

## CONSUMER THEORY\*

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### 1. Introduction

The main objective of consumer theory is to determine the impact on observable demands for commodities of alternative assumptions on the objectives and on the behavioral rules of the consumer, and on the constraints which he faces when making a decision. The traditional model of the consumer takes preferences over alternative bundles to describe the objectives. Its behavioral rule consists of maximization of these preferences under a budget restriction which determines the trading possibilities. The principal results of the theory consist of the qualitative implications on observed demand of changes in the parameters which determine the decision of the consumer.

The historical development of consumer theory indicates a long tradition of interest of economists in the subject, which has undergone substantial conceptual changes over time to reach its present form. A detailed survey of its history can be found in Katzner (1970).

\*We would like to thank Angus Deaton, Michael Intriligator, Andreu Mas-Colell, and Hugo Sonnenschein for valuable suggestions and criticism. Of course, we bear the sole responsibility for any remaining errors.

†Parts of this paper were written while Volker Böhm was at the Center for Operations Research and Econometrics and at the Université Libre de Bruxelles. Financial support under Ford Foundation Grant no. 775/0022 is gratefully acknowledged.

## 2. Commodities and prices

Commodities can be divided into goods and services. Each commodity is completely specified by its physical characteristics, its location, and date at which it is available. In studies of behavior under uncertainty an additional specification of the characteristics of a commodity relating to the state of nature occurring may be added [see Arrow (1953) and Debreu (1959)], which leads to the description of a contingent commodity. With such an extensive definition of a commodity the theory of the consumer subsumes all more specific theories of consumer choice of location and trade, of intertemporal choice over time, and of certain aspects of the theory of decisions under uncertainty. Formally these are equivalent. The traditional theory usually assumes that there exists a finite number  $\ell$  of commodities implying that a finite specification of physical characteristics, location, etc. suffices for the problems studied. Quantities of each commodity are measured by real numbers. A *commodity bundle*, i.e. a list of real numbers  $(x_h)$ ,  $h = 1, \dots, \ell$ , indicating the quantity of each commodity, can be described therefore as an  $\ell$ -dimensional vector  $x = (x_1, \dots, x_\ell)$  and as a point in  $\ell$ -dimensional Euclidean space  $R^\ell$ , the *commodity space*. Under perfect divisibility of all commodities any real number is possible as a quantity for each commodity so that any point of the commodity space  $R^\ell$  is a possible commodity bundle. The finite specification of the number of commodities excludes treatment of situations in which characteristics may vary continuously. Such situations arise in a natural way, for example, in the context of quality choice of commodities, and in the context of location theory when real distance on a surface is the appropriate criterion. As regards the time specification of commodities, each model with an infinite time horizon, whether in discrete or continuous time, requires a larger commodity space. This is also the case for models of uncertainty with sufficiently dispersed random events. These situations lead in a natural way to the study of infinite dimensional commodity spaces taken as vector spaces over the reals. Applications can be found in Gabszewicz (1968) and Bewley (1970, 1972), with an interpretation regarding time and uncertainty. Mas-Colell (1975) contains an analysis of quality differentiation. As far as the theory of the consumer is concerned many of the usual results of the traditional theory in a finite dimensional commodity space can be extended to the infinite dimensional case. This survey restricts itself to the finite dimensional case.

The *price*  $p_h$  of a commodity  $h$ ,  $h = 1, \dots, \ell$ , is a real number which is the amount paid in exchange for one unit of the commodity. With the specification of location and date the theory usually assumes the general convention that the prices of all commodities are those quoted now on the floor of the exchange for delivery at different locations and at different dates. It is clear that other conventions or constructions are possible and meaningful. A price system or a

price vector  $p = (p_1, \dots, p_l)$  can thus be represented by a point in Euclidean space  $R^l$ . The *value* of a commodity bundle given a price vector  $p$  is  $\sum_{h=1}^l p_h x_h = p \cdot x$ .

### 3. Consumers

Some bundles of the commodity space are excluded as consumption possibilities for a consumer by physical or logical restrictions. The set of all consumption bundles which are possible is called the *consumption set*. This is a non-empty subset of the commodity space, denoted by  $X$ . Traditionally, inputs in consumption are described by positive quantities and outputs by negative quantities. This implies, in particular, that all labor components of a consumption bundle  $x$  are non-positive. One usually assumes that the consumption set  $X$  is closed, convex, and bounded below, the lower bound being justified by finite constraints on the amount of labor a consumer is able to perform. A consumer must choose a bundle from his consumption set in order to exist.

Given the sign convention on inputs and outputs and a price vector  $p$ , the value  $p \cdot x$  of a bundle  $x \in X$  indicates the *net outlay*, i.e. expenses minus receipts, associated with the bundle  $x$ . Since the consumer is considered to trade in a market, his choices are further restricted by the fact that the value of his consumption should not exceed his initial wealth (or income). This may be given to him in the form of a fixed non-negative number  $w$ . He may also own a fixed vector of initial resources  $\omega$ , a vector of the commodity space  $R^l$ . In the latter case his *initial wealth* given a price vector  $p$  is defined as  $w = p \cdot \omega$ . The set of possible consumption bundles whose value does not exceed the initial wealth of the consumer is called the *budget set* and is defined by

$$\beta(p, w) = \{x \in X \mid p \cdot x \leq w\}. \quad (3.1)$$

The ultimate decision of a consumer to choose a bundle from the consumption set depends on his tastes and desires. These are represented by his *preference relation*  $\succeq$  which is a binary relation on  $X$ . For any two bundles  $x$  and  $y$ ,  $x \in X$ ,  $y \in X$ ,  $x \succeq y$  means that  $x$  is at least as good as  $y$ . Given the preferences the consumer chooses a most preferred bundle in his budget set as his *demand*. This is defined as

$$\varphi(p, w) = \{x \in \beta(p, w) \mid x' \in \beta(p, w) \text{ implies } x \succeq x' \text{ or not } x' \succeq x\}. \quad (3.2)$$

Much of consumer theory, in particular the earlier contributions, describe consumer behavior as one of utility maximization rather than of preference maximization. Historically there has been a long debate whether the well-being of

an individual can be measured by some real valued function. The relationship between these two approaches has been studied extensively after the first rigorous axiomatic introduction of the concept of a preference relation by Frisch (1926). It has been shown that the notion of the preference relation is the more basic concept of consumer theory, which is therefore taken as the starting point of any analysis of consumer behavior. The relationship to the concept of a utility function, however, as a convenient device or as an equivalent approach to describe consumer behavior, constitutes a major element of the foundations of consumer theory. The following analysis therefore is divided basically into two parts. The first part, Sections 4–9, deals with the axiomatic foundations of preference theory and utility theory and with the existence and basic continuity results of consumer demand. The second part, Sections 10–15, presents the more classical results of demand theory mainly in the context of differentiable demand functions.

#### 4. Preferences

Among alternative commodity bundles in the consumption set, the consumer is assumed to have preferences represented by a binary relation  $\succeq$  on  $X$ . For two bundles  $x$  and  $y$  in  $X$  the statement  $x \succeq y$  is read as “ $x$  is at least as good as  $y$ ”. Three basic axioms are usually imposed on the preference relation which are often taken as a definition of a rational consumer.

*Axiom 1 (Reflexivity)*

For all  $x \in X$ ,  $x \succeq x$ , i.e. any bundle is as good as itself.

*Axiom 2 (Transitivity)*

For any three bundles  $x, y, z$  in  $X$  such that  $x \succeq y$  and  $y \succeq z$  it is true that  $x \succeq z$ .

*Axiom 3 (Completeness)*

For any two bundles  $x$  and  $y$  in  $X$ ,  $x \succeq y$  or  $y \succeq x$ .

A preference relation  $\succeq$  which satisfies these three axioms is a *complete preordering* on  $X$  and will be called a *preference order*. Two other relations can be derived immediately from a preference order. These are the relation of strict preference  $>$  and the relation of indifference  $\sim$ .

*Definition 4.1*

A bundle  $x$  is said to be strictly preferred to a bundle  $y$ , i.e.  $x > y$  if and only if  $x \succeq y$  and not  $y \succeq x$ .

*Definition 4.2*

A bundle  $x$  is said to be indifferent to a bundle  $y$ , i.e.  $x \sim y$  if and only if  $x \succeq y$  and  $y \succeq x$ .

With  $\succeq$  being reflexive and transitive the strict preference relation is clearly irreflexive and transitive. It will be assumed throughout that there exist at least two bundles  $x'$  and  $x''$  such that  $x' \succ x''$ . The indifference relation  $\sim$  defines an *equivalence relation* on  $X$ , i.e.  $\sim$  is reflexive, symmetric, and transitive.

The validity of these three axioms is not questioned in most of consumer theory. They do represent assumptions, however, which are subject to empirical tests. Observable behavior in many cases will show inconsistencies, in particular with respect to transitivity and completeness. Concerning the latter, it is sometimes argued that it is too much to ask of a consumer to be able to order all possible bundles when his actual decisions will be concerned only with a certain subset of his consumption set. Empirical observations or experimental results frequently indicate intransitivities of choices. These may be due to simple errors which individuals make in real life or in experimental situations. On the other hand, transitivity may also fail for some theoretical reasons. For example, if a consumer unit is a household consisting of several individuals where each individual's preference relation satisfies the three axioms above, the preference relation of the household may be non-transitive if decisions are made by majority rule. Some recent developments in the theory of consumer demand indicate that some weaker axioms suffice to describe and derive consistent demand behavior [see, for example, Chipman et al. (1971), Sonnenschein (1971), Katzner (1971), Hildenbrand (1974), and Shafer (1974)]. Some of these we will indicate after developing the results of the traditional theory.

The possibility of defining a strict preference relation  $\succ$  from the weaker one  $\succeq$ , and vice versa, suggests in principle an alternative approach of starting with the strict relation  $\succ$  as the primitive concept and deriving the weaker one and the indifference relation. This may be a convenient approach in certain situations, and it seems to be slightly more general since the completeness axiom for the strict relation has no role in general. On the other hand, a derived indifference relation will generally not be transitive. Other properties are required to make the two approaches equivalent, which then, in turn, imply that the derived weak relation is complete. However, from an empirical point of view the weak relation  $\succeq$  seems to be the more natural concept. The observed choice of a bundle  $y$  over a bundle  $x$  can only be interpreted in the sense of the weak relation and not as an indication of strict preference.

Axioms 1–3 describe order properties of a preference relation which have intuitive meaning in the context of the theory of choice. This is much less so with

the topological conditions which are usually assumed as well. The most common one is given in Axiom 4 below.

*Axiom 4 (Continuity)*

For every  $x \in X$  the sets  $\{y \in X | y \succeq x\}$  and  $\{y \in X | x \succeq y\}$  are closed relative to  $X$ .

The set  $\{y \in X | y \succeq x\}$  is called the *upper contour set* and  $\{y \in X | x \succeq y\}$  is called the *lower contour set*. Intuitively Axiom 4 requires that the consumer behaves consistently in the "small", i.e. given any sequence of bundles  $y^n$  converging to a bundle  $y$  such that for all  $n$  each  $y^n$  is at least as good as some bundle  $x$ , then  $y$  is also at least as good as  $x$ . For a preference order, i.e. for a relation satisfying Axioms 1–3, the intersection of the upper and lower contour sets for a given point  $x$  defines the *indifference class*  $I(x) = \{y \in X | y \sim x\}$  which is a closed set under Axiom 4. For alternative bundles  $x$  these are the familiar indifference curves for the case of  $X \subset R^2$ . Axioms 1–4 together also imply that the upper and the lower contour sets of the derived strict preference relation  $\succ$  are open, i.e.  $\{y \in X | y \succ x\}$  and  $\{y \in X | x \succ y\}$  are open.

Many known relations do not display the continuity property. The most commonly known of them is the lexicographic order which is in fact a strict preference relation which is transitive and complete. Its indifference classes consist of single elements.

*Definition 4.3*

Let  $x = (x_1, \dots, x_\ell)$  and  $y = (y_1, \dots, y_\ell)$  denote two points of  $R^\ell$ . Then  $x$  is said to be lexicographically preferred to  $y$ ,  $x \text{ Lex } y$ , if there exists  $k$ ,  $1 \leq k \leq \ell$ , such that

$$\begin{aligned} x_j &= y_j, \\ x_k &> y_k, \quad j < k. \end{aligned}$$

It can easily be seen that the upper contour set for any point  $x$  is neither open nor closed.

The relationship between the order properties of Axioms 1–3 and the topological property of Axiom 4 has not been studied extensively. Schmeidler (1971), however, indicated that continuity of a preference relation which is transitive implies completeness. The case when transitivity is implied by continuity and completeness is examined by Sonnenschein (1965).

*Theorem 4.1 [Schmeidler (1971)]*

Let  $\succeq$  denote a transitive binary relation on a connected topological space  $X$ . Define the associated strict preference relation  $\succ$  by  $x \succ y$  if and only if  $x \succeq y$  and not  $y \succeq x$ , and assume that there exists at least one pair  $\bar{x}, \bar{y}$  such that  $\bar{x} \succ \bar{y}$ . If for

every  $x \in X$

- (i)  $\{y \in X | y \succsim x\}$  and  $\{y \in X | x \succsim y\}$  are closed  
and  
(ii)  $\{y \in X | y \succ x\}$  and  $\{y \in X | x \succ y\}$  are open,  
then  $\succsim$  is complete.

