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Firms and Market Equilibria in a Private Ownership Economy*

By

Volker Boehm, Bonn

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1. Introduction

The traditional economic concept of a competitive equilibrium of a private ownership economy as e. g. defined in Debreu's Theory of Value [6] is based on the assumptions that (1) the number of the participating firms in the market is fixed, (2) each firm follows the prescribed rule of maximizing profit at given prices, and (3) the percentage share of each consumer in the profit of each firm is fixed. One possible justification of these assumptions is that the underlying institutional structure of the market economy permits and/or imposes a two step procedure. In the first step, all consumers who ultimately exercise control over all productive facilities decide prior to the opening of the market which productive facilities shall be used and how the profit of the selected producers are to be distributed. After such an agreement has been reached each participating producer is told to maximize his profit at the prevailing market price. Then, the market will open and all agents, consumers and producers, carry out their plans.

Economically, assumptions (1) and (3) are rather restrictive. If the procedure of determining the features of (1) and (3) is of the above type, one would expect that there exists some underlying structure

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of ownership control over all productive facilities by the consumers. Since market prices are not known while consumers decide on the set of firms, at the final market equilibrium consumers may want to revise their original decision. This would lead to another round of determining a set of firms and of profit shares which may result again in an unstable market structure at the new market equilibrium, since no real market information is available while the consumers decide on the set of firms and on the profit shares. In such a case it would be much more desirable to make the outcome of such a procedure part of the market mechanism. This way such phenomena as entry and exit of firms, i. e. the selection of profitable and efficient production facilities can be incorporated in the model as well as the determination of the final profit distribution in the market, which, if appropriately done, will reflect the ownership structure of the economy. This paper attempts to describe these phenomena for a typical private ownership economy, i. e. an economy in which all commodity resources are owned by consumers and in which consumers exercise control over all productive facilities. The following section contains the necessary conceptual extensions of the model by Debreu. In section 3 a new equilibrium concept is defined and some preliminary results are given. Section 4 supplies a general existence theorem for such equilibria.

2. The Model

The consumption characteristics of a typical consumer i out of the set of consumers $I=\{1,\ldots,n\}$ are described by the triple $(X_i, e_i, \underset{i}{\succsim})$, i. e. his consumption set X_i, a non-empty subset of the commodity space R^l, his preference relation $\underset{i}{\succsim}$, and his endowment $e_i \in R^l$.

The total set of producers or firms which may participate in the market will be denoted by $\mathscr{J}$, a non-empty subset of the non-negative integers. Each firm $j \in \mathscr{J}$ will be described by a production possibility set Y_j, a non-empty subset of the commodity space R^l. Each firm is owned and controlled by some group of consumers $S \subset I$ which implies that S decides whether its firm j participates in the market. In general a group of consumers S could control more than one firm or none at all. In the first case this would imply that S controls several separate productive units in the economy. Without too much loss of generality one may assume that these units can be combined into one firm, so that each group S owns at most one firm. For the opposite case in which a group of consumers does not own a firm, it will be said for the purpose of the analysis that S owns a firm j with $Y_j=\{0\}$, i. e. S owns a firm whose only activity consists of doing nothing. Finally, as a mathematical convention, the empty group of consumers will be said to own the firm $j=0$ with $Y_0=\{0\}$. Completing notational matters with these assumptions $\mathscr{J}$ will be the set $\{0, 1, \ldots, 2^n-1\}$ and, with an appropriate numbering of the groups of consumers, firm j will be owned by S_j, $j \in \mathscr{J}$. Hence the economy may be described by the list $\mathscr{E}=\{I, (X_i), (e_i), (\underset{i}{\succsim}), (Y_j)\}$.

