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NON-STABLE CORES OF EXCHANGE ECONOMIES

1. During past years criticism has been voiced frequently against the core as a meaningful equilibrium concept for socially stable outcomes.[1] One aspect of this criticism has been that in may cases the core excludes distributions of total welfare which enjoy the same feature as any allocation in the core, namely that if some coalition is made better off at one distribution in the core than at another it cannot enforce the preferable one by using its own resources. A set of such mutually undominating allocations should be one possibility of describing socially stable outcomes, since no coalition has sufficient reason and power to reject one in favor of a different one. Another aspect of the criticism is that in many non-pathological cases the core is very small or even empty whereas there exist large sets of mutually undominating allocations. In most of these cases the blocking mechanism imposes a strong irrationality of behavior of some participants, requiring the rejection of more favourable outcomes for them than any allocation in the core. For this criticism to be valid however, every consumer has to have complete knowledge of the composition of the core. The classical notion of a von Neumann-Morgenstern solution or, as it is sometimes called, a stable set, embodies an additional external stability requirement and it overcomes both of the above arguments. It is the one solution concept proposed frequently as an alternative to the core. Since the core is always contained in every stable set one would like to know in which cases does one actually enlarge the set of socially stable outcomes if one discards the core in favor of the stable sets, i.e. one would like to know how frequently it may occur that the core itself is a stable set.

There has been a series of papers with examples and conjectures on the relationship of the core and the solutions (for the definition of these concepts see Part II), for characteristic function games with side-payments (see e.g. Lucas [4, 5, 6] and the references there). For the class of convex games Shapley [8] has shown that the core coincides with the unique solution. It is well known that the core of such games is large. Shapley

G. Schwödiauer (ed.), Equilibrium and Disequilibrium in Economic Theory, 53–66.

also indicates that the convexity assumption is not necessary for equivalence; hence for some games which have small cores one may still hope to obtain equivalence.

Recently S. Hart [3] has proposed a straightforward extension of the domination relation to exchange economies. He discussed a general class of solutions in this context. In the first part of this paper two examples are given which indicate that, in general, the core will not be stable. In fact it will be argued that, for the definition of domination used by Hart, the core can never be stable except for trivial cases.

Given the usual representation of an exchange economy as a characteristic function game, one may attempt to characterize stable cores by exploiting the relationship of the core of the economy in the commodity space with its counterpart in the utility space. It is shown in the second part of the paper that the utility counterpart of a solution applying Hart's definition is in general much larger than the solution in utility. An alternative notion of domination is proposed which guarantees equivalence between the solutions for the economy and the solutions for the game. In an effort to find sufficient conditions for stable cores using the alternative notion of domination, a natural generalization of the concept of convex games for the non-side-payment case is introduced. Although the question of equivalence of the core with the unique solution is left open as unsolved, it is shown that convexity is a 'cardinal' concept which can be destroyed by some utility transformations and that convexity cannot be generated easily by making natural assumptions for the economy, even in the special case when there exists a representation as a side-payment game.

2. Let $\mathscr{E} = \{I, (X_i, e_i, \underset{i}{\succsim})\}$ denote an exchange economy with a finite set of consumers $I = \{1, \ldots, n\}$, where for each $i \in I$, X_i, a nonempty subset of the commodity space R^l, is consumer i's consumption set, $e_i \in R^l$ is consumer i's initial endowment, and $\underset{i}{\succsim}$ is a preordering on X_i, consumer i's preference relation. An allocation for the economy is a list $x = (x_i)$ of commodity bundles, one for each consumer i, such that $x_i \in X_i$. It is called feasible if $\sum_{i \in I} x_i \leqq \sum_{i \in I} e_i$.

Definition. An allocation x is said to be blocked if there exist a nonempty coalition $S \subset I$ and bundles $y_i \in X_i$, $i \in S$, such that

(1) $y_i \succ_i x_i$ all $i \in S$

(2) $\sum y_i \leqq \sum e_i$.

