days of instruction)
Interviewer: [Reads aloud] "One third and one fourth." Are they equal or is one less?
Bob: One third is less than one fourth because three is smaller than four.
[Interviewer alludes to a manipulative aid used in instruction.]
Bob: [Recalling that 1/4 was blue and 1/3 brown] One fourth is less than one third because four blues cover the whole, so they are smaller [than thirds, which require only three].

From this point on, Bob made no further errors like that in his first statement. He correctly related the order of two fractions to the compensating relation between the size and number of equal-sized parts needed to cover a unit. A typical response from Bob to subsequent questions on order was: "Three fourths is greater [than 3/9]. If you take two units and cut one into fourths and the other into ninths, fourths are gonna be bigger because there is less of them to cover the unit."

Note also Bob's ability to coordinate the inverse relation between the number of parts into which the whole is divided and the resulting size of each part. Bob's ability to interrelate the concrete and symbolic modes of representation was in sharp contrast to that of Jane, who in Excerpt 2 exhibits difficulty in making translations from the descriptive language of manipulative aids to the more formal symbols:

2. (Classroom observation during the fourth day of instruction)
Instructor: One fifth and one tenth, which is less?

Jane: One fifth is less than one tenth.

[Instructor seeks to help by referring to a manipulative aid.]

Jane: One orange [representing 1/5] is greater than one purple [representing 1/10].

Instructor: Say the sentence with fractions [i.e., translate from embodiment mode to verbal descriptive mode].

Jane: One orange is greater than one purple.

[Instructor repeats request and prompts.]

Jane: One fifth equals one orange.

According to our analysis of coordinating translations, for Jane to accomplish the translation from "one orange is greater than one purple" to "one fifth is greater than one tenth," she must (a) translate one orange part to one fifth and store this in STM, (b) translate one purple part to one tenth and store this in STM, (c) observe that one orange part is greater than one purple part, and (d) coordinate (a), (b), and (c) to translate the judgment about the relation between the orange and purple parts (embodiments) to a relational statement about one tenth and one fifth. Jane gave evidence that she was able to accomplish (a), but whether she accomplished (b) is not known. In any case, she was unable to accomplish the coordinated translation.

Her difficulty persisted and was equally evident 3 days later:

3. (Interview after 7 days of instruction)

Interviewer: One fifth and one ninth, which is less?

Jane: One fifth is less . . . because five is less than nine.

[Interviewer directs Jane to use colored parts.]

Jane: [Covers one circular unit with orange (1/5) parts and another with white (1/9) parts] It takes 9 white and 5 orange.

Interviewer: [Draws attention to colored parts] Which is less, one fifth or one ninth?

Jane: One fifth, because it takes five to cover this, and it takes nine to cover this [points to the circular units].

[Interviewer asks about the size of the colored parts.]

Jane: One orange is bigger than one white. One fifth is less than one ninth.

In Excerpt 3, we see two potential explanations for Jane's difficulty in ordering fractions at that time. In her first response is evidence that any mental images based on the manipulative aid that she might have developed to suggest the order of 1/5 and 1/9 were dominated, overpowered, by her knowledge of the ordering of the whole numbers 5 and 9 (see Behr et al., 1984, for a discussion of the whole-number-dominance strategy). The second interaction between the interviewer and Jane suggests that Jane was able to make the 1/5 to (1)/5 and 1/9 to (1)/9 translations. In her third response we observe a possible linguistic interference in her ability to coordinate the translations; she appeared to confuse *more* in reference to number of parts with *bigger* and *greater* in reference to size of parts and fractions, respectively. Even with the embodiment present, Jane did not give evidence that she perceived the compensating relation between the number and size of equal parts in a partition. Whether the interference came from the linguistic similarity of *more* and *greater* or from an overgeneralized schema for ordering whole numbers is not clear (see Behr et al., 1984).

Jane's difficulty in coordinating translations between fraction embodiments and mathematical symbols for fractions persisted, as did her difficulty with ordering fractions. Excerpt 4 is from Lesson 7 (see Behr et al., 1984, for a description of the lesson sequence). The instructor read aloud a pair of fractions, and the children were told to write them and circle the one that was less.