The typical situation in the economy after the market opens will now be as follows. There will be a non-empty subset $J \subset \mathscr{J}$ of firms which participate in the market. Once a firm has entered the market, i. e. its owners decided that their firm should produce, it will decide on some production plan and also on how its profits, at the given market price, will be divided among the consumers. It is always assumed that the actual decision-making within the firm is not costly and that it is independent of the consumption characteristics of its owners. If each firm $j \in J$ decides on a production plan $y_j \in Y_j$, $\sum\limits_{j \in J} y_j$ will be the aggregate supply of the productive sector. Let θ_{ij} be the profit share of consumer i in firm j, where $0 \leqq \theta_{ij} \leqq 1$ and $\sum\limits_{i \in I} \theta_{ij}=1$ for $j \in J$. Let $P=\{p \in R^l{}_+ | \sum\limits_{i=1}^{l} p_i=1\}$ denote the set of possible prices. Then, by the convention of signs for the bundles $y_j \in Y_j$, the scalar product $p \cdot y_j$ will be firm j's profit and consumer i will receive an amount of $t_i = \sum\limits_{j \in J} \theta_{ij}\, p \cdot y_j$. For the remaining analysis it is sufficient to consider the aggregate payments t_i which consumer i receives. Hence the actions of the productive sector in any market situation, i. e. if prices p prevail, are completely specified by the set $J \subset \mathscr{J}$, production bundles $y_j \in Y_j$, $j \in J$, and profit payments $t_i \in R^1$, $i \in I$.

Definition: A triple $[J, (y_j), (t_i)]$ is called a firm structure relative to prices p if

(1) $J \subset \mathscr{J}$

(2) $y_j \in Y_j$, $j \in J$

(3) $\sum\limits_{j \in J} p \cdot y_j = \sum\limits_{i \in I} t_i$

One of the standard assumptions on the properties of the production possibility sets in general equilibrium theory is that $0 \in Y_j$ for all firms. It should be clear that with such an assumption a true distinction of whether a firm participates in the market or not is not possible at all prices. This assumption would actually eliminate the economic problem of selecting profitable firms since any member of the all firm set could always avoid a loss at all prices and still remain in the market. On the other hand $0 \notin Y_j$ seems to be the only way to allow for set-up costs or fixed costs of a firm within such a general framework of production. Since the non-exclusion of $0 \notin Y_j$ is the only way to create situations in which some firm's maximal profit is negative, it is also the only possibility to distinguish firms according to their profitability. Due to the normalization of prices absolute profit levels are not a meaningful criterion. Hence, in a theory, which attempts to explain why not necessarily all firms will participate in the market at all prices, one has to use assumptions which may force a firm to shut down at certain prices. In what follows it will always be assumed that, for some firm j, $0 \notin Y_j$.

3. Stable Firm Structures

Since the ultimate control over productive facilities lies in the hands of the consumers, an equilibrium concept should take into account that any group of consumers which is dissatisfied with its profit payments from a given firm structure and which could actually achieve higher payments for all of its members from its own firm, will always bargain for at least the maximum profit from its own firm. This argument provides the basis for the following definition.

Definition: A list $[J, (y_j), (t_i)]$ is called a stable firm structure relative to prices p if

(1) $[J, (y_j), (t_i)]$ is a firm structure relative to prices p,

(2) $t_i \geqq 0$ for all $i \in I$

(3) $\sum_{i \in S_j} t_i \geqq \sup \{p \cdot y \mid y \in Y_j\}$ for all $S_j \subset I$.

The definition of a stable firm structure and its interpretation describe the production sector of a market economy. All production decisions are decentralized and made by the individual firm. Although the ultimate control over the available production possibilities is exercised by consumers their influence is only traceable with regard to their desire to achieve a high income level. In this respect the definition guarantees a certain "maximal" income to each consumer relative to his ownership of productive facilities. On the other hand, the definition allows for free entry and exit of all possible firms using only minimal assumptions on the cooperation among consumers to guarantee actual participation of any firm. Combining the feature that each consumer maximizes his preference relation subject to his income with the above concept yields the following notion of equilibrium.