*Proof*

The proof makes use of the fact that the only non-empty subset of a connected topological space which is open and closed is the space itself. First it is shown that for any  $x$  and  $y$  such that  $x \succ y$

$$\{z | z \succ y\} \cup \{z | x \succ z\} = X.$$

By definition one has

$$\{z | z \succ y\} \cup \{z | x \succ z\} \subset \{z | z \succsim y\} \cup \{z | x \succsim z\}.$$

The set on the left-hand side of the inclusion is open by assumption (ii), the one on the right-hand side is closed by assumption (i). Therefore equality of the two sets proves the assertion. Suppose  $u \in \{z | z \succsim y\}$  and  $u \notin \{z | z \succ y\}$ . Then  $y \succsim u$ . Since  $x \succ y$ , one obtains  $x \succ u$ . Therefore  $u \in \{z | x \succ z\}$ , which proves equality of the two sets since symmetric arguments hold for all points of the set on the right-hand side.

Now assume that there are two points  $v$  and  $w$  in  $X$  which are not comparable. Since  $\bar{x} \succ \bar{y}$  and  $\{z | z \succ \bar{y}\} \cup \{z | \bar{x} \succ z\} = X$ , one must have  $v \succ \bar{y}$  or  $\bar{x} \succ v$ . Without loss of generality, assume  $v \succ \bar{y}$ . Applying the above result to  $v$  and  $\bar{y}$  yields

$$\{z | z \succ \bar{y}\} \cup \{z | v \succ z\} = X.$$

Since  $v$  and  $w$  are not comparable one has to have  $w \succ \bar{y}$  and  $v \succ \bar{y}$ . By assumption the two sets  $\{z | v \succ z\}$  and  $\{z | w \succ z\}$  are open and so is their intersection.

The intersection is non-empty and not equal to  $X$  by the non-comparability of  $v$  and  $w$ . It will be shown that

$$\{z | v \succ z\} \cap \{z | w \succ z\} = \{z | v \succsim z\} \cap \{z | w \succsim z\},$$

contradicting the connectedness of  $X$ . Suppose  $v \succsim z$  and  $w \succ z$ .  $z \succsim v$  and transitivity implies  $w \succsim v$ , contradicting non-comparability. Similarly,  $z \succ w$  and  $v \succ z$  implies  $v \succ w$ . Therefore  $v \succ z$  and  $w \succ z$ , which proves equality of the two sets and the theorem. Q.E.D.

## 5. Utility functions

The problem of the representability of a preference relation by a numerical function was given a complete and final solution in a series of publications by Eilenberg (1941), Debreu (1954, 1959, 1964), Rader (1963), and Bowen (1968). Historically the concept of a utility function was taken first as a cardinal concept to measure a consumer's well-being. Pareto (1896) seemed to be the first to recognize that arbitrary increasing transformations of a given utility function would result in identical maximization behavior of a consumer. The importance and methodological consequence, however, were only recognized much later by Slutsky (1915) and especially by Wold (1943–44) who gave the first rigorous study of the representation problem.

### *Definition 5.1*

Let  $X$  denote a set and  $\succsim$  a binary relation on  $X$ . Then a function  $u$  from  $X$  into the reals  $R$  is a representation of  $\succsim$ , i.e. a utility function for the preference relation  $\succsim$ , if, for any two points  $x$  and  $y$ ,  $u(x) \geq u(y)$  if and only if  $x \succsim y$ .

It is clear that for any utility function  $u$  and any increasing transformation  $f: R \rightarrow R$  the function  $v = f \circ u$  is also a utility function for the same preference relation  $\succsim$ . Some weaker forms of representability were introduced in the literature [see, for example, Aumann (1962) and Katzner (1970)]. But they have not proved useful in consumer theory.

One basic requirement of a utility function in applications to consumer theory is that the utility function be continuous. It is seen easily that Axioms 1–4 are necessary conditions for the existence of a continuous utility function. That this is true for Axioms 1–3 follows directly from the definition of a representation. To demonstrate necessity of Axiom 4 for the continuity of the function  $u$  one observes that, for any point  $x$ , the upper and lower contour sets of the preference relation coincide with the two sets  $\{z \in X \mid u(z) \geq u(x)\}$  and  $\{z \in X \mid u(x) \geq u(z)\}$  which are closed sets by the continuity of  $u$ . The basic result of utility theory is that Axioms 1–4 combined with some weak assumption on the consumption set  $X$  is also sufficient for the existence of a continuous utility function.

### *Theorem 5.1* (Debreu, Eilenberg, Rader)

Let  $X$  denote a topological space with a countable base of open sets (or a connected, separable topological space) and  $\succsim$  a continuous preference order defined on  $X$ , i.e. a preference relation which satisfies Axioms 1–4. Then there exists a continuous utility function  $u$ .

We will give a proof for the case where  $X$  has a countable base. It divides into two parts. The first is concerned with a construction of a representation function



$v$  given by Rader (1963). The second applies a basic result of Debreu (1964) to show continuity of an appropriate increasing transformation of the function  $v$ .

*Proof*

- (i) *Existence.* Let  $O_1, O_2, \dots$  denote the open sets in the countable base. For any  $x$  consider  $N(x) = \{n | x \succ z, \forall z \in O_n\}$  and define

$$v(x) = \sum_{n \in N(x)} \frac{1}{2^n}.$$

If  $y \succ x$ , then  $N(x) \subset N(y)$ , so that  $v(x) \leq v(y)$ . On the other hand, if  $y \succ x$  there exists  $n \in N(y)$  such that  $x \in O_n$  but not  $n \in N(x)$ . Therefore  $N(x) \subsetneq N(y)$  and  $v(y) > v(x)$ . Hence  $v$  is a utility function.

- (ii) *Continuity.* Let  $S$  denote an arbitrary set of the extended real line which later will be taken to be  $v(X)$ .  $S$  as well as its complement may consist of non-degenerate and degenerate intervals. A *gap* of  $S$  is a maximal non-degenerate interval of the complement of  $S$  which has an upper and a lower bound in  $S$ . The following theorem is due to Debreu (1964).

### Theorem 5.2

If  $S$  is a subset of the extended real line  $\bar{R}$  there exists an increasing function  $g$  from  $S$  to  $\bar{R}$  such that all the gaps of  $g(S)$  are open.

Applying the theorem one defines a new utility function  $u = g \circ v$ . According to the theorem all the gaps of  $u(X)$  are open. For the continuity of  $u$  it suffices to show that for any  $t \in \bar{R}$  the sets  $u^{-1}([t, +\infty])$  and  $u^{-1}([-\infty, t])$  are closed.

If  $t \in u(X)$ , there exists  $y \in X$  such that  $u(y) = t$ . Then  $u^{-1}([t, +\infty]) = \{x \in X | x \succeq y\}$  and  $u^{-1}([-\infty, t]) = \{x \in X | y \succeq x\}$ . Both of these sets are closed by assumption.

If  $t \notin u(X)$  and  $t$  is not contained in a gap, then

- (a)  $t \leq \text{Inf } u(X)$ , or  
 (b)  $t \geq \text{Sup } u(X)$ , or  
 (c)  $[t, +\infty] = \bigcap_{\substack{\alpha < t \\ \alpha \in u(X)}} [\alpha, +\infty]$

and

$$[-\infty, t] = \bigcap_{\substack{t < \alpha \\ \alpha \in u(X)}} [-\infty, \alpha].$$

(a) implies  $u^{-1}([t, +\infty)) = X$  and  $u^{-1}([-\infty, t]) = \emptyset$ ,

(b) implies  $u^{-1}([t, +\infty)) = \emptyset$  and  $u^{-1}([-\infty, t]) = X$ .

Both  $X$  and the empty set  $\emptyset$ , are closed.

In the case of (c),

$$u^{-1}([t, +\infty)) = \bigcap_{\substack{\alpha < t \\ \alpha \in u(X)}} u^{-1}([\alpha, +\infty))$$

and

$$u^{-1}([-\infty, t]) = \bigcap_{\substack{\alpha > t \\ \alpha \in u(X)}} u^{-1}([-\infty, \alpha])$$

are closed as intersections of closed sets.

If  $t$  belongs to an open gap, i.e.  $t \in ]a, b[$ , where  $a$  and  $b$  belong to  $u(X)$ , then

$$u^{-1}([t, +\infty)) = u^{-1}([b, +\infty)) \quad \text{which is closed}$$

and

$$u^{-1}([-\infty, t]) = u^{-1}([-\infty, a]) \quad \text{which is closed. Q.E.D.}$$

Theorem 5.1 indicates that the concepts of utility function and of the underlying preferences may be used interchangeably to determine demand, provided the preferences satisfy Axioms 1–4. In many situations, therefore, it becomes a matter of taste or of mathematical convenience to choose one or the other.

## 6. Properties of preferences and of utility functions

In applications additional assumptions on the preferences and/or on the utility function are frequently made. For some of them there exist almost definitional equivalence of the specific property of preferences and of the corresponding property of the utility function. Others require some demonstration. We will discuss the ones most commonly used in an order which represents roughly an increasing degree of mathematical involvement to demonstrate equivalence.

### 6.1. Monotonicity, non-satiation, and convexity

#### 6.1.1. Monotonicity

##### Definition 6.1

A preference order on  $R^l$  is called monotonic if  $x \succeq y$  and  $x \neq y$  implies  $x \succ y$ .

This property states that more of any one good is preferred, which means that all goods are desired (desirability). The associated utility function of a monotonic preference order is an increasing function on  $R^l$ .

### 6.1.2. Non-satiation

#### Definition 6.2

A point  $x \in X$  is called a satiation point for the preference order  $\succsim$  if  $x \succsim y$  holds true for all  $y \in X$ .

A satiation point is a maximal point with respect to the preference order. Most parts of consumer theory discuss situations in which such global maxima do not exist or, at least for the discussion of demand problems, where an improvement of the consumer can be achieved by a change of his chosen consumption bundle. In other words, the situations under discussion will be points of non-satiation. If at some point  $x$  an improvement can be found in any neighborhood of  $x$ , one says that the consumer is locally not satiated at  $x$ . More precisely:

#### Definition 6.3

A consumer is locally not satiated at  $x \in X$  if for every neighborhood  $V$  of  $x$  there exists a  $z \in V$  such that  $z \succ x$ .

This property excludes the possibility of indifference classes with non-empty interiors and it implies that the utility function is non-constant in the neighborhood of  $x$ .

### 6.1.3. Convexity

#### Definition 6.4

A preference order on  $X \subset R^l$  is called convex if the set  $\{y \in X \mid y \succsim x\}$  is convex for all  $x \in X$ .

This definition states that all upper contour sets are convex. The associated concept for a utility function is that of quasi-concavity.

#### Definition 6.5

A function  $u: X \rightarrow R$  is called quasi-concave if  $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$  for all  $x, y \in X$  and any  $\lambda, 0 \leq \lambda \leq 1$ .

It is easy to see that the upper level sets of a quasi-concave function are convex. Therefore, a utility function  $u$  for a preference order  $\succsim$  is quasi-concave if and only if the preference order is convex. Thus, quasi-concavity is a property which relates directly to the ordering. Therefore it is preserved under increasing transformations. Such properties of a utility function are sometimes called ordinal

properties as opposed to cardinal properties which are related to some specific representation function  $u$ . Concavity, for example, is one such cardinal property. Since the class of concave functions is a subclass of the class of quasi-concave functions, one may ask which additional properties of a convex preference order are required such that there exists a concave utility function. A final answer to this question was given by Kannai (1977). A treatment of this problem would go beyond the scope of this survey.

*Definition 6.6*

A preference order is called strictly convex if for any two bundles  $x$  and  $x'$ ,  $x \neq x'$ ,  $x \succ x'$ , and for  $\lambda$ ,  $0 < \lambda < 1$ ,  $\lambda x + (1 - \lambda)x' \succ x'$ .

Any associated utility function of a strictly convex preference relation is a strictly quasi-concave function. Strict convexity excludes all preference relations whose indifference classes have non-empty interiors.

## 6.2. Separability

Separable utility functions were used in classical consumer theory long before an associated property on preferences had been defined. The problem has been studied by Sono (1945), Leontief (1947), Samuelson (1947a), Houthakker (1960), Debreu (1960), Koopmans (1972), and others. Katzner (1970) presented a general characterization of essentially two notions of separability which we will also use here.

Let  $N = \{N_j\}_{j=1}^k$  denote a partition of the set  $\{1, \dots, l\}$  and assume that the consumption set  $X = S_1 \times \dots \times S_k$ . Such partitions arise in a natural way if consumption is considered over several dates, locations, etc. Loosely speaking, separability then implies that preferences for bundles in each element of the partition (i.e. at each date, location, etc.) are independent of the consumption levels outside. Let  $J = \{1, \dots, k\}$  and, for any  $j \in J$  and any  $x \in X$ , write  $x_{\hat{j}} = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k)$  for the vector of components different from  $x_j$ . For any fixed  $x_{\hat{j}}$  the preference ordering  $\succsim$  induces a preference ordering on  $S_j$  which is defined by  $x_j \succsim_{x_{\hat{j}}} x'_j$  if and only if  $(x_{\hat{j}}^0, x_j) \succsim (x_{\hat{j}}^0, x'_j)$  for any  $x_j$  and  $x'_j$  in  $S_j$ . In general, such an induced ordering will depend on the particular choice of  $x_{\hat{j}}^0$ . The first notion of separability states that these orderings for a particular element  $j$  of the partition are identical for all  $x_{\hat{j}}$ .

*Definition 6.7 (weak separability of preferences)*

A preference order  $\succsim$  on  $X = \prod_{j \in J} S_j$  is called weakly separable if for each  $j \in J$   $x_j \succsim_{x_{\hat{j}}} x'_j$  implies  $x_j \succsim_{x_{\hat{j}}} x'_j$  for all  $x_{\hat{j}} \in \prod_{i \neq j} S_i$ .