4. (Classroom observation after about 25 days of instruction)

Instructor: One third and one sixth.

Jane: [Writes fractions. Long pause. Circles 1/6 . . . changes answer, . . . changes again, . . . continues to vacillate.]

Instructor: Explain your answer.

Jane: In numbers this one [1/3] is smaller . . . but in fractions . . . this one [1/6] is . . . [pause] . . . pieces are bigger. . . .

[Some class discussion takes place.]

Jane: One third has bigger pieces to cover it [the unit]. . .one sixth has smaller pieces . . . so one third is bigger.

Instructor: Two thirds and two sixths . . .?

Jane: [Indicates 2/6 is less, changes to 2/3, pauses, changes again]. . . . Two thirds takes six pieces to cover the whole . . . two sixths takes twelve pieces to cover the whole, so [2/6] has got to be smaller.

Jane's vacillation indicated her uncertainty as to how to determine which of two fractions is less. In her second response she initially made a distinction between whole numbers and fractions, but that was not sufficient. It can, however, be taken as evidence that she was gaining at least implicit knowledge that her thinking about the order of the whole numbers 3 and 6 must be modified to accommodate the ordering of 1/3 and 1/6. In the same response Jane also mentioned "pieces" and hesitantly indicated that (1)/6 pieces are bigger than (1)/3 pieces. It is apparent that she was still having difficulty coordinating the information involved in the translations between symbol and embodiment with the information involved in the physical comparison of parts of a whole.

Some directed reference to embodiments facilitated Jane's thinking, as reflected in her third response. She was able to accomplish the coordinated translation when she began to talk directly in terms of the size of pieces, her memory apparently aided by reference to the embodiment.

When the discussion shifted to the nonunit fractions 2/6 and 2/3, Jane did not visualize the size of the parts. Rather, she adopted a rule of multiplying numerator and denominator so that the larger the product, the smaller the fraction. Although her verbalization of this implicit rule referred to "pieces," the rule had no exact counterpart in the embodiment.

Jane's difficulty in relating and coordinating visual and symbolic information persisted into Lesson 9. In a worksheet problem, the children were asked to order three pictures to show the order of fractions from smallest to largest (see Figure 5). Jane's behavior is presented in condensed form in Excerpt 5.

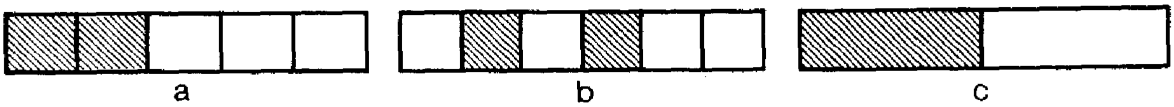

Figure 5. Worksheet task on ordering fractions from smallest to largest.

5. (Classroom observation during the 38th day of instruction)
Jane initially says *a* is less than *b* is less than *c*, reasoning that two fifths is less than two sixths because "five is less than six." When she corrects these errors, she seems to rely only on the physical size of the shaded parts and to ignore the fraction symbol. She still seems unable to translate her statement about the given order relation from embodiment mode to symbol mode.

Thought Flexibility for Transformations Within a Mode of Representation

Differences in the flexibility of thought shown by Bob and Jane for transformations within the embodiment system of representation are evident in a discussion about a paper-folding display (see Figure 6). The discussion is presented in condensed form in Excerpt 6. How this difference affected each child's development of flexibility within the mathematical system of representation can be observed by comparing their progress in subsequent excerpts.

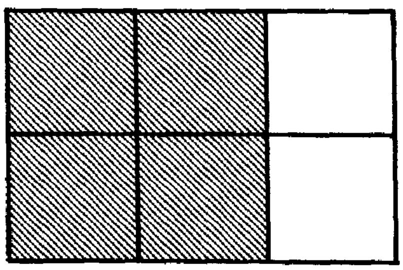

Figure 6. Paper-folding display to show 2/3 = 4/6.

6. (Classroom observation on about the 15th day of instruction)
Bob: [Asked about 2/3 and 4/6 in the context of the paper-folding display, says they are equal and explains] "You don't have any more . . . you have more parts but not more space. . . ."