The core $C(\mathscr{E})$ is the set of feasible, unblocked allocations. In order to define the concept of a solution, Hart [3] introduces the following notion of domination.

DEFINITION. An allocation y dominates an allocation x, written as y Dom^* x if there exists a non-empty coalition $S \subset I$ such that

(1) $y_i \succ_i x_i$ all $i \in S$,

(2) $\sum_{i \in S} y_i \leqq \sum_{i \in S} e_i$

i.e. S prefers y over x and S is effective for y in the sense that it can enforce y restricted to S directly by using its own resources.

DEFINITION. An allocation x is individually rational if

$$x_i \succsim_i e_i \qquad \text{for all } i \in I.$$

Then, as a consequence, one has:

DEFINITION. A set $L^*(\mathscr{E})$ of feasible, individually rational allocations is a von Neumann-Morgenstern solution or, simply, a solution if

(1) for any $x, y \in L^*(\mathscr{E})$ $\quad x\ D\phi m^*\ y,\ y\ D\phi m^*\ x$

(2) for any $z \notin L(\mathscr{E})$ there exists an $x \in L(\mathscr{E})$ such that $x\ Dom^*\ z$, where $x\ D\phi m^*\ y$, denotes not $x\ Dom^*\ y$.

The following two examples suggest that in general any solution will be larger than the core of the economy.

Example 1. Let $l = 2$, $n = 3$, $e_1 = (1, 0)$, $e_2 = e_3 = (0, 1)$ and preferences are identical and representable by $u(x_1, x_2) = \min\{x_1, x_2\}$. It is well known that the core of this economy consists of the allocations

$$\{((1, 1+\alpha), (0, \beta), (0, \gamma)) \mid \alpha, \beta, \gamma \geqq 0, \alpha + \beta + \gamma = 1\}$$

whereas

$$L^*(\mathscr{E}) = \left\{ \begin{array}{c|c} (\alpha_1, \alpha_2) & \alpha_1 \leqq \alpha_2 \quad \alpha_1, \beta_1, \gamma_1 \geqq 0 \\ (\beta_1, \beta_2) & \beta_1 \leqq \beta_2, \quad \alpha_1 + \beta_1 + \gamma_1 = 1 \\ (\gamma_1, \gamma_2) & \gamma_1 \leqq \gamma_2 \quad \alpha_2 + \beta_2 + \gamma_2 = 2 \\ & \beta_1 = \gamma_1 \end{array} \right\}$$

is a solution. It is easy to see that for any allocation with $\beta_1 \neq \gamma_1$ (e.g. $\beta_1 > \gamma_1$) there exists an allocation in $L^*(\mathscr{E})$ with the first commodity distributed in the following way: for some small $\varepsilon > 0$,

$$\frac{\beta_1 + \gamma_1 - \varepsilon}{2} > \gamma_1,$$

$$\alpha_1 + \varepsilon + 2\,\frac{\beta_1 + \gamma_1 - \varepsilon}{2} = 1$$

and

$$\alpha_1 + \varepsilon + \frac{\beta_1 + \gamma_1 - \varepsilon}{2} \leqq 1.$$

With an appropriate choice of α_2', β_2', and of γ_2'

$$\begin{pmatrix} \alpha_1 + \varepsilon, & \alpha_2' \\ \dfrac{\beta_1 + \gamma_1 - \varepsilon}{2}, & \beta_2' \\ \dfrac{\beta_1 + \gamma_1 - \varepsilon}{2}, & \gamma_2' \end{pmatrix} \; Dom^* \; \begin{pmatrix} \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \\ \gamma_1, \gamma_2 \end{pmatrix}$$

via coalition $\{1, 3\}$. Hence the core is strictly smaller than this solution.