The induced ordering can then be denoted by  $\succsim_j$ . The corresponding notion of a weakly separable utility function is given in the next definition.

**Definition 6.8** (weak separability of utility functions)

A utility function  $u: \prod_{j \in J} S_j \rightarrow R$  is called weakly separable if there exist continuous functions  $v_j: S_j \rightarrow R$ ,  $j \in J$  and  $V: R^k \rightarrow R$  such that  $u(x) = V(v_1(x_1), \dots, v_k(x_k))$ .

The following theorem establishes the equivalence of the two notions of weak separability. Its proof is straightforward and can be found in Katzner (1970).

**Theorem 6.1**

Let  $\succeq$  be a continuous preference order. Then  $\succeq$  is weakly separable if and only if every continuous representation of it is weakly separable.

Historically a stronger notion of separability was used first. Early writers in utility theory thought of each commodity as having its own intrinsic utility represented by some scalar function. The overall level of utility then was simply taken as the sum of these functions. With the more general development of preference and utility theory the additive utility function has remained an interesting case used frequently in certain classes of economic problems. Its relationship to the properties of the underlying preferences is now well understood, the essential result being due to Debreu (1960).

**Definition 6.9** (strong separability of utility functions)

A utility function  $u: \prod_{j \in J} S_j \rightarrow R$  is called strongly separable if there exist continuous functions  $v_j: S_j \rightarrow R$ ,  $j \in J$  and  $V: R \rightarrow R$  such that

$$u(x) = V\left(\sum_{j \in J} v_j(x_j)\right).$$

Since  $V$  is continuous and increasing one observes immediately that the function  $V^{-1} \circ u$  is additive, representing the same preference relation. The problem of finding conditions on preference relations which yield strong separability for all of its representations is therefore equivalent to the one of establishing conditions under which an additive representation exists.

Let  $u(x) = \sum_{j \in J} u_j(x_j)$  denote an additive utility function with respect to the partition  $N$ . Consider any non-empty proper subset  $I \subset J$  and two bundles  $x$  and  $x'$  where all components  $j$  belonging to  $J \setminus I$  are kept at the same level  $x_j^0$ ,  $j \in J \setminus I$ . We can write therefore  $x = (x_I, x_{J \setminus I}^0)$  and  $x' = (x'_I, x_{J \setminus I}^0)$ . Since  $u$  is additive it is immediately apparent that the induced utility function on  $\prod_{j \in I} S_j$  is independent of the particular choice of  $x_{J \setminus I}^0$ , which also makes the underlying induced preference order independent. This property holds true for any non-empty proper subset  $I \subset J$ . It also supplies the motivation for the definition of a strongly separable preference relation.

*Definition 6.10* (strong separability of preferences)

A preference order  $\succsim$  on  $X = \prod_{j \in J} S_j$  is called strongly separable if it is weakly separable with respect to all proper partitions of all possible unions of  $N_1, \dots, N_k$ .

Equivalently, a preference order is strongly separable if for any proper subset  $I$  of  $J$ , the induced preference order on  $\prod_{j \in I} S_j$  is independent of the particular choice of  $x_{J \setminus I}^0$ .

*Theorem 6.2* [Debreu (1964)]

Let  $\succsim$  be a continuous preference order. Then  $\succsim$  is strongly separable if and only if every continuous representation is strongly separable.

### 6.3. Smooth preferences and differentiable utility functions

The previous sections on preferences and utility functions indicated the close relationship between the continuity of the utility function and the chosen concept of continuity for a preference order. It was shown that the two concepts are identical. Section 7 uses this fact to demonstrate that, under some conditions, demand behaves continuously when prices and wealth change. When continuous differentiability of the demand function is required, continuity of the preference order will no longer be sufficient. It is clear that some degree of differentiability of the utility function is necessary. Since the utility function is the derived concept the differentiability problem has to be studied with respect to the underlying preferences. The first rigorous attempt to study differentiable preference orders goes back to Antonelli (1886) and an extensive literature has developed studying the so-called integrability problem, which was surveyed completely by Hurwicz (1971). Debreu (1972) has chosen a more direct approach to characterize differentiable preference orders. His results indicate that sufficiently "smooth", i.e. differentiable preference orders, are essentially equivalent to sufficiently differentiable utility functions and that a solution to the integrability problem can be found along the same lines. [See also Debreu (1976) and Mas-Colell (1975).]

To present the approach of Debreu we will assume, for the purpose of this section as well as for all later sections which deal with differentiability problems, that the consumption set  $X$  is the interior of the positive cone of  $R^l$  which will be denoted  $P$ . The preference order  $\succsim$  is considered as a subset of  $P \times P$ , i.e.  $\succsim \subset P \times P$ , and it is assumed to be continuous and monotonic. To describe a smooth preference order, differentiability assumptions will be made on the graph of the indifference relation in  $P \times P$ .

Let  $C^k$ ,  $k \geq 1$ , denote the class of functions which have continuous partial derivatives up to order  $k$ , and consider two open sets  $X$  and  $Y$  in  $R^n$ . A function  $h$

from  $X$  onto  $Y$  is a  $C^k$ -diffeomorphism if  $h$  is one-to-one and both  $h$  and  $h^{-1}$  are of class  $C^k$ . A subset  $M$  of  $R^n$  is a  $C^k$ -hypersurface if for every  $z \in M$ , there exists an open neighborhood  $U$  of  $z$ , a  $C^k$ -diffeomorphism of  $U$  onto an open set  $V \subset R^n$ , and a hyperplane  $H \subset R^n$  such that  $h(M \cap U) \subset V \cap H$ . Up to the diffeomorphism  $h$ , the hypersurface  $M$  has locally the structure of a hyperplane. The hypersurface of interest here with respect to the preference order is the indifference surface  $\tilde{I}$  defined as  $\tilde{I} = \{(x, y) \in P \times P \mid x \sim y\}$ . *Smooth preference orders* are those which have an indifference hypersurface of class  $C^2$ , and we will say that  $\succeq$  is a  $C^2$ -preference order. The result of Debreu states that  $C^2$ -utility functions are generated by preference orders of class  $C^2$  and vice versa.

*Theorem 6.3 [Debreu (1972)]*

Let  $\succeq$  denote a continuous and monotonic preference order on the interior of the positive cone of  $R^l$ . There exists a monotonic utility function of class  $C^2$  with no critical point for  $\succeq$  if and only if  $\tilde{I}$  is a  $C^2$ -hypersurface.

## 7. Continuous demand

Given a price vector  $p \neq 0$  and the initial wealth  $w$  the consumer chooses the best bundle in his budget set as his demand. For preference orders satisfying Axioms 1–3 any best element with respect to the preference relation is also a maximizer for any utility function representing it, and vice versa. Thus, preference maximization and utility maximization lead to the same set of demand bundles. We now study the dependence of demand on its two exogenous parameters, price and wealth.

The budget set of a consumer was defined as  $\beta(p, w) = \{x \in X \mid p \cdot x \leq w\}$ . Let  $S \subset R^{l+1}$  denote the set of price–wealth pairs for which the budget set is non-empty. Then  $\beta$  describes a correspondence (i.e. a set valued function) from  $S$  into  $R^l$ . The two notions of continuity of correspondences used in the sequel are the usual ones of upper hemi-continuity and lower hemi-continuity [see, for example, Hildenbrand (1974), Hildenbrand and Kirman (1976), or Green and Heller, Chapter 1 in Volume I of this Handbook].

### *Definition 7.1*

A correspondence  $\psi$  from  $S$  into  $T$ , a compact subset of  $R^l$ , is upper hemi-continuous (u.h.c.) at a point  $y \in S$  if, for all sequences  $z^n \rightarrow z$  and  $y^n \rightarrow y$  such that  $z^n \in \psi(y^n)$ , it follows that  $z \in \psi(y)$ .

With  $T$  being compact this definition says that  $\psi$  is upper hemi-continuous if it has a closed graph. Furthermore, one immediately observes that any upper

hemi-continuous correspondence which is single valued is in fact a continuous function.

*Definition 7.2*

A correspondence  $\psi$  from  $S$  into an arbitrary subset  $T$  of  $R^l$  is lower hemi-continuous (l.h.c.) at a point  $y \in S$  if for any  $z^0 \in \psi(y)$  and for any sequence  $y^n \rightarrow y$  there exists a sequence  $z^n \rightarrow z^0$  such that  $z^n \in \psi(y^n)$  for all  $n$ .

A correspondence is continuous if it is both lower and upper hemi-continuous. With these notions of continuity the following two lemmas are easily established. [For proofs see, for example, Debreu (1959) or Hildenbrand (1974).]

*Lemma 7.1*

The budget set correspondence  $\beta: S \rightarrow X$  has a closed graph and is lower hemi-continuous at every point  $(p, w)$  for which  $w > \min\{p \cdot x \mid x \in X\}$  holds.

The condition  $w > \min\{p \cdot x \mid x \in X\}$  is usually referred to as the *minimal wealth condition*.

It was argued above that preference maximization and utility maximization lead to the same set of demand bundles if the preference relation is reflexive, transitive, and complete. Therefore, if  $u: X \rightarrow R$  is a utility function the *demand* of a consumer can be defined as

$$\varphi(p, w) = \{x \in \beta(p, w) \mid u(x) \geq u(x'), x' \in \beta(p, w)\}, \quad (7.1)$$

which is equivalent to definition (3.2). With a continuous utility function the demand set  $\varphi(p, w)$  will be non-empty if the budget set is compact. In this case an application of a fundamental theorem by Berge (1966) yields the following lemma on the continuity of the demand correspondence.

*Lemma 7.2*

For any continuous utility function  $u: X \rightarrow R$  the demand correspondence  $\varphi: S \rightarrow X$  is non-empty valued and upper hemi-continuous at each  $(p, w) \in S$ , where  $\beta(p, w)$  is compact and  $w < \min\{p \cdot x \mid x \in X\}$ .

From the definition of the budget correspondence and the demand correspondence it follows immediately that  $\varphi(\lambda p, \lambda w) = \varphi(p, w)$  for any  $\lambda > 0$  and for any price-wealth pair  $(p, w)$ . This property states in particular that demand is homogeneous of degree zero in prices and wealth. For convex preference orders the demand correspondence will be convex-valued, a property which plays a crucial role in existence proofs of competitive equilibrium. If the preference order is strictly convex then the demand correspondence is single-valued, i.e. one



obtains a demand function. Upper hemi-continuity then implies the usual continuity. To summarize this section, we indicate in the following lemma the weakest assumptions of traditional demand theory which will generate a continuous demand function.

*Lemma 7.3*

Let  $\succsim$  denote a strictly convex and continuous preference order. Then the demand correspondence  $\varphi: S \rightarrow X$  is a continuous function at every  $(p, w) \in S$  for which  $\beta(p, w)$  is compact and  $w > \min\{p \cdot x \mid x \in X\}$  holds. Moreover, for all  $\lambda > 0$ ,  $\varphi(\lambda p, \lambda w) = \varphi(p, w)$ , i.e.  $\varphi$  is homogeneous of degree zero in prices and wealth.

For the remainder of this survey, following the traditional notational convention, the letter  $f$  will be used to denote a demand function.

## 8. Demand without transitivity

Empirical studies of demand behavior have frequently indicated that consumers do not behave in a transitive manner. This fact has been taken sometimes as evidence against the general assumption that preference maximization subject to a budget constraint is the appropriate framework within which demand theory should be analyzed. Sonnenschein (1971) indicates, however, that the axiom of transitivity is unnecessary to prove existence and continuity of demand. A related situation without transitivity is studied by Katzner (1971) where preferences are defined locally and thus "local" results for demand functions are obtained.

Recall that without transitivity a utility function representing the preference relation cannot be defined. Let  $\succsim$  denote a preference relation which is complete but not necessarily transitive. Then the definition of demand as in Definition 3.2 can be given in the following form:

*Definition 8.1*

The demand correspondence  $\varphi: S \rightarrow X$  is defined as  $\varphi(p, w) = \{x \in \beta(p, w) \mid x \succsim x' \text{ for all } x' \in \beta(p, w)\}$ .

*Theorem 8.1 (Sonnenschein)*

Let  $\varphi(p, w) \neq \emptyset$  for all  $(p, w) \in S$  and assume that  $\beta$  is continuous at  $(p^0, w^0) \in S$ . If the preference relation is continuous, then the demand correspondence  $\varphi$  is u.h.c. at  $(p^0, w^0)$ .

The assumption that  $\varphi(p, w) \neq \emptyset$  for all  $(p, w) \in S$  is implied by some modified convexity assumption on the strict preference relation, as indicated by the next theorem.

**Theorem 8.2 (Sonnenschein)**

Let  $\succsim$  denote a continuous preference relation on  $X$  such that the set  $\{x' \in X \mid x' \succ x\}$  is convex for all  $x \in X$ . Then  $\varphi(p, w) \neq \emptyset$  whenever  $\beta(p, w) \neq \emptyset$ .

Therefore, the two results by Sonnenschein indicate that continuous demand functions will be obtained if the transitivity is replaced by the convexity of preferences.

A further result in the theory of the non-transitive consumer was given by Shafer (1974). This approach formulates the behavior of a consumer as one of maximizing a continuous numerical function subject to a budget constraint. This function, whose existence and continuity does not depend on transitivity, can be considered as an alternative approach to represent a preference relation.

Let  $X = R_+^l$  and let  $\succsim$  denote a preference relation on  $X$ . Since any binary relation uniquely defines a subset of  $R_+^l \times R_+^l$ , and vice versa, one writes  $\succsim \subset R_+^l \times R_+^l$ , such that  $(x, y) \in \succsim$  if and only if  $x \succsim y$ . Define:

$$\begin{aligned}\succsim(x) &= \{y \mid (y, x) \in \succsim\}, \\ \succsim^{-1}(x) &= \{y \mid (x, y) \in \succsim\}, \\ \succ(x) &= \{y \mid (y, x) \in \succsim \text{ and } (x, y) \notin \succsim\}.\end{aligned}$$

With this notation the usual properties of completeness, continuity, and strict convexity which will be needed below are easily redefined.

