On Cores and Equilibria of Productive
Economies with a Measure Space of Consumers:
An Example

In a market economy where production and consumption decisions are decentralized and where coalitions exercise control over production possibilities, an equilibrum concept has to supply a satisfactory solution as to how profits are distributed. If all production possibility sets are cones, maximal profits at any equilibrium will be zero. Hence, the budget constraint of any consumer will be determined by the value of his endowment alone. A problem of profit distribution does not exist. In the general case, stability considerations with respect to possible actions of coalitions do require that at the equilibrium the profit distribution should guarantee to each coalition a payment at least as high as the maximal profit relative to its production possibility set. These considerations led to the formulation of an equilibrium with a stable profit distribution as proposed in [2] and [4].

In his 1968 article [3] Hildenbrand showed that for an economy with an atomless measure space of consumers the core and the set of equilibrium allocations coincide if production possibility sets are additive. This was the natural extension of Aumann's earlier result in [1]. It is easily seen that the notion of equilibrium used by Hildenbrand is a special case of an equilibrium with a stable profit distribution. However, as the following example shows, one cannot hope to obtain an equivalence result even in the zero profit case if the production correspondence is strictly superadditive.

Let the set of consumers A be the interval $[0, 1]$ and $\mathfrak{A}$ all its Borel subsets with Lebesgue measure μ. Let (S_1, S_2) be a partition of $[0, 1]$ such that $\mu(S_1) = \mu(S_2) = 1/2$. The commodity space is R_+^2, and endowments are constant on A such that

$$\int_A e \, d\mu = (2, 0)$$

The preference relations for the members of the two coalitions S_1 and S_2 are depicted in Figs. 1 and 2 and are defined by

$$u_a(x) = \begin{cases} \min\{(3/2) x_1, x_2 + 1/2\} & \text{for} \quad x_1 \geqslant 1/2, \quad x_2 \geqslant 1/4 \\ 3. \min\{(1/2) x_1, x_2\} & \text{otherwise} \end{cases}$$

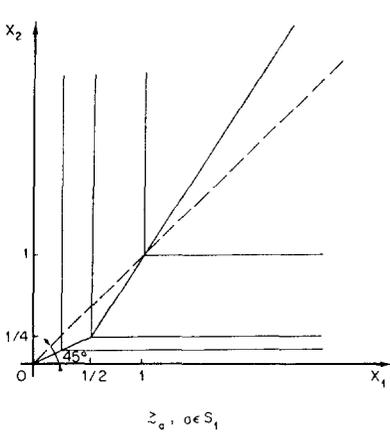<div align="center">409</div>

FIGURE 1   FIGURE 2
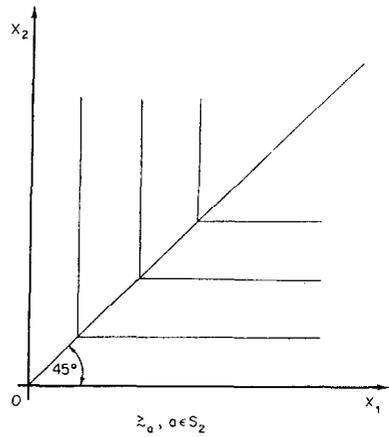
for $a \in S_1$ and by $u_a(x) = \min\{x_1, x_2\}$ for $a \in S_2$. The production correspondence $Y(.)$ is given by $Y(S) = \{y \in R^2 \mid y \leqslant (-z_1, z_2),\ z_1 \geqslant 0,$ $z_2 \geqslant 0,\ z_2 = 2\mu(S \cap S_2)z_1\}$. Y is strictly super-additive and only those coalitions which contain subcoalitions of S_2 with positive measure are able to produce positive amounts of commodity two.

It is easy to verify that the allocation $f: A \to R^2_+$ with $f(a) = (1, 1)$ for all $a \in A$, $p = (1, 1)$, and $y = (-1, 1)$ is the only competitive equilibrium with $\sup p.Y(S) = 0$ for all $S \subset A$. On the other hand, it will be shown that the allocation $g: A \to R^2_+$ such that $g(a) = (2, 2)$ for $a \in S_2$ and $g(a) = (0, 0)$ for $a \in S_1$ is unblocked.