Example 2. Let $l = 3$, $n = 3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Preferences are indentical and can be represented by the utility function $u(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$. It is immediate that for a Pareto-optimal allocation a consumer has to receive equal amounts of all commodities. The core is given by

$$C(\mathscr{E}) = \left\{ \begin{array}{c|c} (\alpha, \alpha, \alpha) & \alpha + \beta + \gamma = 1 \\ (\beta, \beta, \beta) & \\ (\gamma, \gamma, \gamma) & \frac{1}{9} \leqq \alpha, \beta, \gamma \leqq \frac{5}{9} \end{array} \right\}.$$

The unique solution in this case is the set of all individually rational and Pareto-optimal allocations

$$L^*(\mathscr{E}) = \left\{ \begin{matrix} (\alpha, \alpha, \alpha) \\ (\beta, \beta, \beta) \\ (\gamma, \gamma, \gamma) \end{matrix} \middle| \begin{matrix} \alpha + \beta + \gamma = 1 \\ \\ \frac{1}{9} \leqq \alpha, \beta, \gamma \end{matrix} \right\} .$$

In fact no Pareto-optimal allocation is dominated by any other, i.e. the set of undominated allocations is

$$L^{**}(\mathscr{E}) = \left\{ \begin{matrix} (\alpha, \alpha, \alpha) \\ (\beta, \beta, \beta) \\ (\gamma, \gamma, \gamma) \end{matrix} \middle| \begin{matrix} \alpha + \beta + \gamma = 1 \\ \\ 0 \leqq \alpha, \beta, \gamma \end{matrix} \right\} .$$

In this particular case, no coalition S except the set of all consumers is effective for any $x \in L^{**}(\mathscr{E})$ which is clearly sufficient for a set of mutually undominating allocations.

It is clear that the lack of effectiveness of subcoalitions for Pareto-optimal allocations may be responsible for large and/or many solutions. The following reasoning is designed to give further indications that equivalence of $C(\mathscr{E})$ and $L^*(\mathscr{E})$ cannot be expected, except for a small class of economies.

Clearly, $C(\mathscr{E}) = L^*(\mathscr{E})$ if all individually rational and Pareto-optimal allocations are in the core, a truly exceptional situation. If this is not the case and if there were equivalence, any Pareto-optimal allocation x not in the core has to be dominated by some allocation y in the core, which implies $\sum_S y_i \leqq \sum_S e_i$. Since y is Pareto-optimal for the economy as a whole, it is clearly Pareto-optimal with respect to the two subeconomies S and $I \backslash S$. Such decomposability of core allocations has to occur for nearly all two-element partitions of I, considering the size of the set of Pareto-optimal allocations. Decomposability, however, is equivalent to the fact that there are no gains from trade among the two subeconomies. Hence one may conjecture that the set of economies for which $C(\mathscr{E}) = L^*(\mathscr{E})$ is not much larger than the set for which the core coincides with the set of individually rational Pareto-optimal allocations.

3. A comparison of the utility counterpart of $L^*(\mathscr{E})$ and of the appropriate

von Neumann-Morgenstern solution of the game associated with an economy indicates another undesirable property. Let v: $2^I \to R^n$ be the characteristic function associated with the economy $\mathscr{E}$, where $v(S)$, the set of utility assignments enforceable by S, is defined by

$$v(S) = \{u \in R^n \mid u_i = 0,\ i \notin S;\ \exists\ x_i \in X_i,\ i \in S \text{ such that } u_i = \tilde{u}_i(x_i) \text{ and } \sum_S x_i \leqq \sum_S e_i\}$$

where $\tilde{u}_i$: $X_i \to R$ is consumer i's utility function. It is well known that the core of such a game $\mathscr{C}(v)$ is defined as

$$\mathscr{C}(v) = \{u \in v(I) \mid \text{ for no } S\ \exists\, z \in v(S) \text{ s.t. } z \gneqq_S u\}$$

where $\geqq_S$ denotes the usual order on the subspace associated with S. It is also well known that, if v is derived from an economy $\mathscr{E}$, then for every $u \in \mathscr{C}(v)$ there exists an allocation $x \in C(\mathscr{E})$ such that $u_i = \tilde{u}_i(x_i)$, and vice versa.