**Definition 8.2**

The relation  $\succsim$  is complete if and only if  $(x, y) \in \succsim$  or  $(y, x) \in \succsim$  for any  $x$  and  $y$ .

**Definition 8.3**

The relation  $\succsim$  is continuous if for all  $x$  the sets  $\succsim(x)$  and  $\succsim^{-1}(x)$  are closed.

**Definition 8.4**

The relation  $\succsim$  is strictly convex if for  $(x, z) \in \succsim(y)$  and  $0 < \alpha < 1$  it follows that  $\alpha x + (1 - \alpha)z \in \succ(y)$ .

We can now state Shafer's representation result.

**Theorem 8.3 (Shafer (1974))**

Let  $\succsim \subset R_+^l \times R_+^l$  denote a continuous, complete, and strictly convex preference relation. Then there exists a continuous function  $k: R_+^l \times R_+^l \rightarrow R$  satisfying

- (i)  $k(x, y) > 0$  if and only if  $x \in \succ(y)$ ,
- (ii)  $k(x, y) < 0$  if and only if  $y \in \succ(x)$ ,
- (iii)  $k(x, y) = 0$  if and only if  $x \in \succsim(y)$  and  $y \in \succsim(x)$ ,
- (iv)  $k(x, y) = -k(y, x)$ .

The assumptions of the theorem are the traditional ones except that the transitivity axiom is excluded. If the latter were assumed then a utility function  $u$  exists and the function  $k$  can be defined as  $k(x, y) = u(x) - u(y)$ .

As before, let  $\beta(p, w)$  denote the budget set of the consumer. Then the demand of the consumer consists of all points in the budget set which maximize  $k$ . More precisely, the demand is defined as

$$\varphi(p, w) = \{x \in \beta(p, w) \mid k(x, y) \geq 0, \text{ all } y \in \beta(p, w)\}$$

or, equivalently,

$$\varphi(p, w) = \{x \in \beta(p, w) \mid (x, y) \in \succeq, \text{ all } y \in \beta(p, w)\}.$$

The strict convexity assumption guarantees that there exists a unique maximal element. The following theorem makes the maximization argument precise and states the result of the existence of a continuous demand function by Sonnenschein in an alternative form, which may be used to derive demand functions for a non-transitive consumer explicitly.

*Theorem 8.4* (Shafer)

Under the assumptions of Theorem 8.2 and for each strictly positive price vector  $p$  and positive wealth  $w$ , the demand  $x = f(p, w) = \{x \in \beta(p, w) \mid k(x, y) \geq 0, \text{ all } y \in \beta(p, w)\}$ , exists and the function  $f$  is continuous at  $(p, w)$ .

## 9. Demand under separability

Separability of the preference order and the utility function, whether weak or strong, has important consequences for the demand functions. Using the notation and definitions of section 6.2, under separability the utility function can be written as

$$u(x) = V(v_1(x_1), \dots, v_k(x_k)), \quad (9.1)$$

where the  $x_j$ ,  $j = 1, \dots, k$ , are vectors of quantities of commodities in  $S_j$  and  $X = S_1 \times \dots \times S_k$ . The  $v_j(x_j)$  are utility functions defined on  $S_j$ . We will use the vector  $p_j$  for the prices of the commodities in  $N_j$ .

*Definition 9.1*

For any  $w_j \in R_+^l$  define a sub-budget set

$$\beta^j(p_j, w_j) \equiv \{x_j \in S_j \mid p_j' x_j \leq w_j\}. \quad (9.2)$$

We are now in a position to introduce the concept of conditional demand functions  $f_j^j(p_j, w_j)$  describing the  $x_j$  which maximize  $v_j(x_j)$  over the sub-budget set.

*Definition 9.2*

Conditional demand functions are given as

$$f_j^j(p_j, w_j) \equiv \{x_j \in \beta^j(p_j, w_j) \mid v_j(x_j) > v_j(x_j^0), x_j^0 \in \beta^j(p_j, w_j)\}. \quad (9.3)$$

These conditional demand functions share all the properties of the usual demand functions, except that their domain and range is limited to  $p_j$ ,  $w_j$ , and  $S_j$ . Given  $v_j(x_j)$ ,  $p_j$ , and  $w_j$ , then demand  $x_j$  is known. However,  $w_j$  is not given exogenously, but as part of the overall optimization problem. Let  $f_j(p, w)$  be the  $j$ -subvector of the demand function  $f(p, w)$ . Then,  $w_j$  is given by

$$w_j^*(p, w) = p_j' f_j(p, w). \quad (9.4)$$

Note that in general the full price vector  $p$  is needed to determine  $w_j^*$ .<sup>1</sup> When using the  $w_j^*$  generated by  $w_j(p, w)$  in the conditional demand functions one would expect to obtain the same demand vector as the one given by  $f_j(p, w)$ . Indeed, one has

*Theorem 9.1*

Under separability of the utility function

$$f_j^j(p_j, w_j^*(p, w)) = f_j(p, w), \quad \text{for all } j. \quad (9.5)$$

*Proof*

Consider a particular  $(p^0, w^0)$ . Let  $x_j^* = f_j^j(p_j^0, w_j^*(p^0, w^0))$  for some  $j$  and  $x^0 = f(p^0, w^0)$ . Clearly,  $x_j^0 \in \beta^j(p_j^0, w_j^*(p^0, w^0))$ . Assume  $x_j^* \neq x_j^0$ . Then  $v_j(x_j^*) > v_j(x_j^0)$  and

$$\begin{aligned} &V(v_1(x_1^0), \dots, v_j(x_j^*), \dots, v_k(x_k^0)) \\ &> V(v_1(x_1^0), \dots, v_j(x_j^0), \dots, v_k(x_k^0)) = u(x^0) \end{aligned} \quad (9.6)$$

because  $V$  is monotone increasing in  $v_j(x_j)$ . Since  $(x_j^0, x_j^*)$  is an element of the

<sup>1</sup>Gorman (1959) has investigated under what condition one can replace the full  $l$ -vector  $p$  in (9.4) by a  $k$ -vector of price indexes of the type  $P_j(p_j)$ . These conditions are restrictive.

budget set  $\beta(p, w)$ , inequality cannot hold. Equality, however, means  $v_j(x_j^*) = v_j(x_j^0)$  and  $x_j^* = x_j^0$  since  $x_j$  is the unique vector maximizing  $v_j(x_j)$  over  $x_j \in \beta_j^j(p_j^0, w_j^*(p^0, w^0))$ . Therefore (9.5) holds for  $(p^0, w^0)$ . Since  $(p^0, w^0)$  is arbitrary, it holds for all admissible  $(p, w)$  and the theorem is proved.

The interest of Theorem 9.1 is twofold. First, it shows that the other prices affect the demand for  $x_j$  only by way of the scalar function  $w_j^*(p, w)$ , implying a considerable restriction on the scope of the impact of  $p_j$ . Second, if one can observe the  $w_j$  empirically one can concentrate on the conditional demand functions for which only the  $p_j$  are needed. An example of the latter is to consider demand behavior for a certain period, say a year. Under the (usually implicit) assumption of separability over different time periods one only needs to know total expenditure for that period ( $w_j$ ) and the corresponding price vector ( $p_j$ ). In this context (9.4) may then be considered as the consumption function, relating total consumer expenditure to total wealth and prices for all periods.

## 10. Expenditure functions and indirect utility functions

An alternative approach in demand analysis, using the notion of an expenditure function, was suggested by Samuelson (1947a). It was only fully developed after 1957 [see, for example, Karlin et al. (1959) and McKenzie (1957)]. It later became known as the duality approach in demand analysis [see also Diewert (1974) and Diewert, Chapter 12 in Volume II of this Handbook]. In certain cases it provides a more direct analysis of the price sensitivity of demand and enables a shorter and more transparent exposition of some classical properties of demand functions. Without going too much into the details of this approach we will describe its basic features and results for a slightly more restrictive situation than the general case above. These restrictions will be used in all later sections.

From now on it will be assumed that the consumption set  $X$  is equal to the positive orthant  $R^l_+$  and that all prices and wealth are positive. This implies that the budget set is compact and that the minimum wealth condition is satisfied. Therefore for any continuous utility function the demand correspondence  $\varphi$  is upper hemi-continuous. Furthermore, the assumption of local non-satiation will be made on preferences or on the utility function, respectively. This implies that the consumer spends all his wealth when maximizing preferences.

Given an attainable utility level  $v = u(x)$ ,  $x \in X$ , the *expenditure function* is the minimum amount necessary to be spent to obtain a utility level at least as high as  $v$  at given prices  $p$ . Hence, the expenditure function  $E: R^l_{++} \times R \rightarrow R$  is defined as

$$E(p, v) = \min \{ p \cdot x \mid u(x) \geq v \}. \quad (10.1)$$

The following properties of the expenditure function are easily established.

*Lemma 10.1*

If the continuous utility function satisfies local non-satiation then the expenditure function is:

- (i) strictly increasing and continuous in  $v$  for any price vector  $p$ , and
- (ii) non-decreasing, positive linear homogeneous, and concave in prices at each utility level  $v$ .

Let  $y = E(p, v)$  denote the minimum level of expenditures. Since  $E$  is continuous and strictly increasing in  $v$ , its inverse  $v = g(p, y)$  with respect to  $v$  expresses the utility level as a function of expenditures and prices which is called the *indirect utility function*. It is easy to see that

$$g(p, y) = \max \{u(x) \mid p \cdot x = y\}. \quad (10.2)$$

Due to the properties of the expenditure function the indirect utility function is

- (i) strictly increasing in  $y$  for each price vector  $p$ , and
- (ii) non-increasing in prices and homogeneous of degree zero in income and prices.

From the definitions of  $E$  and  $g$  the following relations hold identically:

$$v \equiv g(p, E(p, v)) \quad \text{and} \quad y \equiv E(p, g(p, y)). \quad (10.3)$$

Given a price vector  $p$  and a utility level  $v$  the expenditure minimum  $E(p, v)$  will be attained on some subset of the plane defined by  $E(p, v)$  and  $p$ . If preferences are strictly convex there will be a unique point  $x \in X$  minimizing expenditures and we denote the function of the minimizers by  $x = h(p, v)$ . By definition one has therefore

$$E(p, v) = p \cdot h(p, v). \quad (10.4)$$

The function  $h$  is the so-called *Hicksian "income-compensated" demand function*.  $h$  is continuous in both of its arguments and homogeneous of degree zero in prices.

Considering the original problem of the maximization of utility subject to the budget constraint  $p \cdot x \leq w$ , our assumptions of local non-satiation and strict convexity imply that one obtains a continuous function of maximizers  $f(p, w)$ . This function is the so-called *Marshallian "market" demand function* which satisfies the property

$$p \cdot f(p, w) = w. \quad (10.5)$$

From these definitions one obtains a second pair of identities which describe the fundamental relationship between the Hicksian and the Marshallian demand functions:

$$\begin{aligned} f(p, w) &= h(p, g(p, w)), \quad \text{for all } (p, w), \\ h(p, v) &= f(p, E(p, v)), \quad \text{for all } (p, v). \end{aligned} \tag{10.6}$$

One important property of the Hicksian demand function can be obtained immediately. For a fixed utility level  $v$ , consider two price vectors,  $p$  and  $p'$ , and the associated demand vectors,  $x = h(p, v)$  and  $x' = h(p', v)$ . Using the property that  $x$  and  $x'$  are expenditure minimizers, one obtains

$$(p - p')(x - x') \leq 0. \tag{10.7}$$

For the change  $\Delta p_k = p_k - p'_k$  of the price of a single commodity  $k$  with all other prices held constant, i.e.  $\Delta p_h = 0$ ,  $h \neq k$ , (10.7) implies

$$\Delta p_k \Delta x_k \leq 0. \tag{10.8}$$

In other words, a separate price increase of one particular commodity will never result in a larger demand for this commodity. The Hicksian demand function of any commodity is therefore never upward sloping in its own price. This property is commonly known as the *negativity (non-positivity) of the own substitution effect*. A detailed discussion in the differentiable context can be found in Section 13.

## 11. Properties of differentiable utility functions

The following sections treat utility and demand in the context of differentiability which is the truly classical approach to the theory of consumer demand [see, for example, Slutsky (1915), Hicks (1939), and Samuelson (1947a)].

Let  $u: X \rightarrow \mathbb{R}$  be a  $C^2$  utility function with no critical point representing a complete and continuous preference order of class  $C^2$  on  $X$  characterized by monotonicity and strict convexity. Then this function is (i) continuous, (ii) increasing, i.e.  $u(x) > u(y)$  for  $x \succ y$  and  $x \neq y$ , and (iii) strictly quasi-concave, i.e.  $u(\alpha x + (1 - \alpha)y) > u(y)$  for  $\alpha \in (0, 1)$  and  $u(x) \geq u(y)$ . It is twice continuously differentiable with respect to  $x$ , i.e. all its second-order partial derivatives exist and are continuous functions of  $x$ . It will be assumed that all first-order derivatives, namely  $\partial u / \partial x_i = u_i$ ,  $i = 1, \dots, \ell$ , are positive. They are called *marginal utilities*. In the following the symbol  $u_x$  will be used to denote the  $\ell$ -vector of marginal utilities. Since the second-order derivatives are continuous functions of

their arguments, applying Young's theorem yields the *symmetry property*

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i} = u_{ji}.$$

Let  $U_{xx}$  be the  $\ell \times \ell$  Hessian matrix of the utility function, i.e. the matrix of the second partial derivatives of  $u$  with typical element  $u_{ij}$ . The symmetry property means that  $U_{xx}$  is a symmetric matrix, i.e.  $U_{xx} = U'_{xx}$  (where the prime denotes transposition).