Jane is asked the same question and does not respond. Asked if 2/3 covers more than 4/6, she still does not respond. Instructor prompts by pointing to the shaded part of the figure and counting—again, no response. Jane seems confused by the simultaneous presence in the display of 3 parts with 2 shaded and also 6 parts with 4 shaded.

In Bob's response we see evidence of his ability to observe that an embodiment for 2/3 can be transformed into an embodiment for 4/6 and vice versa. He is able, because of two interpretations of the partition (2 of 3 and 4 of 6), to associate both fractions to the display. Moreover, the equivalence of the shaded areas is more salient for Bob than the nonequivalence of the partition. Therefore, he is able to conclude that 2/3 and 4/6 are equal. Understandings such as this very likely provide the cognitive structure that allows him to begin to perceive that each of the symbols 2/3 and 4/6 is transformable to the other.

Jane's confusion suggests that with respect to an imagined or physical transformation of the embodiment, her thought is inflexible. This inflexibility inhibits her in perceiving that interpretations of 2 of 3 parts and 4 of 6 parts are simultaneously (or successively) possible. In turn, she is unable to simultaneously attach the two fractions 2/3 and 4/6 to the display. Jane's lack of flexible thought for this transformation is also manifested by her inability to imagine the removal (unpartitioning) or addition (repartitioning) of partitioning lines in the display. Her perception of the partition lines may dominate her perception of the equivalence of the areas shaded (Behr, Lesh, Post, & Silver, 1983). These difficulties—each a manifestation of inflexible thought for embodiment transformations—inhibit the important element of area equivalence from being salient for Jane. As a result, she is unable to deal with the question of the equivalence of 2/3 and 4/6 by using the embodiment as the basis for her thought.

The following observations (Excerpts 7 and 8) demonstrate clearly Bob's thought flexibility for transformations between two discrete (chip) embodiments and his progressively more flexible thought for transformations on mathematical symbols for fractions.

7. (Classroom observation on about the 33rd day of instruction)

A worksheet asks the student to use a 12-chip unit to show the same fraction as the rod display (Figure 7). Bob arranges 12 white chips and covers 9 with red chips (display A in Figure 8). Then he changes the display (display B in Figure 8).

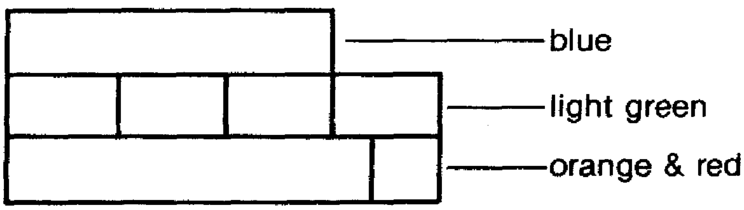

Figure 7. Rod display to show 3/4.

Bob: [Explaining to observer] Both show three fourths and nine twelfths.

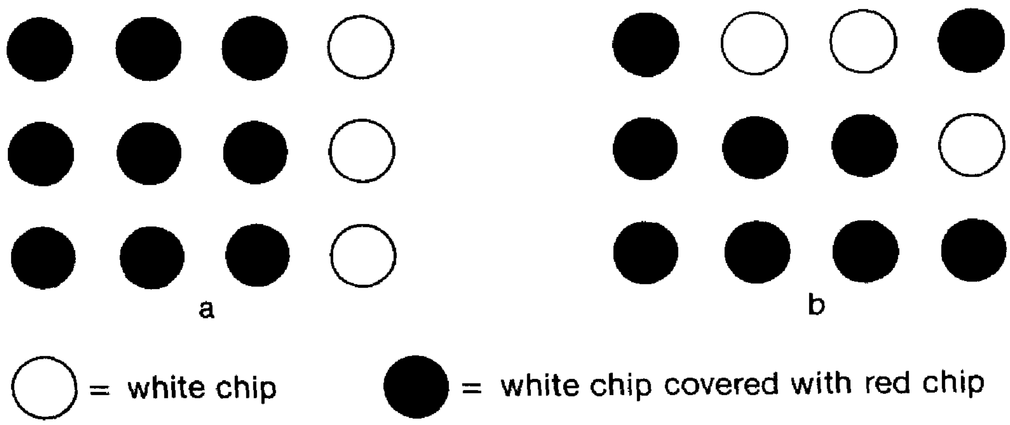

Figure 8. Bob's 12-chip displays in response to rod display.