DEFINITION. A utility assignment $u \in R^n$ dominates a utility assignment $z \in R^n$ if for some $S \subset I$, $S \neq \emptyset$, $u_i > z_i$, $i \in S$, and $u_S \in v(S)$ where u_S is obtained from u by substituting zero for every u_i, $i \notin S$.

A utility assignment u is feasible if $u \in v(I)$, and it is called individually rational if for every $i \in I$, $u \geqq_{\{i\}} v(\{i\})$.

DEFINITION. A set $\mathscr{L}(v)$ of feasible, individually rational utility assignments is a von Neumann-Morgenstern solution for v if

(1) $u, z \in \mathscr{L}(v)$ implies $u\ d\phi m\ z$ and $z\ d\phi m\ u$

(2) $z \notin \mathscr{L}(v)$ implies $\exists\, u \in \mathscr{L}(v)$ such that $u\ dom\ z$.

This notion of a solution is the usual one. However, it is not possible to establish the same relationship between $\mathscr{L}(v)$ and $L^*(\mathscr{E})$ as there exists between $\mathscr{C}(v)$ and $C(\mathscr{E})$. Consider Example 2 again, for which

$$C(\mathscr{E}) \subsetneqq L^*(\mathscr{E}).$$

On the other hand $\mathscr{L}(v) = \mathscr{C}(v)$ for this economy. One only needs to show that every assignment not in the core is dominated by some assignment in the core. For example:

$$u = \begin{pmatrix} \tilde{u}\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\ \tilde{u}\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\ \tilde{u}\left(\frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right) \end{pmatrix}$$

is not in the core since it is blocked by $\{1, 2\}$ using $x^1 = (\frac{1}{2}, \frac{1}{2}, 0)$, $x^2 = (\frac{1}{2}, \frac{1}{2}, 0)$. Yet

$$\tilde{u}(\tfrac{1}{2}, \tfrac{1}{2}, 0) = \tilde{u}(\tfrac{2}{9}, \tfrac{2}{9}, \tfrac{2}{9}) = \sqrt{2}$$

and

$$z = \begin{pmatrix} \tilde{u}\left(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right) \\ \tilde{u}\left(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right) \\ \tilde{u}\left(\frac{5}{9}, \frac{5}{9}, \frac{5}{9}\right) \end{pmatrix}$$

is in the core.

Hence

$$z = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{5} \end{pmatrix} \text{ dom } \begin{pmatrix} 1 \\ 1 \\ \sqrt{7} \end{pmatrix} = u \qquad \text{via } \{1, 2\}.$$

Similarly, it can be shown that all other utility assignments not in the core are dominated by elements in the core. Hence the core is stable. In fact, transforming the utility function $\tilde{u}$ into

$$\tilde{\tilde{u}}(x_1, x_2, x_3) = (\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3})^2,$$

the corresponding characteristic function v' describes a side-payment game with

$$\begin{aligned} &v'(\{i\}) = 1 \qquad i = 1, 2, 3 \\ &v'(\{1, 2\}) = v'(\{1, 3\}) = v'(\{2, 3\}) = 4 \\ &v'(\{1, 2, 3\}) = 9. \end{aligned}$$

This game is convex, hence $\mathscr{L}(v') = \mathscr{C}(v')$ due to Shapley's result.

An alternative way of defining a solution which overcomes the difficulties above can be taken directly from the definition of the characteristic function and the dom relation.