The property of strict quasi-concavity of the utility function implies some further restrictions on the first- and second-order partial derivatives of the utility function.

### *Theorem 11.1*

Under strict quasi-concavity of the utility function

$$z'U_{xx}z \leq 0 \quad \text{for every element of } \{z \in R^\ell \mid u'_x z = 0\},$$

where the derivatives are evaluated at the same but arbitrarily selected bundles  $x \in X$ .

### *Proof*

Take an arbitrary bundle  $x \in X$ . Select another bundle  $y \neq x$  such that  $u(y) = u(x)$ . For scalar real  $\alpha$  consider the bundle  $m = \alpha y + (1 - \alpha)x = \alpha z + x$  with  $z = y - x$ . For fixed  $x$  and  $y$  and varying  $\alpha$ , define the function  $f(\alpha) = u(m)$ , which is a differentiable and strictly quasi-concave function, such that  $f(\alpha) > f(0) = f(1)$ ,  $\alpha \in (0, 1)$  and  $f'(0) > 0$ ,  $f'(1) < 0$ . Thus, there exists a value  $\hat{\alpha} \in (0, 1)$  such that  $f'(\hat{\alpha}) = 0$  corresponding to a maximum of  $f(\alpha)$  implying  $f''(\hat{\alpha}) \leq 0$ . This maximum is unique since  $f$  is *strictly* quasi-concave. Now

$$f'(\hat{\alpha}) = \sum_i \frac{\partial u}{\partial m_i} \frac{dm_i}{d\alpha} = u'_x z = 0,$$

$$f''(\hat{\alpha}) = \sum_i \sum_j \frac{dm_i}{d\alpha} \frac{\partial^2 u}{\partial m_i \partial m_j} \frac{dm_j}{d\alpha} = z'U_{xx}z \leq 0,$$

with the derivatives evaluated at  $\hat{w} = \hat{\alpha}z + x$ . Since  $y$  can be chosen arbitrarily close to  $x$ , the property holds also when the derivatives are evaluated at bundles in a small neighborhood of  $x$  and by virtue of continuity at  $x$  itself.

The property of strict quasi-concavity of the utility function is not strong enough to obtain everywhere differentiable demand functions. As a regularity



condition the weak inequality of the theorem above is changed into a strong one. The result is known as strong quasi-concavity.

*Definition 11.1*

A strictly quasi-concave utility function is said to be strongly quasi-concave if

$$z'U_{xx}z < 0 \quad \text{for every element of } \{z \in R^l \mid u'_x z = 0, z \neq 0\}.$$

This additional condition is equivalent to non-singularity of the so-called *bordered Hessian matrix*

$$H = \begin{bmatrix} U_{xx} & u_x \\ u'_x & 0 \end{bmatrix}. \quad (11.1)$$

*Theorem 11.2*

The bordered Hessian matrix  $H$  associated with a strictly quasi-concave monotone increasing utility function is non-singular if and only if the utility function is strongly quasi-concave.

*Proof*<sup>2</sup>

(i) *Sufficiency.* Assume that  $H$  is singular. Then there exist an  $l$ -vector  $z$  and a scalar  $r$  such that simultaneously

$$U_{xx}z + u_x r = 0; \quad u'_x z = 0; \quad (z', r) \neq 0. \quad (11.2)$$

The case of  $z=0, r \neq 0$  can be ruled out because then  $u_x r = 0$ , which implies  $u_x = 0$ , contradicting the monotonicity of the utility function. The case of  $z \neq 0$  can also be ruled out, since premultiplying (11.2) by  $z'$  would result in  $z'U_{xx}z = 0, u'_x z = 0, z \neq 0$ , contradicting the property of strong quasi-concavity. Consequently, no non-zero vector  $(z', r)$  exists such that  $(z', r)H = 0$  and thus  $H$  is non-singular.

(ii) *Necessity.* We will proceed in three steps. First it will be shown that if  $H$  is non-singular there exist real numbers  $\alpha < \alpha^*$  such that the matrix  $A(\alpha) = U_{xx} + \alpha u_x u'_x$  is non-singular, where the typical element of  $A(\alpha)$  is  $u_{ij} + \alpha u_i u_j$ . Secondly, we will show that under strict quasi-concavity there exist real numbers  $\beta \leq \beta^*$  such that  $A(\beta)$  is a negative semi-definite matrix. The third step will combine the first two steps.

*Step 1.* Non-singularity means that for all  $l$ -vectors  $c_1$  such that  $u'_x c_1 = 0, A(\alpha)c_1 \neq 0$ , for all  $\alpha$ . Consider next all vectors  $c_2$  such that  $u'_x c_2 \neq 0$ , and

<sup>2</sup>See also Debreu (1952), Dhrymes (1967), and Barten et al. (1969).

normalize  $c_2$  such that  $u'_x c_2 = 1$ .  $A(\hat{\alpha})c_2 = 0$  means  $\hat{\alpha} = -c'_2 U_{xx} c_2$ . Let  $\alpha^* = \min_{c_2} \{-c'_2 U_{xx} c_2 \mid c'_2 u_x = 1\}$ . For  $\alpha < \alpha^*$ ,  $A(\alpha)c_2 \neq 0$  and  $A(\alpha)$  is non-singular.

*Step 2.* If  $A(\beta)$  is negative semi-definite, i.e.  $c'A(\beta)c \leq 0$ , then clearly for all  $\beta$ ,  $z'U_{xx}z \leq 0$  for all  $z$  such that  $u'_x z = 0$ . Furthermore, if  $z'U_{xx}z \leq 0$  for all  $z$  such that  $u'_x z = 0$ , then also  $z'A(\beta)z \leq 0$  for all  $\beta$ . Consider next all vectors  $c$  such that  $u'_x c \neq 0$  and normalize these such that  $u'_x c = 1$ .  $c'A(\beta)c \leq 0$  requires  $\beta \leq -c'U_{xx}c$ . Let  $\beta^* = \min_c \{-c'U_{xx}c \mid c'u_x = 1\}$ . Therefore,  $A(\beta)$  is negative semi-definite if  $\beta \leq \beta^*$ .

*Step 3.* There exist real numbers  $\gamma < \beta^* = \alpha^*$  such that  $A(\gamma)$  is both non-singular and negative semi-definite. A negative semi-definite matrix which is non-singular is a negative definite matrix. Therefore,  $z'A(\gamma)z = z'U_{xx}z < 0$  for all  $u'_x z = 0$ ,  $z \neq 0$ , i.e.  $u$  is strongly quasi-concave. Q.E.D.

What has been said about the property of the derivatives of  $u$  is also true for any differentiable increasing transformation of  $u$ . This is evident in the case of the positive sign of the marginal utilities and the consequences of strict quasi-concavity which are based directly on properties of the preference ordering, i.e. on monotonicity and convexity, respectively. Still, it is useful to write down explicitly the consequences of such transformations for the derivatives. Let  $F$  be a twice continuously differentiable increasing transformation  $F: R \rightarrow R$ , i.e.  $F' > 0$  and  $F''$  continuous. Define  $v(x) = F(u(x))$ . The following relations hold between the first- and second-order partial derivatives of  $v(x)$  and  $u(x)$ :

$$\partial v / \partial x_i = F' \partial u / \partial x_i \quad \text{or} \quad v_x = F' u_x,$$

$$\partial^2 v / \partial x_i \partial x_j = F' \partial^2 u / \partial x_i \partial x_j + F'' (\partial u / \partial x_i) (\partial u / \partial x_j),$$

or

$$V_{xx} = F' U_{xx} + F'' u_x u'_x.$$

Since  $F'$  is positive,  $v_x$  has the same sign as  $u_x$ . The elements of  $V_{xx}$  do not necessarily have the same sign as those of  $U_{xx}$ . However,  $z'U_{xx}z < 0$  for every element of  $\{z \in R^l \mid u'_x z = 0\}$  implies  $z'V_{xx}z < 0$  for every element of  $\{z \in R^l \mid v'_x z = 0\}$ . Indeed,  $v'_x z = F' u'_x z = 0$  and thus  $z'V_{xx}z = F' z'U_{xx}z + F'' (z'u_x)^2 = F' z'U_{xx}z < 0$ . Note that this is not a new result, but simply another demonstration of the fact that strict and strong quasi-concavity reflect properties of the preference ordering.

Since the marginal utilities  $\partial u / \partial x_i$  are not invariant under monotone increasing transformations, one sometimes reasons in terms of ratios of a pair of marginal utilities, for instance  $(\partial u / \partial x_i) / (\partial u / \partial x_j)$  which is clearly invariant. Keeping the

level of utility constant and varying  $x_i$  and  $x_j$  alone, one has, locally,

$$(\partial u / \partial x_i) dx_i^* + (\partial u / \partial x_j) dx_j^* = 0 \quad (11.3)$$

or

$$R_{ij} = \frac{\partial u / \partial x_i}{\partial u / \partial x_j} = - \frac{dx_j^*}{dx_i^*}, \quad (11.4)$$

where  $R_{ij}$  is the *marginal rate of substitution* (MRS) between commodities  $i$  and  $j$ . It represents the amount of commodity  $j$  to be sacrificed in exchange for an increase in commodity  $i$ , keeping utility constant.

$R_{ij}$  is assumed to be a decreasing function of  $x_i$ , i.e. at the same level of utility less of  $x_j$  has to be sacrificed to keep utility constant when  $x_i$  is large than when  $x_i$  is small. This assumption of *diminishing marginal rate of substitution* for any pair  $(i, j)$  follows from strong quasi-concavity of the utility function.

Decreasing MRS means that

$$\frac{\partial R_{ij}}{\partial x_i} - R_{ij} \frac{\partial R_{ij}}{\partial x_j} < 0, \quad (11.5)$$

which yields

$$\frac{1}{u_j^3} (u_{ii}u_j^2 - 2u_iu_ju_{ij} + u_{jj}u_i^2) < 0.$$

The term in parentheses is equal to  $z'U_{xx}z$  for  $z_k = 0, k \neq i, j$  and  $z_i = -u_j$  and  $z_j = u_i$ . Since  $u_j > 0$  and  $u'_x z = 0$ , strong quasi-concavity implies the negativity of (11.5). The reverse implication holds under some additional assumptions which are discussed by Arrow and Enthoven (1961).

Traditionally the concept of MRS has been used in connection with weak and strong separability. Before turning to this issue it is useful to investigate the consequences of the differentiability of  $u(x)$  in the case of (weak) separability. Given that under separability

$$u(x) = V(v_1(x_1), \dots, v_k(x_k)), \quad (11.6)$$

differentiability implies that for  $i \in N_j$

$$\frac{\partial u}{\partial x_i} = \frac{\partial V}{\partial v_j} \frac{\partial v_j}{\partial x_i} \quad (11.7)$$

exists and hence  $\partial V/\partial v_j$  and  $\partial v_j/\partial x_i$  exist. Since  $v_j(x_j)$  has all the properties of a utility function, hence  $\partial v_j/\partial x_i > 0$ ; also  $\partial V/\partial v_j$  is positive because  $\partial u/\partial x_i > 0$ . Next for  $i, k \in N_j$

$$\frac{\partial^2 u}{\partial x_i \partial x_k} = \frac{\partial V}{\partial v_j} \frac{\partial^2 v_j}{\partial x_i \partial x_k} + \frac{\partial^2 V}{\partial v_j^2} \frac{\partial v_j}{\partial x_i} \frac{\partial v_j}{\partial x_k}, \quad (11.8)$$

and for  $i \in N_j, k \in N_g, j \neq g$

$$\frac{\partial^2 u}{\partial x_i \partial x_k} = \frac{\partial^2 V}{\partial v_j \partial v_g} \frac{\partial v_j}{\partial x_i} \frac{\partial v_g}{\partial x_k}. \quad (11.9)$$

Therefore, the existence and symmetry of  $U_{xx}$  implies the existence and symmetry of the Hessian matrix  $V_{vv}$ .

In the case of strong separability  $\partial V/\partial v_j = V'$ , i.e. equal for all  $j$ . Then

$$\frac{\partial u(x)}{\partial x_i} = V' \frac{\partial v_j(x_j)}{\partial x_i}, \quad (11.10)$$

while the Hessian matrix  $V_{vv}$  has all elements equal.

Separability and properties of the MRS are related to each other by the following two theorems.

*Theorem 11.3*

The MRS between two commodities  $i$  and  $k$  within the same element  $N_j$  of the partition is independent of consumption levels outside of  $N_j$  if and only if the utility function is weakly separable.

This means that for all  $i \in N_j$  the  $\partial u/\partial x_i$  consist of the product of a common factor  $\alpha_j(x)$  and a specific factor  $\beta_{ji}(x_j)$  which is a function of  $x_j$  only, i.e.

$$\partial u(x)/\partial x_i = \alpha_j(x) \beta_{ji}(x_j).$$

This corresponds to (11.7) for  $\alpha_j(x) = \partial V/\partial v_j$  and  $\beta_{ji}(x_j) = \partial v_j/\partial x_i$ .

*Theorem 11.4*

The MRS between a commodity  $i \in N_j$  and another commodity  $f \in N_g, g \neq j$ , in different elements of the partition, can be written as the ratio of two functions  $\beta_{ji}(x_j)$  and  $\beta_{gf}(x_g)$ , respectively, if and only if the utility function is strongly separable.