Definition: A list $[(\bar{x}_i), J, (\bar{y}_j), (\bar{t}_i), \bar{p}]$ is a market equilibrium with a stable firm structure if

(1) $\bar{x}_i$ maximizes $\underset{i}{\succsim}$ in the budget set
$\{x_i \in X_i | \bar{p} \cdot x_i \leqq \bar{p} \cdot e_i + \bar{t}_i\}$ for all $i \in I$

(2) $[J, (\bar{y}_j), (\bar{t}_i)]$ is stable at $\bar{p}$

(3) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i + \sum_{j \in J} \bar{y}_j.$

Thus an equilibrium has the two main properties that no group of consumers through independent action can increase its total income and no consumer can achieve a higher level of satisfaction using his own income. The concept represents a generalization of the usual competitive equilibrium. In fact, one can show under traditional assumptions that, for an economy where all firms have been formed, the competitive equilibrium is also one with a stable firm structure if each firm distributes profits only to its owners. In general, however, the concept is independent of any behavioral assumption for firms; in particular, profit maximization of firms will in general not be present at the equilibrium point.

There exists a second relationship between stable firm structures and competitive behavior of firms which is stated in the following lemma.

Lemma 1: Let $[J, (y_j) (t_i)]$ be a stable firm structure relative to p such that the set J defines a partition of I, i. e., for any j' and j'' contained in J, $S_{j'} \cap S_{j''} = \phi$, and $\bigcup_{j \in J} S_j = I$. Then for all $j \in J$,

(1) $p \cdot y_j = \text{Max}\ \{p \cdot y | y \in Y_j\}$

(2) $\sum_{i \in S_j} t_i = p \cdot y_j.$

Proof: Consider the partition $\{S_j\}$. The stability implies that for all $j \in J$,

$$\sum_{i \in S_j} t_i \geqq \text{Max}\ \{p \cdot y | y \in Y_j\} \geqq p \cdot y_j.$$

Hence

$$\sum_{j \in J} p \cdot y_j = \sum_{i \in I} t_i = \sum_{j \in J} \sum_{i \in S_j} t_i \geqq \sum_{j \in J} p \cdot y_j$$

which yields (1) and (2). Q. E. D.

Finally, if the ownership structure for the firms in the market economy is interpreted in the sense of a coalition production economy (see e. g. [2] and [8]) one obtains the following result.

Lemma 2: Let $[(\bar{x}_i), J, (\bar{y}_j), (\bar{t}_i), \bar{p}]$ be a market equilibrium with a stable firm structure. Then $(\bar{x}_i)$ is an allocation in the core.

Proof: Suppose the statement were false. Then there would exist a non-empty coalition S_j which could block $(\bar{x}_i)$, i. e., there exist (x_i), $i \in S_j$, $y \in Y_j$ such that

(1) $x_i \succ_i \bar{x}_i, \; i \in S_j$

(2) $\sum_{i \in S_j} x_i = \sum_{i \in S_j} e_i + y.$

Yet (1) implies $\bar{p} \cdot x_i > \bar{p} \cdot e_i + \bar{t}_i$, $i \in S_j$. Hence

$$\bar{p} \cdot \sum_{i \in S_j} e_i + \bar{p} \cdot y > \bar{p} \cdot \sum_{i \in S_j} e_i + \sum_{i \in S_j} \bar{t}_i \geqq \bar{p} \cdot \sum_{i \in S_j} e_i + \text{Max} \{\bar{p} \cdot y | y \in Y_j\}$$

implying

$$\bar{p} \cdot y > \text{Max} \{\bar{p} \cdot y | y \in Y_j\}$$

which is a contradiction. Q. E. D.