DEFINITION. An allocation x is said to dominate an allocation y, x Dom y, if for some non-empty coalition $S \subset I$

$$x_i \underset{i}{\succ} y_i, \qquad \text{all } i \in S$$

and if there exist $z_i \underset{i}{\succsim} x_i$, $i \in S$, such that

$$\sum_{i \in S} z_i \leqq \sum_{i \in S} e_i.$$

It is easy to verify that there exists a one-one relationship between 'dom' and 'Dom', i.e. for any x and y such that x Dom y it follows that $\tilde{u}(x)$ dom $\tilde{u}(y)$, and conversely. Hence it is immediate that any solution $L(\mathscr{E})$ associated with the 'Dom'-relation has a utility counterpart $\mathscr{L}(v)$ and vice versa. Finally, it can be verified easily that for Example 2 above $L(\mathscr{E}) = C(\mathscr{E})$. For Example 1, however, the solution given there is also a solution with respect to the 'Dom'-relation.

4. This final section is devoted to some aspects of the problem of obtaining equivalence of $C(\mathscr{E})$ and $L(\mathscr{E})$. For the class of economies which are representable as convex side-payment games one may use Shapley's result now since the relationship between $\mathscr{L}(v)$ and $L(\mathscr{E})$ has been established.

A straightforward extension of convexity to the non-side-payment case is the following[1].

DEFINITION. A game in characteristic function form is convex if for every non-empty S and T, $S \subset I$, $T \subset I$,

$$v(S) + v(T) \subseteq v(S \cup T) + v(S \cap T).$$

Intuitively, the notion of a convex game is very convincing, implying that "the incentive for any consumer to join a coalition increases as the coalition grows, so that one might expect a 'snow-balling' or a 'band-wagon' effect when the game is played cooperatively" (Shapley [2], p. 11). It remains an open problem at this time whether for convex games $\mathscr{L}(v) = \mathscr{C}(v)$ holds. However, it will be argued for the remainder of the paper that, even if the answer to the above question were positive, it would be of little use since convexity does not seem to be related to any known set of assumptions one may impose on $\mathscr{E}$. Furthermore, the additive structure in the definition embodies an element of cardinality which is easily destroyed by some arbitrary utility transformation. This is indicated in Lemma 1. Finally, the set of economies for which there exists a convex representation is very small in the case where the economy is representable as a side-payment game.

LEMMA 1. If for an exchange economy $\mathscr{E}$ there exists a consumer k and two disjoint coalitions S_1 and S_2 not containing k such that k can benefit from trading with S_1 and with S_2, then the utility function representing k's preferences can be chosen in such a way that the associated characteristic function v is not convex.

Proof. Let $S = S_1 \cup S_2$ and choose any utility function for $i \in S \cup \{k\}$ such that $\tilde{u}_i(e_i) = 0$. Denote by $T_1 = S_1 \cup \{k\}$ and by $T_2 = S_2 \cup \{k\}$ and define

$$m_1 = \text{Max } \{\tilde{u}_k(x_k) | \sum_{i \in T_1} x_i \leqq \sum_{i \in T_1} e_i, \tilde{u}_i(x_i) = 0, \text{ i} \in S_1\}$$

$$m_2 = \text{Max } \{\tilde{u}_k(x_k) | \sum_{i \in T_2} x_i \leqq \sum_{i \in T_2} e_i, \tilde{u}_i(x_i) = 0, \text{ i} \in S_2\}$$

$$m = \text{Max } \{\tilde{u}_k(x_k) | \sum_{i \in T_1 \cup T_2} x_i \leqq \sum_{i \in T_1 \cup T_2} e_i, \tilde{u}_i(x_i) = 0, i \in S\}.$$

Denote by M_1, M_2, and M the associated utility vectors in R^n having zero components everywhere except for $i = k$. Then $M_1 \in v(T_1)$, $M_2 \in v(T_2)$, and $M \in v(T_1 \cup T_2)$. Clearly, by assumption, m_1, m_2, and m are all positive. If $m_1 + m_2 > m$ then $M_1 + M_2 \notin v(T_1 \cup T_2) + v(\{k\})$, since $v(\{k\}) = \{0\}$. Hence v is not convex. If $m_1 + m_2 \leqq m$, there exists an integer $n > 1$ such that

$$m_1^{1/n} + m_2^{1/n} > m^{1/n}.$$

Using $\tilde{\tilde{u}}_k(x_k) = \{\tilde{u}_k(x_k)\}^{1/n}$ as k's utility function will yield the required result. Q.E.D.