Observer: Could you find out how many eighths are the same?
Bob: That will be six . . . six eighths.

8. (Classroom observation during the 30th day of instruction)
Bob shows 3/4 using a 24-chip unit (display A in Figure 9). Without explaining, he says the display also shows 1 1/2 halves. He transforms the display (display B in Figure 9) and says that it is also 4 1/2 sixths.

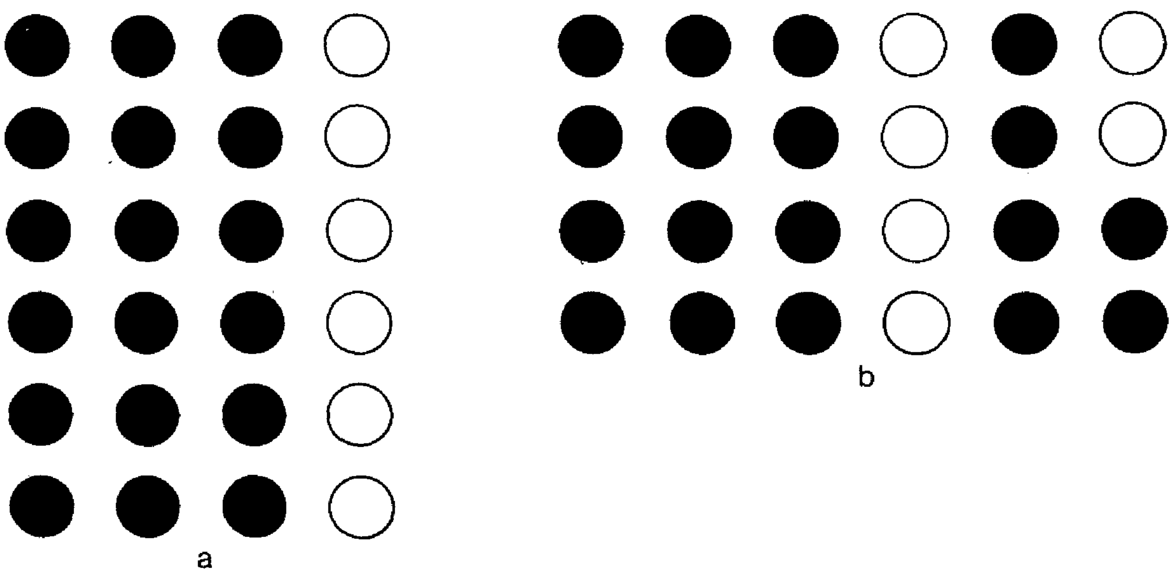

Figure 9. Bob's 24-chip displays to show 3/4.

In Excerpt 7, Bob solved the question about eighths abstractly—evidence that he was acquiring progressively more flexibility for making transformations of mathematical symbols. The extent of Bob's thought flexibility can be seen in Excerpt 8, which also presents evidence about how his flexibility for embodiment representations facilitates this thought flexibility for mathematical symbol variation. It seems unlikely that his generation of the mathematical symbols of 1 1/2 halves and 4 1/2 sixths as equivalents of 3/4 would have occurred in the absence of an embodiment basis for his thought.

Bob's flexible thought and consequent success with fraction equivalence is in contrast to Jane's thought. Her inability to accomplish transformations on embodiments clearly inhibited her progress on fraction equivalence tasks.

9. (Classroom observation during the 33rd day of instruction)

Jane writes "3/12" as the fraction shown on a worksheet (Figure 10). This leads her to the open sentence 3/12 = □/4 on the worksheet; the directions are to solve the sentence using chips. She makes a chip display (Figure 11) and writes "3/12 = 3/4" as the solution. She is unable to do any regrouping of her chips to show fourths.