4. Existence of Equilibria with Stable Firm Structures

This section contains a main existence theorem the proof of which is a straightforward extension of the existence proof for competitive equilibria given by Debreu in [6]. His notation and definitions will be followed as closely as possible. The major differences between Debreu's proof and the one presented here are a consequence of the different equilibrium concept. Since his method of proof is only directly applicable to an economy with a fixed set of firms where each firm can always produce at a non-negative profit, it was necessary to find a procedure which determines a set of firms and supply bundles at each price. More precisely, for every price a stable firm structure had to be found. The crucial argument is taken directly from the definition of a stable firm structure which has an immediate interpretation as a solution of a side-payment game for each price. Its core defined in an appropriate way yields the necessary continuity property of the payoffs to show existence of an equilibrium. Lemma 1 represents the crucial step. It also supplies the basic argument for the construction of the set of firms, defined for each price by the dual variables of a linear program, which is an application of the result on cores of balanced games.

Definition: A set of firms J is called balanced if the collection of coalitions controlling J is a balanced family, i. e. if there exist weights $d_j > 0$, $j \in J$, such that

$$\sum_{\substack{j \in J \\ S_j \ni i}} d_j = 1 \text{ for all } i \in I.$$

Let $Y = \bigcup_{J \subset \mathcal{J}} \sum_{j \in J} Y_j$ denote the aggregate production possibility set.

Theorem:

Let the economy $\mathscr{E}$ be described by

$$\mathscr{E} = \{I, (X_i), (e_i), (\underset{i}{\gtrsim}), (Y_j)\}.$$

Then $\mathscr{E}$ has a market equilibrium with a stable firm structure if for all $i \in I$

(C1) $X_i \subset R^l$ is closed, convex, and bounded from below,

(C2) i is locally not satiated,

(C3) $\underset{i}{\gtrsim}$ is a complete, transitive, and continuous preordering on X_i such that the set $\{x_i \in X_i | x_i \underset{i}{\gtrsim} x_i'\}$ is convex for every $x_i' \in X_i$,

(C4) there exists $x_i^0 \in X_i$ such that $x_i^0 \ll e_i$,

and if

(P1) $0 \in Y_j$ for $S_j = \{i\}$ for all $i \in I$

(P2) Y_j is closed for all $j \in \mathcal{J}$

(P3) Y is closed

(P4) $Y \cap (-Y) \subset \{0\}$

(P5) $Y \supset R_-^l$

and for every balanced set of firms J and weights (d_j)

(P6) $\sum_{j \in J} d_j Y_j \subset \sum_{j \in J} Y_j$

(P7) $\sum_{j \in J} Y_j$ is convex.

Assumptions (C1)—(C4) are standard for any existence proof in general equilibrium theory. On the production side (P2)—(P5) are the appropriate generalizations of the assumptions usually made in a competitive model. (P1) assures that actual profit payments to each consumer will be non-negative. (P6) and (P7) describe a specific ownership distribution and a specific separation of the total productive possibilities for which it seems difficult to give a direct and complete economic interpretation or characterization in terms of the individual sets Y_j. However, as the following two examples demonstrate, assumptions (P6) and (P7) describe many typical cases including the traditional model with $0 \in Y_j$ for every firm as well as situations in which firms have set-up costs.

Clearly, $0 \in Y_j$ for all $j \in \mathscr{J}$ and (P7) imply (P6) since

$$\sum_{j \in J} d_j Y_j \subset \sum_{j \in J} \operatorname{conv} Y_j = \sum_{j \in J} Y_j.$$

On the other hand, consider an economy in which "most" of the firms have the same convex cone as production possibility set. If the remaining firms are controlled by disjoint, proper subsets of I which do not form a partition of I and if their production possibility sets are any arbitrary subsets of this cone then (P6) and (P7) will hold.

Definition: The set of attainable states of the economy $\mathscr{E}$ is an $(n+1)$-list of vectors $(x_1, \ldots, x_n, y) \in R^{l(n+1)}$ such that for all $i \in I$, $x_i \in X_i$, $y \in Y$, and $\sum_{i \in I} x_i = \sum_{i \in I} e_i + y$.