It should be observed that the only real assumption is that there are some gains from trade for k with two disjoint coalitions in the economy. This same phenomenon was used on a larger scale as the basis for the argument in Section II which indicated that if there are gains from trade in the economy, then equivalence of $L^*(\mathscr{E})$ and $C(\mathscr{E})$ cannot hold. It is clear that gains from trade imply super-additivity of the associated characteristic function. On the other hand, convexity seems to require a strong form of super-additivity which, if sufficient for stable cores, would be somewhat counterintuitive to the above results.

It is well-known that the game generated by an exchange economy is one with side-payments if preferences are identical for all consumers and representable by a linear-homogeneous, concave utility function. If $\tilde{u}: R^l_+ \to R_+$ is this utility function then $v(S) = \tilde{u}(\sum_{i \in S} e_i)$, all $S \subset I$, gives the characteristic function of the associated side-payment game.

DEFINITION. A game with side-payments is called convex if for all coalitions S and T

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T).$$

LEMMA 2. Let $\tilde{u}: R^l_+ \to R_+$ be the common utility function for an economy in which all consumers have identical preferences. If $\tilde{u}$ is linear-homogeneous and strictly concave for any x and y such that $x \neq \lambda y$, for all $\lambda \geqq 0$, and if there exist two coalitions S and T, $S \cap T \neq \emptyset$ such that

$$(1) \qquad \sum_{i \in S} e_i = \sum_{i \in T} e_i$$

$$(2) \qquad \sum_{S \cup T} e_i \neq \lambda \sum_{S \cap T} e_i \qquad \text{all } \lambda \geqq 0$$

then the associated characteristic function is not convex.

Proof. $\tilde{u}$ linear-homogeneous and strictly concave for non-proportional vectors x and y implies

$$\tilde{u}(x) + \tilde{u}(y) < \tilde{u}(x+y) \qquad x \neq \lambda y, \text{ all } \lambda \geqq 0.$$

(1) implies

$$v(S) = \tilde{u}(\sum_{i \in S} e_i) = v(T).$$

Hence, using linear-homogeneity and (2) one obtains

$$\begin{aligned} v(S)+v(T) &= \tilde{u}(2\sum_{i\in S} e_i) \\ &= \tilde{u}(\sum_{i\in S\cup T} e_i + \sum_{i\in S\cap T} e_i) \\ &> \tilde{u}(\sum_{i\in S\cup T} e_i) + \tilde{u}(\sum_{i\in S\cap T} e_i) \\ &= v(S\cup T) + v(S\cap T). \quad \text{Q.E.D.} \end{aligned}$$

Assumption (1) will clearly be satisfied if there are two identical consumers in the economy. On the other hand one observes that the proof will also carry through if two consumers are sufficiently similar. In fact one may allow some ε-difference in assumption (1) and the conclusions of the Lemma will still follow. In other words if the distribution of endowments is sufficiently rich one cannot hope to obtain convexity. The following two Lemmata exploit the full richness of endowments which is present in atomless economies. They indicate that (1) strict convexity cannot hold for a large class of coalitions and (2) for non-proportional endowments and strictly concave utility functions convexity cannot hold.

Let $(A, \mathscr{A}, \mu)$ be an atomless space of consumers with identical preferences representable by a linear-homogenous, concave utility function $\tilde{u}$: $R^l_+ \to R_+$. Denote by e: $A \to R^l_+$ the integrable function describing the distribution of initial resources of the consumers. The associated characteristic function v: $\mathscr{A} \to R_+$ is given by

$$v(S) = \tilde{u}(\int_S e d\mu) \qquad S \in \mathscr{A}$$

with the usual convention $v(\emptyset) = 0$.