In the literature, for example Goldman and Uzawa (1964), strong separability is identified with independence of the MRS between  $i$  and  $f$ ,  $i \in N_j$ ,  $f \in N_g$ ,  $g \neq j$ , of the consumption levels of commodities in other groups. This definition requires a partition into at least three groups, which is unsatisfactory since strong separability can exist also for two groups. The present theorem covers also the case of a partition into two groups and is inspired by Samuelson (1947a). Note that in general weak separability into two groups does not imply strong separability.

## 12. Differentiable demand

In Lemma 7.3 conditions are given for the existence of continuous demand functions  $f(p, w)$ , which are moreover homogeneous of degree zero in prices and wealth. This section will focus on the consequences of the assumption of differentiability of the utility function for the demand functions. In particular, the differentiability of the demand functions will be studied.

We will confine ourselves to the case where the consumption set  $X$  is the open positive cone  $P$  of  $R^l$ . To obtain demand bundles in  $P$  we will further assume that preferences are monotonic, of class  $C^2$ , and that the closures of the indifference hypersurfaces are contained in  $P$ . Then, under positive prices and positive wealth the demand function is well defined and it maps into the positive cone of  $R^l$ . Moreover, the consumer will spend all his wealth maximizing preferences. Therefore his choice can be considered as being limited to all bundles  $x \in P$  which satisfy  $p'x = w$ .

If  $u(x)$  is twice continuously differentiable then demand  $x = f(p, w)$ , as defined in (3.2) or in (7.1), can be determined as the solution of a classical maximization problem: maximize  $u(x)$  with respect to  $x$  subject to  $p'x = w$ . One forms the Lagrangean

$$L(x, \lambda, p, w) = u(x) - \lambda(p'x - w), \quad (12.1)$$

where  $\lambda$  is a Lagrange multiplier. First-order conditions for a stationary value of  $u(x)$  are

$$\partial L / \partial x = u_x - \lambda p = 0, \quad (12.2a)$$

$$\partial L / \partial \lambda = w - p'x = 0. \quad (12.2b)$$

As in the previous section it will be assumed that  $u_i > 0$ ,  $i = 1, \dots, n$ . Then, with  $u_x$  and  $p$  both strictly positive, the first condition implies a positive value of  $\lambda$ . A

necessary second-order condition for a relative maximum is that

$$z' L_{xx} z \leq 0 \quad \text{for every } z \in R^l \text{ such that } p'z = 0, \quad (12.3)$$

where  $L_{xx} = \partial^2 L / \partial x \partial x'$ , evaluated at a solution of (12.2). Under strict quasi-concavity of the utility function (see Theorem 11.1) this condition is satisfied since  $L_{xx} = U_{xx}$  and  $p'z = 0$  implies  $u'_x z = 0$  in view of (12.2a) and positive  $\lambda$ .

System (12.2) is a system of  $\ell + 1$  equations in  $2(\ell + 1)$  variables: the  $\ell$ -vectors  $x$  and  $p$  and the scalars  $\lambda$  and  $w$ . For our purpose  $p$  and  $w$  are taken as given, and  $x$  and  $\lambda$  are the "unknown" variables. Lemma 7.3 guarantees the existence of a unique solution for  $x = f(p, w)$ . Then there also exists a unique solution for  $\lambda$ , namely  $\theta(p, w) = u'_x f(p, w) / w$ , with  $u'_x$  evaluated at  $x = f(p, w)$ .

It can easily be verified that the solution of (12.2) for  $x$  is invariant under monotone increasing transformations of  $u(x)$  but the one for  $\lambda$  is not. For such a transformation  $F$  the first-order conditions (12.2) are changed into  $F' u_x - \lambda^* p = 0$ ,  $w - p'x = 0$ . The first one becomes (12.2a) again after division by  $F' > 0$ , with  $\lambda = \lambda^* / F'$ . The solution for  $x$  is thus invariant, while  $\lambda^* = F' \lambda$  is the solution for the Lagrange multiplier for the transformed problem.

We now turn to the matter of differentiability of  $f(p, w)$  and  $\theta(p, w)$ . To prepare the ground we start by writing down the differential form of system (12.2) at  $(x^0, \lambda^0, p, w)$  with  $x^0 = f(p, w)$  and  $\lambda^0 = \theta(p, w)$ :

$$U_{xx}^0 dx - \lambda^0 dp - p d\lambda = 0, \quad (12.4a)$$

$$dw - p' dx - x^{0'} dp = 0, \quad (12.4b)$$

where  $U_{xx}^0$  is  $U_{xx}$  evaluated at  $x^0$ . After some rearrangement of terms one obtains the *Fundamental Matrix Equation of Consumer Demand*:

$$\begin{bmatrix} U_{xx}^0 & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} dx \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \lambda^0 I & 0 \\ -x^{0'} & 1 \end{bmatrix} \begin{bmatrix} dp \\ dw \end{bmatrix}, \quad (12.5)$$

where  $I$  is the  $\ell \times \ell$  identity matrix. One may write formally

$$dx = X_p dp + x_w dw; \quad d\lambda = \lambda'_p dp + \lambda_w dw \quad (12.6)$$

or

$$\begin{bmatrix} dx \\ -d\lambda \end{bmatrix} = \begin{bmatrix} X_p & x_w \\ -\lambda'_p & -\lambda_w \end{bmatrix} \begin{bmatrix} dp \\ dw \end{bmatrix}. \quad (12.7)$$

It is clear that  $f(p, w)$  and  $\theta(p, w)$  are continuously differentiable if and only if

the matrix on the right-hand side of (12.7) is the unique matrix of derivatives. Combining (12.5) and (12.7) one obtains

$$\begin{bmatrix} U_{xx}^0 & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} X_p & x_w \\ -\lambda_p & -\lambda_w \end{bmatrix} = \begin{bmatrix} \lambda^0 I & 0 \\ -x^{0'} & 1 \end{bmatrix} \quad (12.8)$$

for arbitrary  $(dp, dw)$ . We can now state the result on the differentiability of demand, first given by Katzner (1968), who also gives an informative counterexample.

*Theorem 12.1 (Differentiability of Demand)*

The system of demand functions  $f(p, w)$  is continuously differentiable with respect to  $(p, w)$  if and only if the matrix

$$\begin{bmatrix} U_{xx}^0 & p \\ p' & 0 \end{bmatrix} \quad (12.9)$$

is non-singular at  $x^0 = f(p, w)$ .

*Proof*

- (i) *Sufficiency.* Applying the Implicit Function Theorem to the system (12.2) requires the non-singularity of the Jacobian matrix with respect to  $(x, \lambda)$ , which is equal to the matrix (12.9).
- (ii) *Necessity.* If  $f(p, w)$  is differentiable, the chain rule implies that  $\theta(p, w)$  is differentiable and that the matrix on the right-hand side of (12.7) is the matrix of derivatives. The  $(\ell + \ell) \times (\ell + 1)$  matrix on the right-hand side of (12.8) is of full rank. Thus, the two  $(\ell + 1) \times (\ell + 1)$  matrices in the product on the left-hand side of (12.8) are of full rank. Therefore (12.9) is non-singular.

*Lemma 12.2*

The matrix (12.9) is non-singular if and only if the matrix

$$H = \begin{bmatrix} U_{xx} & u_x \\ u'_x & 0 \end{bmatrix}$$

is non-singular.

We have now established that differentiability of the demand functions derived from strictly quasi-concave monotone increasing differentiable utility functions is equivalent to non-singularity of  $H$ . This latter condition is implied by strong

quasi-concavity of these utility functions (Theorem 11.2), which, in turn, is equivalent to the assumption of diminishing marginal rate of substitution. As far as these two latter properties exclude singular points, the four properties mentioned here are equivalent and interchangeable.

Thus far we have studied the relationship between differentiability of the (direct) utility function and the Marshallian demand functions  $f(p, w)$ . The next theorems establish differentiability properties of the indirect utility function (10.2), the expenditure function (10.1), and the Hicksian demand function  $h(p, v)$  in (10.4).

*Theorem 12.3 (Differentiability of the Indirect Utility Function)*

Assume that the direct utility function  $u(x)$  is twice continuously differentiable and that the demand function  $f(p, w)$  is continuously differentiable. Then, the indirect utility function

$$g(p, w) = \max\{u(x) \mid p'x = w\}$$

is twice continuously differentiable with respect to  $(p, w)$ .

*Proof*

From the differentiability properties of  $u$  and  $f$  and from the identity

$$g(p, w) = u(f(p, w)) \tag{12.10}$$

and from  $p'f(p, w) = w$  one obtains

$$\frac{\partial g}{\partial p'} = u'_x \frac{\partial f}{\partial p'} = \lambda p' \frac{\partial f}{\partial p'} = -\theta(p, w) f(p, w)', \tag{12.11}$$

since

$$p' \frac{\partial f}{\partial p'} = -f(p, w)'$$

Similarly, differentiation with respect to  $w$  yields

$$\frac{\partial g}{\partial w} = u'_x \frac{\partial f}{\partial w} = \lambda p' \frac{\partial f}{\partial w} = \theta(p, w), \tag{12.12}$$

since

$$p' \frac{\partial f}{\partial w} = 1.$$



Since  $\theta(p, w)$  and  $f(p, w)$  are both continuously differentiable with respect to  $(p, w)$  the indirect utility function is twice continuously differentiable.

Two comments can be made. Note that  $\lambda$ , the Lagrange multiplier associated with the budget constraint, is the derivative of the indirect utility function with respect to wealth: the *marginal utility of wealth (income, money)*, i.e.

$$\lambda = \frac{\partial g}{\partial w} = \frac{\partial u(f(p, w))}{\partial w}.$$

Next, one has a result due to Roy (1942):

*Corollary 12.4 (Roy's Identity)*

$$f(p, w) = - \frac{1}{\partial g / \partial w} \partial g / \partial p. \quad (12.13)$$

*Proof*

Combine (12.11) and (12.12).

According to (10.3) the expenditure function  $E(p, v)$  is the solution of  $v = g(p, w)$  with respect to  $w$ .

*Theorem 12.5 (Differentiability of the Expenditure Function)*

The expenditure function is continuously differentiable if and only if the indirect utility function is continuously differentiable.

*Proof*

$g(p, w)$  is the inverse of  $E(p, v)$  with respect to  $v$ , i.e.

$$E(p, v) = g^{-1}(p, v), \quad (12.14)$$

$$g(p, w) = E^{-1}(p, w). \quad (12.15)$$

Differentiation of  $E(p, E^{-1}(p, w)) = w$  then yields

$$\frac{\partial E}{\partial p} \cdot \frac{1}{\partial E / \partial v} = - \frac{\partial E^{-1}}{\partial p} = - \frac{\partial g}{\partial p} \quad (12.16)$$

and

$$\frac{1}{\partial E / \partial v} = \frac{\partial E^{-1}}{\partial w} = \frac{\partial g}{\partial w}. \quad (12.17)$$

Since  $\partial E/\partial v > 0$ , differentiability of  $E$  implies differentiability of  $g$ . Conversely, differentiation of  $g(p, g^{-1}(p, v)) = v$  yields

$$\frac{\partial g}{\partial p} \cdot \frac{1}{\partial g/\partial w} = -\frac{\partial g^{-1}}{\partial p} = -\frac{\partial E}{\partial p} \quad (12.18)$$

and

$$\frac{1}{\partial g/\partial w} = \frac{\partial g^{-1}}{\partial v} = \frac{\partial E}{\partial v}. \quad (12.19)$$

Since  $\partial g/\partial w > 0$ , differentiability of  $g(p, w)$  implies differentiability of  $E(p, v)$ . Q.E.D.

*Theorem 12.6 (Differentiability of the Hicksian Demand Function)*

If the expenditure function  $E(p, v)$  is differentiable then

$$\frac{\partial E}{\partial p}(p, v) = h(p, v). \quad (12.20)$$

Consequently, if  $E(p, v)$  is twice continuously differentiable then

$$\frac{\partial(\partial E/\partial p)}{\partial p'} = \frac{\partial h}{\partial p'} \quad (12.21)$$

which implies that  $\partial h/\partial p'$  is a symmetric matrix.

*Proof*

One only has to establish (12.20).

Consider an arbitrary commodity  $k$  and let  $t_k$  denote the  $\ell$ -vector with  $t > 0$  in the  $k$ th position and zero elsewhere. Then

$$\begin{aligned} \frac{E(p+t_k, v) - E(p, v)}{t} &= \frac{p'[h(p+t_k, v) - h(p, v)]}{t} + h_k(p+t_k, v) \\ &\geq h_k(p+t_k, v). \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{E(p-t_k, v) - E(p, v)}{-t} &= \frac{p'[h(p-t_k, v) - h(p, v)]}{-t} + h_k(p-t_k, v) \\ &\leq h_k(p-t_k, v). \end{aligned}$$

Taking the limit for  $t \rightarrow 0$  yields

$$\frac{\partial E}{\partial p_k}(p, v) = h_k(p, v). \quad \text{Q.E.D.}$$

### 13. Properties of first-order derivatives of demand functions

Since  $u(x)$  is twice continuously differentiable, the Hessian matrix  $U_{xx}$  is symmetric. Then also matrix (12.9) is symmetric, and the same is true for its inverse. Let  $Z$  be an  $\ell \times \ell$  matrix,  $z$  be an  $\ell$ -vector, and  $\zeta$  a scalar defined by

$$\begin{bmatrix} Z & z \\ z' & \zeta \end{bmatrix} = \begin{bmatrix} U_{xx}^0 & p \\ p' & 0 \end{bmatrix}^{-1}. \quad (13.1)$$

Note that  $Z = Z'$ , i.e. matrix  $Z$  is symmetric. Premultiply both sides of (12.8) by this inverse to obtain

$$\begin{bmatrix} X_p & x_w \\ -\lambda'_p & -\lambda_w \end{bmatrix} = \begin{bmatrix} \lambda Z - zx' & z \\ \lambda z' + \zeta x' & \zeta \end{bmatrix}, \quad (13.2)$$

where  $zx'$  denotes the matrix  $(z_i x_j)$ . Clearly  $z = x_w$  and  $\zeta = -\lambda_w$ . Furthermore, writing  $K = \lambda Z$  one has

$$X_p = K - x_w x_w', \quad (13.3a)$$

with  $x_w x_w'$  denoting the matrix  $[(\partial f_i / \partial w) x_j]$  and  $x = f(p, w)$ . In scalar form this equation, sometimes named the *Slutsky equation*, can be written as

$$\frac{\partial f_i(p, w)}{\partial p_j} = k_{ij}(p, w) - \frac{\partial f_i(p, w)}{\partial w} f_j(p, w). \quad (13.3b)$$

The matrix  $K = (k_{ij}(p, w))$  is known as the *Slutsky matrix*. Its properties are of considerable interest.