Proof: First, one observes that (C1), (P3)—(P5) imply that the set of attainable states of the economy is closed and bounded (Debreu [6], Theorems 1 and 2, p. 77). Hence, most arguments can be carried out in a well-chosen compact cube in the commodity space (Debreu [6], proof of Theroem 1, p. 83). Let K_1 be a closed cube of R^l with center at the origin containing in its interior the set of all attainable consumption and production plans. For $i \in I$, define $X_i^1 = X_i \cap K_1$ and for $S_j \subset I$, define $Y_j^1 = Y_j \cap K_1$. Following Debreu one can show the existence of an equilibrium for the economy $\mathscr{E}_1 = \{I, (X_i \cap K_1), (e_i), (\underset{i}{\succsim}), (Y_j \cap K_1)\}$. Although any equilibrium will be contained in this truncated economy, one cannot conclude that any equilibrium with a stable firm structure for $\mathscr{E}_1$ is also an equilibrium with a stable firm structure for $\mathscr{E}$. Therefore, an increasing sequence of cubes K_q with the associated truncated economies $\mathscr{E}_q$ will be constructed, where K_q becomes infinitely large. Arguments similar to the ones used by Debreu ([4], Section 3) and by Hildenbrand ([9], proof of Theorem 2) will then establish that there exists an equilibrium for the unrestricted economy $\mathscr{E}$.

The proof will now be carried out in several steps Let

$$v_j(p) = \operatorname{Max} \{p \cdot y \mid y \in Y_j \cap K_1\}, \quad j \in \mathscr{J}$$

Lemma 1: If for all $j \in \mathscr{J}$, Y_j^1 is compact, non-empty and if for $j \in \mathscr{J}$, such that $S_j = \{i\}$, $0 \in Y_j$, then for each $p \in P$ there exists a payoff vector $h \in R^n$, $h \geqq 0$, and a generalized characteristic vector $d \in R^{2^n}$, $0 \leqq d \leqq 1$, such that

(1) $\sum_{i \in S_j} h_i \geqq v_j(p)$ for all $S_j \subset I$

(2) $\sum_{i \in I} h_i = \sum_{j \in \mathscr{J}} d_j v_j(p)$

(3) $\sum_{S_j \ni i} d_j = 1$ for all $i \in I$.

Proof: Since Y_j^1 is compact, $v_j(p)$ exists for all j at any p. In particular, $v_j(p) \geqq 0, S_j = \{i\}$. Denote by $e_j \in R^n$ the characteristic vector of coalition S_j, i. e., $(e_j)_i = 1$, if $i \in S_j$, and zero otherwise; and $e_\phi = (0, \ldots, 0)$. Let $E = (e_j)$ be the matrix of all 2^n vectors. Arranging the elemetns in I and E in the appropriate order, one can rewrite (1) as

$$Eh \geqq v(p).$$

Consider the following linear program and its dual.

Primal: Min $\sum_{i \in I} h_i$
Subject to $Eh \geqq v(p)$

Dual: Max $d \cdot v(p)$
Subject to $dE = 1 \quad d \geqq 0$.

1 denotes a vector of appropriate dimension each element of which is equal to one.

Since $v(p)$ is finite both problems are feasible. Then, by standard duality arguments, both have optimal solutions (d^*, h^*) such that

$$d^* \cdot v(p) = \sum_{i \in I} h_i^*.$$

Hence (d^*, h^*) satisfy (1), (2), and (3). Q. E. D.

Clearly, for a given p, d^* and h^* will not be unique. Define

$$\delta(p) = \{d | d \text{ a solution of the dual at } p\}$$

$$\tau(p) = \{h | h \text{ a solution of the primal at } p\}.$$

Then Lemma 1 states that $\delta(p) \neq \phi$ and $\tau(p) \neq \phi$.

Lemma 2: If Y_j^1 is compact for all $j \in \mathscr{J}$, then δ and τ are upper hemi-continuous and convex valued correspondences, and τ admits a continuous selection.