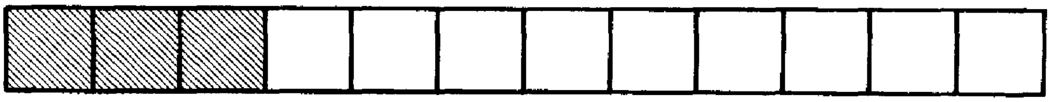

Figure 10. Worksheet task on fraction equivalence.

Figure 11. Jane's chip display to solve 3/12 = □/4.

Although in a previous problem Jane had been able to find abstractly an equivalent fraction in higher terms (the number of 12ths equivalent to 4/6), she was unable here to make any progress toward solving the lower terms problem 3/12 = □/4. Elsewhere, rigidity was observed in Jane's interpretation and production of chip displays. She could group a set of chips to show a fraction in lower terms when it was the one given and then reinterpret the grouping to determine the higher terms fraction; but when given a higher terms fraction, she was unable to regroup the chips to determine the equivalent lower terms fraction.

What specifically do the observations suggest about Bob's and Jane's prospects for developing skill in using transformation algorithms for generating fractions equivalent to the one given or for testing the equivalence of two fractions; that is, for performing abstract fraction-equivalence tasks? Embedded in Bob's realization that 3/4, 6/8, 9/12, 1 1/2 halves, and 4 1/2 sixths are numerous symbolic representations for three fourths is the information that he has developed firm cognitive structures for fraction equivalence. At a minimum, the structures allow for thinking of a fraction in many symbolic forms. Surely this type of structure for the concept of equivalent fractions will provide the necessary base on which meaningful algorithmic skills can be developed. The absence of evidence for such cognitive structures suggests the opposite conclusion for Jane *at this point in her development.* We emphasize that our data do not suggest, nor is it our interpretation, that Jane will not be able to develop and perform at Bob's level. The point to be made is that thought flexibility for transformations on embodiments and for transformations on mathematical symbols are basic cognitive skills for success with tasks on the order and equivalence of fractions. Moreover, it is likely that thought flexibility for transformations on embodiments is a precursor to thought flexibility for transformations of mathematical symbols.

Progressive Independence of Thought from Embodiments

In Lesson 6, Bob's thinking about equivalent fractions was still embodiment dependent:

10. (Classroom observation during the 19th day of instruction)

Instructor: How are three fifths and six tenths alike, and how are they different?

Bob: They are alike because we [Bob and his working partner] used the same pieces [orange rod units], except I used red to make pieces [partition], and you used white. I covered the same amount as you even though it took more pieces. [Writes on his paper, *Alike:* "Because they cover the same amount." *Different:* "Because they cover more pieces."]

Evidence that Bob's thinking was becoming less embodiment dependent is provided by an activity that occurred later in the same lesson:

11. (Classroom observation during the 21st day of instruction)

The class is given the following task: A dark green rod [6 cm] is what fraction of an orange rod [10 cm] *and* a red rod [2 cm] train [i.e., a 12-cm length]? Bob writes, "6/12 3/6 1/2."

Thus the notion of equivalent fractions seems to be evident in Bob's thinking at this point, but it was still tied to the concrete. On writing the fractions in Excerpt 11, he said that more equivalent fractions exist. He did not, however, mention or write the fraction 4/8, which was not represented by the rods—evidence of the concrete nature of his thinking.

In Lesson 7 it was possible to see Bob's progression to embodiment independence.

12. (Classroom observation during the 28th day of instruction)

Bob is working on a worksheet that asks him to compare 4/5 and 8/10. Without comment, he shows 8/10 using an orange-orange (20 cm) rod train as a unit and 10 red (2 cm) rods to partition. To show 4/5, he takes another orange-orange rod train.

Bob: Has to be twice as big as the reds . . . [as he selects 5 purple (4 cm) rods to partition the unit, showing a priori knowledge of needing to cover the unit with pieces twice as long but half as many in number].

Bob's behavior in Excerpt 12 suggests that he knew that 4/5 is obtainable from 8/10 by taking 1/2 of 8 as the numerator and 1/2 of 10 as the denominator. He seemed close to abstracting an algorithm for equivalent fractions. The algorithm later became more stable, as can be seen from a subsequent interview.