Suppose there exists a coalition S, $\mu(S) > 0$, which blocks g using an allocation h and production bundle $y \in Y(S)$. Clearly, $S \neq A$, $S \not\subset S_1$, $S \not\subset S_2$. The remaining case which has to be considered is $\mu(S \cap S_1) > 0$ and $\mu(S \cap S_2) > 0$. Without loss of generality one may assume that

$$\int_S h = \int_S e + y, \quad \text{i.e.,} \quad \int_S h_1 + \frac{1}{2\mu(S \cap S_2)} \int_S h_2 = 2\mu(S)$$

Since $h(a) \succ_a g(a)$ a.e. $a \in S$

$$h_1(a) > 2$$
$$\qquad\qquad \text{a.e.} \quad a \in S \cap S_2$$
$$h_2(a) > 2$$

and

$$h(a) \gg 0 \quad \text{a.e.} \quad a \in S \cap S_1$$

Hence

$$\int_S h_1 > 2\mu(S \cap S_2)$$

$$\int_S h_2 > 2\mu(S \cap S_2)$$

However,

$$\int_S h_1 = 2\mu(S) - \frac{1}{2\mu(S \cap S_2)} \int_S h_2 > 2\mu(S \cap S_2)$$

implies $2\mu(S \cap S_1)\, 2\mu(S \cap S_2) > \int h_2$.

Since $2\mu(S \cap S_1) \leqslant 1$, $\int_S h_2 < 2\mu(S \cap S_2)$. Hence h cannot be a blocking allocation for S.

With a few minor changes in the assumptions the example has a clear geometric representation which has been given in Fig. 3. Assume that $A = [0, 2]$ instead of the unit interval, that for the fixed partition (S_1, S_2) $\mu(S_1) = \mu(S_2) = 1$ holds, and that the production correspondence is given by $Y(S) = \{y \in R^2 \mid y \leqslant (-z_1, z_2), \; z_1 \geqslant 0, \; z_2 \geqslant 0, \; z_2 = \mu(S \cap S_2)z_1\}$
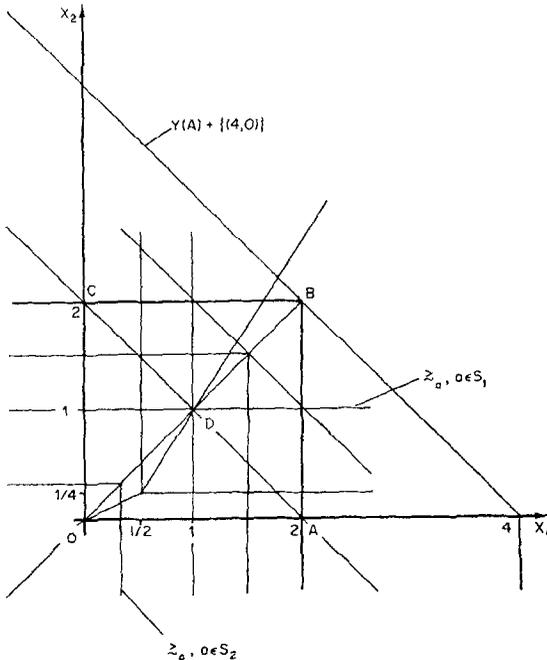

FIGURE 3

Furthermore, $\int e\,d\mu = (4, 0)$ where e is constant on A. The only equilibrium allocation is again $f(a) = (1, 1)$ for all $a \in A$ at prices $p = (1, 1)$ and production bundle $y = (-2, 2)$. In Fig. 3 the set $\{Y(A) + \{(4, 0)\}\} \cap R_+^2$ is the set of aggregate, attainable consumption vectors. $0ABC$ is the appropriate 'Edgeworth box' for the production $y = (-2, 2)$ represented by $B \cdot 0$ and B have been chosen as the origins of the typical consumption sets for $a \in S_1$ and for $a \in S_2$, respectively. D represents the equilibrium allocation. On the other hand, the allocation $g : A \to R_+^2$ such that $g(a) = (0, 0)$ for $a \in S_1$ and $g(a) = (2, 2)$ for $a \in S_2$ represented by point 0 is unblocked.

References

1. R. J. AUMANN, Markets with a Continuum of Traders, *Econometrica* **32** (1964), 39–50.
2. V. BOEHM, Stable Firm Structures and the Core of an Economy with Production, Working Paper No. CP 345, Center for Research in Management Science, University of California, Berkeley, 1972.
3. W. HILDENBRAND, The Core of an Economy with a Measure Space of Economic Agents, *Rev. Econ. Studies* **35** (1968), 443–452.
4. D. SONDERMANN, Existence of Stable Profit Distributions and Equilibria in Coalition Production Economies, CORE Discussion Paper No. 7141, Université Catholique de Louvain, 1972.

VOLKER BOEHM

Institut für Gesellschafts- und Wirtschaftswirssenschaften Universität Bonn, Bonn, Germany