Before proceeding to derive a set of properties for  $x_w$ ,  $X_p$ , and  $K$ , it is useful to give the following implications of (13.1) and (13.2):

$$U_{xx}^0 Z + p x_w' = I, \quad (13.4a)$$

$$U_{xx}^0 x_w = \lambda_w p, \quad (13.4b)$$

$$p' Z = 0, \quad (13.4c)$$

$$p' x_w = 1. \quad (13.4d)$$

**Theorem 13.1**

Let  $f(p, w)$  denote the set of  $\ell$  differentiable demand functions resulting from maximizing the utility function  $u(x)$  subject to  $p'x = w$ . Let  $\partial f/\partial w = x_w$ ,  $\partial f/\partial p' = X_p$ , and  $K = X_p + x_w x'$ . Then one has the following properties:

- (i) Adding up  $p'x_w = 1$  (Engel aggregation),  
 $p'X_p = -x'$  (Cournot aggregation),  
 $p'K = 0$ .
- (ii) Homogeneity  $X_p p + x_w w = 0$  or  $Kp = 0$ .
- (iii) Symmetry  $K = K'$ .
- (iv) Negativity  $y'Ky < 0$ , for  $\{y \in R^\ell \mid y \neq \alpha p\}$ .
- (v) Rank  $r(K) = \ell - 1$ .

**Proof**

- (i) *Adding-up.* Follows from (13.4d),  $K = \lambda Z$ , (13.4c), and (13.3).
- (ii) *Homogeneity.* Apply Euler's theorem to  $f(p, w)$  which is homogeneous of degree zero in prices and wealth. Use (13.3) and  $p'x = w$ .
- (iii) *Symmetry.*  $K = \lambda Z$  is symmetric since  $Z$  is symmetric.
- (iv) *Negativity.* Since  $K = \lambda Z$  and  $\lambda > 0$ , the negativity condition for  $K$  is equivalent to the negativity condition for  $Z$ . Premultiplication of (13.4a) by  $Z$  gives  $ZU_{xx}^0 Z = Z$ . Consider the quadratic form  $y'ZU_{xx}^0 Zy = y'Zy$  for some  $\ell$ -vector  $y \neq 0$ . Let  $q = Zy$ . Note that  $p'q = 0$  or for  $u_x = \lambda p$ ,  $u'_x q = 0$ . The property of strong quasi-concavity (Definition 11.1) thus implies

$$y'Zy = q'U_{xx}^0 q < 0, \text{ for } q \neq 0.$$

The case of  $q = 0$  occurs when  $y = \alpha p$ . Hence  $Z$  and  $K$  are negative semi-definite matrices. Moreover, all diagonal elements of  $K$  are negative, since  $e'_i K e_i = K_{ii} < 0$ , where  $e_i \neq \alpha p$  denotes the  $i$ th unit vector in  $R^\ell$ .

- (v) *Rank.* Since  $Kp = 0$ , matrix  $K$  cannot have full rank, i.e.  $r(K) \leq \ell - 1$ . It follows from (13.4a) that  $U_{xx} Z = I - px'_w$ , a matrix of rank  $\ell - 1$ . Thus,  $r(Z) = r(K) \geq \ell - 1$ . Therefore  $r(K) = \ell - 1$ .

The Slutsky matrix  $K$  deserves some further discussion. It is invariant under monotone increasing transformations of the utility function. This follows from the fact that  $f(p, w)$  is invariant and so are its derivatives  $X_p$  and  $x_w$ . Consequently,  $K = X_p + x_w x'$  is invariant, too.

Next it can be shown that  $K$  is the matrix of price derivatives of the Hicksian demand function. Differentiation of (10.6) yields

$$\frac{\partial h}{\partial p'} = \frac{\partial f}{\partial p'} + \frac{\partial f}{\partial w} \cdot \frac{\partial E}{\partial p'}$$

Then (12.20) and (13.3) imply

$$\frac{\partial h}{\partial p'} = X_p + x_w x' = K.$$

Therefore, for small price changes  $dp$ ,  $K dp$  describes the changes in the composition of the demand bundle in response to price changes, if utility is kept constant. Since Hicks (1934) and Allen (1934) this is known as the *substitution effect*.

The remaining part of the price effect, namely  $-x_w x' dp$ , is known as the *income effect*. Its effect is similar to that of a change in  $w$  (wealth, income). This effect can be neutralized by an appropriate change in  $w$  equal to  $x' dp$ . Obviously, the level of utility is then kept constant. For this reason the substitution effect is also called the *income compensated price effect*.

Finally, using (12.20) again, it is easily seen that

$$K = \frac{\partial^2 E}{\partial p \partial p'}, \quad (13.6)$$

i.e.  $K$  is the Hessian matrix of the expenditure function when  $v$  is kept constant. According to Lemma 10.1  $E$  is concave in prices. Then, its Hessian matrix  $K$  is negative semi-definite. This result may suffice to indicate that the properties of the matrix  $K$ , derived in the context of Marshallian demand functions, can also be obtained in the context of expenditure functions and Hicksian demand functions.

Concluding this section, it should be pointed out that only very weak properties of a consumer's market demand function have been established. In particular, without any further assumptions on preferences no general statement can be made about the direction of change of the demand of a particular commodity when its price changes. From Theorem 13.1 one only has

$$\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial w} < 0. \quad (13.7)$$

If  $\partial x_i / \partial w$  is positive (the case of a *superior good*) the own-price effect is obviously negative. If commodity  $i$  is an *inferior good*, i.e. when  $\partial x_i / \partial w$  is negative, then the own-price effect may be positive. Commodities which display a positive own-price effect are known as *Giffen goods* – see Marshall (1895) and Stigler (1947). It all depends on the sign of the derivative with respect to  $w$ . On the basis of the adding-up condition one knows that at least one  $\partial x_i / \partial w$  has to be positive. Therefore, all theory has to offer is that for at least one commodity the own-price effect is negative.

#### 14. Separability and the Slutsky matrix

Separability of the utility function implies restrictions on the Slutsky matrix  $K$  because of the structured impact of  $p_j$  on demand for commodities in other elements of the partition than the one to which  $j$  belongs. Let  $N_j$  and  $N_g$  be two elements of the partition  $N_1, \dots, N_k$ , and let  $K$  be partitioned into  $k^2$  blocks of the type  $K_{jg}$ . From (13.3) one has

$$K_{jg} = \frac{\partial x_j}{\partial p'_g} + \frac{\partial x_j}{\partial w} x'_g, \quad (14.1)$$

where, as before,  $x_j = f_j(p, w)$  is the vector of Marshallian demand functions for the commodities in  $N_j$ . Symmetry of  $K$  yields

$$K_{jg} = K'_{gj}. \quad (14.2)$$

For any  $j \in J = \{1, \dots, k\}$ , let  $x_j^j = f_j^j(p_j, w_j)$  as defined above in (9.3).

##### *Theorem 14.1*

Under weak separability of the utility function

$$K_{jg} = \psi_{jg}(p, w) \frac{\partial x_j^j}{\partial w_j} \frac{\partial x_g^g}{\partial w_g} \quad (14.3)$$

where  $\psi_{jg}(p, w) = \psi_{gj}(p, w)$ .

##### *Proof*

Differentiate both sides of (9.5) with respect to  $w$  to obtain

$$\frac{\partial x_j^j}{\partial w_j} \frac{\partial w_j^*}{\partial w} = \frac{\partial x_j}{\partial w}. \quad (14.4)$$

Next, differentiate both sides of (9.5) with respect to the vector  $p'_g$ . Using (14.1) and (14.4) yields

$$\frac{\partial x_j^j}{\partial p'_g} = \frac{\partial x_j^j}{\partial w_j} \frac{\partial w_j^*}{\partial p'_g} = K_{jg} - \frac{\partial x_j^j}{\partial w_j} \frac{\partial w_j^*}{\partial w} x'_g.$$

Therefore,

$$K_{jg} = \frac{\partial x_j^j}{\partial w_j} \left[ \frac{\partial w_j^*}{\partial p'_g} + \frac{\partial w_j^*}{\partial w} x'_g \right] = \frac{\partial x_j^j}{\partial w_j} z'_{jg}. \quad (14.5)$$

where  $z'_{jg}$  is the row vector between square brackets. One also has

$$K_{gj} = \frac{\partial x_g^g}{\partial w_g} z'_{gj}.$$

Since  $K_{gj} = K'_{jg}$ , it follows that

$$z_{jg}(p, w) = \psi_{jg}(p, w) \frac{\partial x_g^g}{\partial w_g}$$

which, when inserted into (14.5), gives (14.3). The equality of  $\psi_{jg}(p, w)$  and  $\psi_{gj}(p, w)$  follows directly from the symmetry of the matrix  $K$ .

One sometimes finds in the literature the derivation of a property similar to (14.3), namely:

*Corollary 14.2*

Under weak separability of the utility function and for  $\partial w_j^*/\partial w \neq 0$ ,  $\partial w_g^*/\partial w \neq 0$ ,

$$K_{jg} = \sigma_{jg}(p, w) \frac{\partial x_j}{\partial w} \frac{\partial x_g'}{\partial w}, \quad j \neq g, \quad (14.6)$$

where

$$\sigma_{jj}(p, w) = \sigma_{gj}(p, w) = \psi_{jg}(p, w) / \left( \frac{\partial w_j^*}{\partial w} \frac{\partial w_g^*}{\partial w} \right).$$

*Proof*

Use (14.4) in (14.3).

The additional conditions of this corollary are of an empirical and not of a theoretical nature. If the conditions are not both true,  $\sigma_{jg}(p, w)$  is not defined. Theorem 14.1 remains valid, however.

Before turning to the structure of the diagonal blocks of  $K$ , i.e. the  $K_{jj}$ , it is useful to note that also for the case of conditional demand equations one has a Slutsky matrix,  $K^j$ , say, which is formally analogous to the matrix  $K$  of the full set of the unconditional demand function. This  $K^j$ , defined by

$$K^j = \frac{\partial x_j^j}{\partial p_j} + \frac{\partial x_j}{\partial w_j} x_j^j, \quad (14.7)$$

has all the properties given earlier for  $K$  in Theorem 13.1.

The first matrix on the right-hand side of (14.7) does not represent the complete effect of a change in  $p_j$  on  $x_j$ , because  $w_j$  is taken to be fixed. For the full impact, one also has to take into account the effect which  $p_j$  has on  $x_j$  via a change in  $w_j^*(p, w)$ . From (9.4) it follows that

$$\frac{\partial w_j^*}{\partial p_j'} = x_j + p_j' \frac{\partial x_j}{\partial p_j'}. \quad (14.8)$$

Thus, for the complete effect one has, using (14.7), (14.8), and (9.5),

$$\frac{\partial x_j'}{\partial p_j'} = K^j + \frac{\partial x_j'}{\partial w_j} p_j' \frac{\partial x_j}{\partial p_j'}. \quad (14.9)$$

In view of (9.5) and (14.4) this has to be equal to

$$\frac{\partial x_j}{\partial p_j'} = K_{jj} - \frac{\partial x_j'}{\partial w_j} \frac{\partial w_j^*}{\partial w} x_j'. \quad (14.10)$$

This prepares the ground for

*Theorem 14.3*

Under weak separability of the utility function

$$K_{jj} = K^j + \psi_{jj}(p, w) \frac{\partial x_j'}{\partial w_j} \frac{\partial x_j'}{\partial w_j}, \quad (14.11)$$

where  $\psi_{jj}(p, w)$  is a scalar function.

*Proof*

Equating the right-hand sides of (14.9) and (14.10) gives

$$K_{jj} = K^j + \frac{\partial x_j'}{\partial w_j} \left[ p_j' \frac{\partial x_j}{\partial p_j'} + \frac{\partial w_j^*}{\partial w} x_j' \right].$$

Symmetry of  $K_{jj}$  implies that the term in square brackets is proportional to  $\partial x_j' / \partial w_j$ , with the scalar factor of proportionality being a function of  $(p, w)$ .

*Corollary 14.4*

$$\psi_{jj}(p, w) = - \sum_{g \neq j} \psi_{jg}(p, w). \quad (14.12)$$



*Proof*

First note that for all  $g$

$$\frac{\partial x_g^{g'}}{\partial w_g} p_g = 1$$

as a consequence of the adding-up condition for the conditional demand functions. Furthermore,  $K^j p_j = 0$ . It then follows from (14.3) and (14.11) that for all  $j$  and  $g$

$$p_j' K_{jg} p_g = \psi_{jg}(p, w).$$

Since  $Kp = 0$ ,

$$\sum_{g=1}^k K_{jg} p_g = 0$$

and thus

$$\sum_{g=1}^k \psi_{jg}(p, w) = 0.$$

Therefore, (14.12) follows immediately.