LEMMA 3. For any two coalitions S and T such that $\mu(S\cap T) > 0$, $\mu(S\backslash T) > 0$, and $\mu(T\backslash S) > 0$ there exist two coalitions S' and T' such that $S\cup T = S'\cup T'$, $S\cap T = S'\cap T'$, and

$$v(S') + v(T') \geqq v(S'\cup T') + v(S'\cap T').$$

Proof. Since the measure is non-atomic using Lyapunov's Theorem one can find partitions (S_1, S_2) of $S\backslash T$ and (T_1, T_2) of $T\backslash S$ such that

$$\int_{S_1} e d\mu = \int_{S_2} e d\mu$$

and $$\int_{T_1} e\,d\mu = \int_{T_2} e\,d\mu.$$

Define $S' = (S \cap T) \cup S_1 \cup T_1$ and $T' = (S \cap T) \cup S_2 \cup T_2$. Clearly $\int_{S'} e\,d\mu = \int_{T'} e\,d\mu$ and $S' \cup T' = S \cup T$ and $S' \cap T' = S \cap T$. Hence concavity and linear homogeneity of u imply

$$\begin{aligned} v(S') + v(T') &= 2u(\int_{S'} e\,d\mu) = u(2 \int_{S'} e\,d\mu) \\ &= u(\int_{S' \cup T'} e\,d\mu + \int_{S' \cap T'} e\,d\mu) \\ &\geqq u(\int_{S' \cup T'} e\,d\mu) + u(\int_{S' \cap T'} e\,d\mu) \\ &= v(S' \cup T') + v(S' \cap T') \end{aligned}$$ Q.E.D.

It should be noted that in general the coalitions S' and T' may be quite different from S and T, so that no direct information on $v(S)$ and $v(T)$ can be drawn from the Lemma. However, using the same method of proof one can show the same result for two coalitions arbitrarily similar in size to a coalition which they both contain.

LEMMA 3'. For any coalition $\bar{S} \neq A$ and for any $\varepsilon > 0$ there exist two coalitions S and T such that

(1) $\bar{S} = S \cap T$

(2) $\mu(S) - \mu(\bar{S}) \leqq \varepsilon$
$\mu(T) - \mu(\bar{S}) \leqq \varepsilon$

(3) $v(S) + v(T) \geqq v(S \cup T) + v(S \cap T)$.

LEMMA 4. If u is strictly concave for any x and y with $x \neq \lambda y$, all $\lambda \geqq 0$, and if endowments are not proportional component wise, i.e. if there exists a non-empty coalition $\bar{S} \neq A$ such that

$$\int_{\bar{S}} e\,d\mu \neq \lambda \int_A e\,d\mu, \quad \text{all } \lambda \geqq 0,$$

then v is not convex.

Proof. According to Lyapunov's Theorem there exists a partition (S_1, S_2) of $A \backslash \bar{S}$ such that

$$\int_{S_1} e d\mu = \int_{S_2} e d\mu.$$

Clearly,

$$\int_{S_1} e d\mu \neq \lambda \int_{A} e d\mu, \quad \text{all } \lambda \geqq 0.$$

Define $S = \bar{S} \cup S_1$ and $T = \bar{S} \cup S_2$. Then

$$\begin{aligned} v(S) + v(T) &= u(2 \int_S e d\mu) > (\int_{S \cup T} e d\mu) + u(\int_{S \cap T} e d\mu) \\ &= v(S \cup T) + v(S \cap T). \quad \text{Q.E.D.} \end{aligned}$$

The assumption of non-proportional endowments excludes the case where the economy could be considered as one with a single commodity. In such a case the core would coincide a.e. with the initial distribution of resources. On the other hand most perturbations of the endowment function will create non-proportionality for all multiple commodity economies with proportional endowments. Hence, given the utility function, the set of economies for which Lemma 4 is not applicable is small.

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NOTES

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[1] This notion of a convex-set correspondence has been applied in production economies by D. Sondermann [9].

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