13. (Interview after 40 days of instruction)

Interviewer: [Writes "3/4 = 9/□"] Find the number which goes in the box so the fractions are equal.

Bob: [Writes "12"] . . .Three goes into nine three times, and four goes into twelve three times.

Bob's progession to embodiment-independent thinking can be contrasted with Jane's thinking, which remained embodiment dependent.

14. (Classroom observation during the 33rd day of instruction)

A worksheet asks the student to write numbers in the boxes to find equivalent fractions shown by the picture (Figure 12). Jane marks lightly on the picture (Figure 13), then writes "2/5." She draws partitioning lines on the picture (Figure 14).

Jane: One, two [pointing at groups], three, . . . , ten; one, two, three, four [writes "4/10"]. One, two [pointing to single dots], three, . . . , twenty; one, two, three, . . . , eight [writes "8/20"].

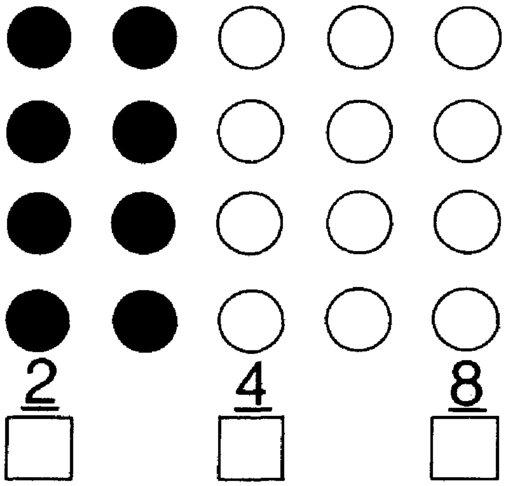

Figure 12. Worksheet task on fraction equivalence.

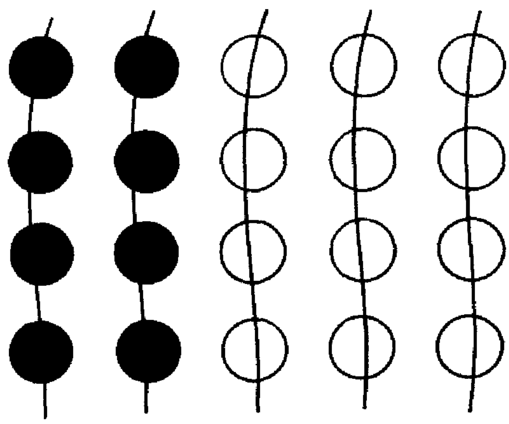

Figure 13. Jane's marks to find 2/5.

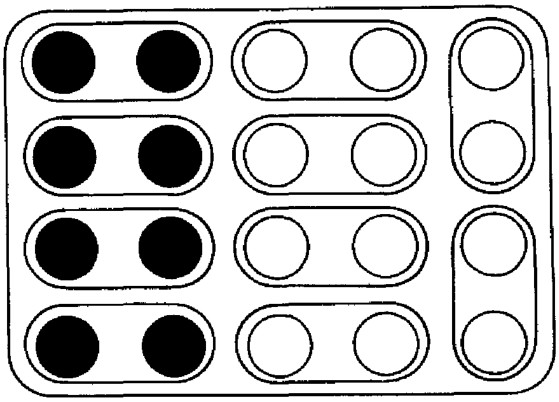

Figure 14. Jane's marks to find 4/10.

Jane seemed to need the deliberate partitioning and counting to organize her thinking in this stepwise fashion. Her actions on the display directed her thought. There was no evidence of a plan to reach the solutions 2/5, 4/10, and 8/20 with the manipulative aid to verify her answer; she used the picture to generate each equivalent fraction.

During Lesson 9, a worksheet was given that required the children to match pictures embodying fractions with fraction symbols. If several fractions could be matched with a picture, the children were to state the equivalence of these fractions. Jane's thinking about equivalent fractions continued to be embodiment dependent, as the following condensed excerpt exhibits.