Result (14.11) deserves some further discussion. One can also write it as

$$K^j = K_{jj} - \psi_{jj}(p, w) \frac{\partial x_j^j}{\partial w_j} \frac{\partial x_j^j}{\partial w_j}, \quad (14.13)$$

where now  $K_{jj}$  is a negative definite matrix. The Slutsky matrix of the conditional demand equations for  $x_j$  can be decomposed into a negative definite matrix and into the outer vector product of the derivatives with respect to  $w_j$  multiplied by a scalar. A similar decomposition was derived by Houthakker (1960) for the Slutsky matrix  $K$  of the full unconditional system, namely

$$K = \theta(p, w) U_{xx}^{-1} + \sigma(p, w) \frac{\partial x}{\partial w} \frac{\partial x'}{\partial w} \quad (14.14)$$

with

$$\sigma(p, w) = -1 / \frac{\partial \ln \theta(p, w)}{\partial w}.$$

The first matrix on the right-hand side of (14.14) is called the *specific substitution matrix* and the second the *general substitution matrix*. Unlike (14.13), decomposition (14.14) is not invariant under monotone increasing transformations of the utility function, because neither  $\sigma(p, w)$  nor  $\theta(p, w)U_{xx}^{-1}$  are invariant. If one considers a "full unconditional system" as a conditional component of a still wider system, decomposition (14.13) is the appropriate one, where one may employ the same interpretation of the two components as in the case of (14.14). Both decompositions do not seem particularly restrictive. Indeed, further restrictions on  $K_{jj}$  in (14.13) or on  $\theta(p, w)U_{xx}^{-1}$  in (14.14) are needed to give the decomposition operational significance. Note that if any symmetric pair of non-diagonal elements of  $U_{xx}^{-1}$  in (14.14) are set equal to zero, the arbitrariness (and cardinal nature) of the decomposition is removed.

As one expects intuitively, *strong* separability is more restrictive than weak separability. Indeed, one has the following theorem.

*Theorem 14.5*

If the utility function is strongly separable with respect to a partition  $N_1, \dots, N_k$  and if the Hessian matrix of its form  $u(x) = \sum_{j \in J} v_j(x_j)$  is non-singular, then, for any pair  $j, g \in J$ ,

$$K_{jg} = \sigma(p, w) \frac{\partial x_j}{\partial w} \frac{\partial x'_g}{\partial w}, \quad j \neq g, \quad (14.15)$$

i.e. the  $\sigma_{jg}(p, w)$  of Corollary 14.2 is not specific for the pair  $j, g$ .

*Proof*

The Hessian matrix  $U_{xx}$  is block diagonal and so is its inverse. For this matrix (14.14) applies. The  $j, g$ -th off-diagonal block is given by (14.15).

Some remarks are in order. First, since  $K$  and  $\partial f / \partial w$  are ordinal concepts,  $\sigma(p, w)$  is invariant under monotone increasing transformations of the utility function. Second, the condition of a non-singular block diagonal  $U_{xx}$  is perhaps special, because  $r(U_{xx}) = \ell - 1$  is also acceptable, as can be seen from (13.4a). However, a singular block diagonal Hessian matrix has rather unrealistic implications, for example, that for all except one element of the partition,  $w_j^*(p, w)$ , the amount to be spent on the partition does not depend on  $w$ . This will not be investigated further. Third, under strong separability the *marginal budget shares*  $\partial w_j^* / \partial w$  are either all positive or all but one negative, the positive one being larger than unity. This property will be derived for the case of strong separability in elementary commodities.

### 15. Additive utility and demand

Consider the case that the utility function is separable in elementary commodities. This is clearly a limiting situation. It is known as *additive utility* and has very restrictive implications. In spite of this, it is not only of historical interest but also frequently used in empirical applications. Therefore, some attention will be paid to the consequences of additive utility for demand, although we will limit ourselves to the case of non-singular Hessian matrices.

Strong separability in elementary commodities clearly means that for all off-diagonal elements of the Slutsky matrix, Theorem 14.5 applies and thus, for any pair  $i, j \in \{1, \dots, \ell\}$ ,  $i \neq j$

$$k_{ij} = \sigma(p, w) \frac{\partial x_i}{\partial w} \frac{x_j}{\partial w}. \quad (15.1)$$

There is no specific substitution effect for any pair of goods. From (14.14) one can derive that

$$k_{ii} = \theta(p, w)/u_{ii} + \sigma(p, w) \left( \frac{\partial x_i}{\partial w} \right)^2. \quad (15.2)$$

Since  $\sum_j k_{ij} p_j = 0$ , it follows that

$$\begin{aligned} k_{ii} &= -\sigma(p, w) \frac{1}{p_i} \frac{\partial x_i}{\partial w} \sum_{j \neq i} b_j(p, w) \\ &= -\sigma(p, w) \frac{1}{p_i^2} b_i(p, w) [1 - b_i(p, w)], \end{aligned} \quad (15.3)$$

where  $b_i(p, w) = p_i \partial x_i / \partial w$  is the marginal propensity to spend on commodity  $i$  (the marginal budget share for  $i$ ). Note that

$$\sum_i b_i(p, w) = 1 \quad (15.4)$$

and that combining (15.1) and (15.2) results in

$$b_i(p, w) = \zeta p_i^2 / u_{ii}, \quad (15.5)$$

where  $\zeta = -\theta/\sigma$ .

The own-price derivative of demand is thus seen to be

$$\begin{aligned}\frac{\partial x_i}{\partial p_i} &= k_{ii} - \frac{\partial x_i}{\partial w} x_i \\ &= \frac{1}{p_i^2} b_i [-\sigma(1-b_i) - p_i x_i].\end{aligned}\quad (15.6)$$

In terms of *price elasticities*  $\epsilon_{ii} = \partial \ln x_i / \partial \ln p_i$  and *wealth (or income) elasticities*  $\eta_i = \partial \ln x_i / \partial \ln w$

$$\epsilon_{ii} = \eta_i [-\sigma^*(1-b_i) - a_i], \quad (15.7)$$

where  $\sigma^* = \sigma/w$  and  $a_i = p_i x_i / w$ , the *average budget share*.

It is argued by Deaton (1974) that, for large  $\ell$ ,  $a_i$  and  $b_i$  will be very small on the average. Then one has *Pigou's Law*  $\epsilon_{ii} \sim -\sigma^* \eta_i$ , i.e. the own-price elasticities are virtually proportional to the wealth (or income) elasticities. This is empirically a rather restrictive implication.

Another consequence is given in

#### *Theorem 15.1*

If the utility function is strongly separable in elementary commodities then the marginal budget shares  $b_i(p, w)$  are either (i) all positive and smaller than one, or (ii) all but one negative, the only positive one being larger than unity.

#### *Proof*

According to the negativity condition,  $k_{ii} < 0$ . Then also  $p_i^2 k_{ii} < 0$  or  $-\sigma(p, w) b_i(p, w) [1 - b_i(p, w)] < 0$ . Consider first the case  $\sigma(p, w) > 0$ . Then,  $b_i(p, w) [1 - b_i(p, w)] > 0$  or  $b_i(p, w) > b_i(p, w)^2$ . Hence,  $0 < b_i(p, w) < 1$ . Next consider the case  $\sigma(p, w) < 0$ . Then  $b_i(p, w) [1 - b_i(p, w)] < 0$ . This means that either  $b_i(p, w) < 0$  or  $b_i(p, w) > 1$ . In view of (15.4) not all  $b_i(p, w)$  can be negative; nor can all be larger than one. Thus, some  $b_i(p, w)$  are negative while others are positive. Since  $\theta$  is positive,  $\zeta$  in (15.5) is also positive. Therefore, the sign of  $u_{ii}$  is equal to the sign of  $b_i(p, w)$ . For the type of utility function under discussion, the strong quasi-concavity condition specializes to  $\sum_j z_j^2 u_{jj} < 0$ ,  $\sum_j z_j u_j = 0$ . This implies that at most one of the  $u_{jj}$  can be positive, and all others have to be negative. Thus, one and not more than one  $b_i(p, w)$  is positive and hence larger than unity. All others are negative.

For practical purposes one can take the  $b_i$  to be all positive. The other alternative is clearly pathological. The theorem is of importance because it is one of the few statements of a qualitative nature that can be made in demand analysis. Under additive utility inferior commodities ( $b_i < 0$ ) are ruled out as

exceptions. Actually, inferiority of a commodity occurs usually only when close substitutes, of a superior nature, are available. Strong separability of the utility function does not allow for the existence of such close substitutes. Another consequence of a qualitative nature is given by

*Corollary 15.2*

If the utility function is strongly separable in the elementary commodities, the cross-substitutions  $k_{ij}$ ,  $i \neq j$ , are either (i) all positive or (ii) negative when  $i$  and  $j$  are inferior commodities and positive when  $i$  or  $j$  is the only superior commodity.

*Proof*

Use (15.1) and the proof of Theorem 15.1.

A discussion of the *Linear Expenditure System (L.E.S.)* may serve to illustrate some of the issues mentioned in this section. Assume that the preference ordering can be adequately represented by the *Stone-Geary utility function*

$$u = \sum_i \beta_i \log(x_i - \gamma_i) \quad (15.8)$$

which was first proposed in Geary (1949) and used in Stone (1954). In this utility function  $\beta_i$  and  $\gamma_i$  are constants – at least they do not depend on the quantities of commodities. The existence of a real-valued utility function requires that  $\gamma_i \leq x_i$ . To be differentiable with respect to the  $x_i$ ,  $\gamma_i = x_i$  has to be ruled out. For the utility function to be increasing the  $\beta_i$  have to be positive. Without loss of generality one may assume that  $\sum_i \beta_i = 1$  and thus  $\beta_i \leq 1$ .

Utility is maximized under the budget condition  $\sum_i p_i x_i = w$  if and only if the  $x_i$  satisfy

$$\frac{\beta_i}{x_i - \gamma_i} = \lambda p_i, \quad (15.9)$$

where  $\lambda = \theta(p, w)$  is a positive factor of proportionality. Solving (15.9) for  $x_i$  and eliminating  $\lambda$  to take the budget condition into account, one finds

$$x_i = \gamma_i + \frac{\beta_i}{p_i} \left( w - \sum_j p_j \gamma_j \right) \quad (15.10)$$

as the typical demand equation. It is usually presented as

$$p_i x_i = p_i \gamma_i + \beta_i \left( w - \sum_j p_j \gamma_j \right), \quad (15.11)$$

where the left-hand side is the expenditure on commodity  $i$ . This expenditure depends linearly on the prices and  $w$ . This explains its name – linear expenditure function – which is attributed to Klein and Rubin (1947). Note that it is not linear in the  $\beta$ 's and  $\gamma$ 's.

As is clear from (15.11) the marginal budget share,  $b_i(p, w) = \beta_i$ , i.e. is constant and positive. Apply (13.3) to obtain

$$k_{ii} = -\sigma(p, w) \frac{(1 - \beta_i)\beta_i}{p_i^2}, \quad (15.12a)$$

and for  $i \neq j$

$$k_{ij} = \sigma(p, w) \frac{\beta_i \beta_j}{p_i p_j}, \quad (15.12b)$$

where

$$\sigma(p, w) = \left( w - \sum_j p_j \gamma_j \right) > 0.$$

Since  $\beta_i/p_i = \partial x_i / \partial w$ , implication (15.1) is verified by (15.12b), i.e. there is no specificity in the relationship among the demands for various goods. The own-price elasticity (15.7) can be expressed as

$$\begin{aligned} \epsilon_{ii} &= - (1 - \beta_i) \beta_i \frac{\sigma}{p_i \gamma_i + \beta_i \sigma} - \beta_i \\ &= - \beta_i \frac{p_i \gamma_i + \sigma}{p_i \gamma_i + \beta_i \sigma} \end{aligned} \quad (15.13)$$

indicating that the own-price elasticity is, in absolute value, larger than  $\beta_i$ . For sufficiently small  $\gamma_i$ ,  $\epsilon_{ii} \sim -1$ . The implied income elasticity is given by

$$\eta_i = \beta_i \frac{w}{p_i \gamma_i + \beta_i \sigma}. \quad (15.14)$$

Consequently,

$$\epsilon_{ii} = -\eta_i \frac{p_i \gamma_i + \sigma}{w}.$$

For sufficiently small  $\gamma_i$ , Pigou's Law is clearly satisfied.

An interesting interpretation of (15.11) is due to Samuelson (1947b). Let  $\gamma_i$  be the a priori decided amount of commodity  $i$  to be acquired anyway. It is the committed quantity of  $i$ . The committed quantities require  $\sum_j p_j \gamma_j$  of the budget. What remains, i.e.  $w - \sum_j p_j \gamma_j$ , is *supernumerary income* to be allocated according to the constant fractions  $\beta_i$ . This interpretation suggests that the  $\gamma_i$  are positive, which is theoretically not a necessary condition.

In applying the L.E.S. or any other demand system to actual data on an aggregated level one should realize that the theory outlined here is a comparative static one referring to the individual consumer deciding on quantities of elementary commodities. This theory may suggest certain properties and interpretations of demand systems, extremely useful for their empirical application. The additional requirements for the empirical validity of these systems are of considerable importance and fall outside the scope of this chapter. Consequently, a further discussion of demand systems, which are first of all tools of empirical analysis, is not undertaken here.<sup>4</sup>

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<sup>4</sup>See Brown and Deaton (1972), Philips (1974), Powell (1974), Theil (1975, 1976), Barten (1977), and Deaton (forthcoming).

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