15. (Classroom observation on about the 40th day of instruction)

Jane makes several correct symbol-picture matchings, one of which results in her writing "4/12 = 1/3." In seeking a symbol to match a picture (picture A in Figure 15), she identifies the picture as representing 4/12 (12 unshaded, 4 shaded) and writes "4/12" by the picture. She considers matching the picture with 1/3 but reconsiders. After making a physical grouping with pencil marks (picture B in Figure 15), she decides the picture represents 1/4 and writes "4/12 = 1/4."

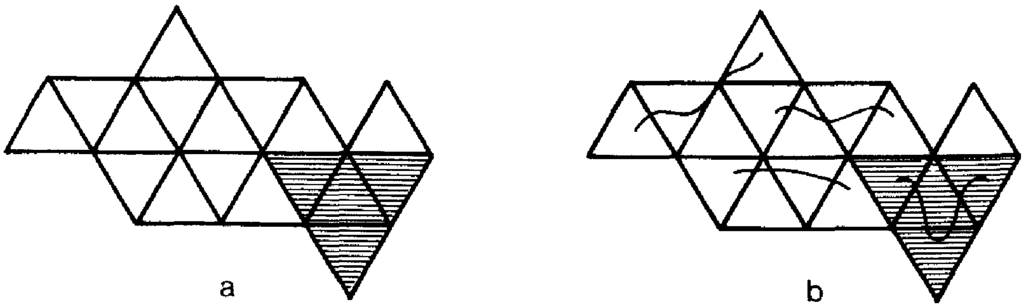

Figure 15. Pictures from worksheet on matching symbols and pictures.

Jane's dependence on the display resulted in an incorrect symbolic statement, one that contradicted the statement 4/12 = 1/3 that she had written earlier. When the instructor queried her about the picture, she recognized it as 4/16. She changed the label she had written under the picture by crossing it out and writing "4/16." The statement 4/12 = 1/4 remained unchanged.

During the entire course of the teaching experiment, Jane never reached the embodiment-independent thought processes achieved by Bob.

16. (Final interview after 18 weeks of instruction; no pictures or manipulative aids were present)

Interviewer: [Writes "3/5   6/10"] Three fifths and six tenths—are they equal, or is one less?

Jane: Six tenths is greater. . . . If you have ten pieces, six covered, and five pieces with *only* three covered [emphasis ours] . . . .

Interviewer: [Writes "3/4 = 9/□"] Find the number which goes in the box so the fractions are equal.

Jane: [Writes "16"] Well, three times three is nine, and four times four is sixteen.

[Interviewer writes "4/2 = 6/□"]

Jane: [Writes "4"] Four? I don't know. . . . Four plus two equals six; two plus two equals four.

DISCUSSION

We open the discussion with two remarks concerning important phenomena identified in the present paper and in Behr et al. (1984) concerning children's performance on tasks concerning the order and equivalence of rational numbers.

First, children's understandings about ordering *whole* numbers often adversely affect their early understandings about ordering fractions. For some children, these misunderstandings persist even after relatively intense instruction based on the use of manipulative aids (see Jane's first response in Excerpt 16). With whole numbers, children can deal with order tasks in two ways that correspond to the cardinal and ordinal aspects of number, respectively: (a) they can compare the "bigness" of two numbers by matching elements of finite sets, or (b) they can make use of the counting sequence, deciding which number is smaller according to which comes first (Resnick, 1983). The direct comparison methods for ordering whole numbers are inadequate for dealing with fraction-ordering tasks because fraction and fraction order require the following complex understandings: (a) fraction size depends on the relation between the two whole numbers in the fraction symbol (a ratio); (b) there is an inverse relation between the number of parts into which the whole is divided and the resulting size of each part; (c) when fractions have like denominators, there is a direct relation between the number of "distinguished" parts and the order of the fractions; (d) when fractions have different numerators and denominators, judgments about their order require an extensive and flexible use of fraction equivalence; and (e) the density of the rational numbers implies the counterintuitive notion that there is no "next" fraction.

Second, linguistic considerations contribute to some children's misunderstanding of the ordering and equivalence of fractions. The words *more* and *greater* (and their counterparts *less* and *fewer*) cause difficulty for some children because *more* can mean *more parts* in the partitioned whole or *more area* covered by *each* part. Similarly, *greater* can mean a greater number of parts in the partitioned whole or a greater fraction size. A similar confusion exists with respect to *size* and *amount,* as illustrated by children who, when asked which of two fractions is less, reply, "Do you mean in size [e.g., size of each subdivision] or in amount [e.g., number of subdivisions]?"

In this paper we have identified three characteristics of thought that are hypothesized to be important for successful performance on tasks dealing with the order and equivalence of fractions. The data we have presented do not necessarily suggest that these characteristics represent distinct or hierarchical levels of thought. Nevertheless, some observations from the data do suggest hypotheses about hierarchical relations.

The progression in Bob's thought indicated that he appeared to acquire various abilities related to thought flexibility in coordinating translations between modes, to thought flexibility for transformations within modes, and to progressive independence from embodiments. He appeared to acquire the following abilities, in approximately the order given:

1. Ability to make single bidirectional translations of the form $m/n \longleftrightarrow (m)/n$
2. Ability to make transformations on embodiments and to make their related transformations on fraction symbols
3. Ability to coordinate bidirectional translations; that is, to translate a judgment about embodiments to a judgment about the represented fractions (Transitive inference is frequently involved in the coordination process.)
4. Ability to "preplan" a manipulative display, which represents the emergence of embodiment-independent thought
5. Emerging ability to identify sequences of symbols that correspond to sequences of physical manipulations of fraction embodiments
6. Ability to consistently and correctly apply transformation algorithms for generating equivalent fractions, which reflects well-developed embodiment-independent thought.

Abilities 1 and 2 may be related and may emerge in parallel rather than sequentially, but together they likely lead to Ability 3. There seems strong support in the data for the hypotheses that Ability 3 is a prerequisite to Abilities 4 and 5 and that Abilities 4 and 5 develop into Ability 6.

We have hypothesized that thought flexibility in coordinating translations between the representational systems of fraction embodiments and mathematical symbols for fractions is a prerequisite to more abstract embodiment-independent thought. The data indicate that at one point a child can make a single bidirectional translation but is unable to keep this information in STM while making a second bidirectional translation. Later, the child is able to make the two bidirectional translations and the relational judgment between embodiments but cannot coordinate this information to make a relational inference from the embodiments to the fraction symbols.

Whether a bidirectional translation is accomplished and stored in STM as one or two separate cognitive units (i.e., two unidirectional maps) is not determinable from our data. If such translations are stored as two units, however, the whole sequence of coordinating the translations may exceed STM capacity. Whatever the cause, the child who cannot coordinate such translations is seriously handicapped in abstracting information from the embodiment system of representation that could be used in making judgments, performing transformations, or doing operations in the mathematical symbol system of representation. This handicap is especially severe if one expects a meaningful performance. Such a child might need more practice in making paired unidirectional translations between modes of representation until the translations became habituated, automated, and schematized.

Thought flexibility for transformations within the fraction-embodiment mode of representation seems to facilitate thought flexibility for transformations within the mathematical symbol representational system. Children who have difficulty with transformations on embodiments (see Excerpt 9) almost surely will have difficulty making meaningful transformations on mathematical symbols. Some children have difficulty in modifying partitions by either actual or imagined physical actions. How children might be aided in overcoming this difficulty has been addressed in part by Cramer, Post, and Behr (1984), but questions remain for further research.

REFERENCES

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 92–126). New York: Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*, 323–341.

Cramer, K., Post, T. R., & Behr, M. J. (1984). *Perceptual cues and the quality of children's thinking in rational number situations.* Unpublished manuscript.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109–151). New York: Academic Press.

[Received March 1983; revised May 1984]

AUTHORS

THOMAS R. POST, Professor of Mathematics Education, College of Education, 240 Peik Hall, University of Minnesota, Minneapolis, MN 55455

IPKE WACHSMUTH, Wissenschaftlicher Assistent, Fachbereich Mathematik, Universität Osnabrück, Albrechtstrasse 28, D4500 Osnabrück, West Germany

RICHARD LESH, Director, Division of Mathematics and Science, WICAT Systems, P.O. Box 539, Orem, UT 84057

MERLYN J. BEHR, Professor, Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60015