Proof: Y_j^1 compact implies that $v(p)$ is a continuous function. The dual as the following maximization problem

$$\Pi(p) = \text{Max}\ \{d \cdot v(p) \mid dE = 1,\ d \geqq 0\}$$

yields δ upper hemi-continuous by standard maximization results since $d \cdot v(p)$ is a continuous function from $\{d|dE=1,\ d\geqq 0\} \times P$ into R and $\{d|dE=1,\ d\geqq 0\}$ is trivially continuous in p. Let $d^1 \in \delta(p)$ and $d^2 \in \delta(p)$ and $0 \leqq \lambda \leqq 1$. Then $\lambda d^1 + (1-\lambda)\ d^2 \in \{d|dE=1,\ d\geqq 0\}$, a convex set. Furthermore, $d^1 \cdot v(p) = d^2 \cdot v(p)$ implies $[\lambda d^1 + (1-\lambda)\, d^2] \cdot v(p) = \lambda d^1 \cdot v(p) + (1-\lambda)\ d^2 \cdot v(p) = d^1 \cdot v(p)$. Hence, $\delta(p)$ is convex valued. Similarly, for the primal, one knows that $\beta(p) = \{h|Eh \geqq v(p)\}$ is convex valued. Take $h^1 \in \beta(p)$, $h^2 \in \beta(p)$. Then $E\ [\lambda h^1 + (1-\lambda) h^2] = \lambda Eh^1 + (1-\lambda)\ Eh^2 \geqq \lambda v(p) + (1-\lambda)\ v(p) = v(p)$.

Furthermore, from the duality property, the objective function of the primal is continuous in p since $\Pi(p) = \{h \cdot \mathbf{1} \mid h \in \tau(p)\}$ where $\Pi(p)$ was shown to be continuous. Hence τ maps P into some compact subset of R^n. For τ to be upper hemi-continuous, it is sufficient to show that its graph is closed. Consider $p^n \to p$, $h^n \to h$, $h^n \in \tau(p^n)$. Then the continuity of $\Pi(p)$ and $h^n \in \tau(p^n)$ implies $h \cdot \mathbf{1} = \Pi(p)$. Hence $h \in \tau(p)$. To show convexity, let $h^1 \in \tau(p)$ and $h^2 \in \tau(p)$. Then $[\lambda h^1 + (1-\lambda)\ h^2] \cdot \mathbf{1} = \lambda h^1 \cdot \mathbf{1} + (1-\lambda)\ h^2 \cdot \mathbf{1} = h^1 \cdot \mathbf{1}$.

It remains to be shown that τ admits a continuous selection. Consider the following linear program

$$\begin{aligned} &\text{Min } h_1 \\ &\text{Subject to } \ Eh \geqq v(p) \\ &\qquad\qquad\quad\ \mathbf{1} \cdot h \leqq \Pi(p). \end{aligned}$$

Clearly, the feasible set for this program is $\tau(p)$, a non-empty, compact, and convex subset of R^n of dimension at most equal to $n-1$, which implies that the program has an optimal solution. Let

$$f_1(p) = \text{Min}\ \{h_1 \mid Eh \geqq v(p),\ \mathbf{1} \cdot h \leqq \Pi(p)\}$$

and

$$\tau_1(p) = \{h|h \in \tau(p),\ h_1 = f_1(p)\}.$$

Using the same arguments as before for the correspondence τ, it follows immediately that f_1 is a continuous function, τ_1 is upper hemi-continuous, and $\tau_1(p)$ is non-empty, compact, and of dimension at most $n-2$. Proceeding in the same fashion, define for $i = 2, \ldots, n$

$$f_i(p) = \text{Min}\ \{h_i \,|\, Eh \geqq v(p), \mathbf{1} \cdot h \leqq \Pi(p), e_{\{i-k\}} \cdot h \leqq f_{i-k}(p), k = 1, \ldots, i-1\}$$

and

$$\tau_i(p) = \{h \mid Eh \geqq v(p), \mathbf{1}\cdot h \leqq \Pi(p), e_{\{i-k\}}\cdot h \leqq f_{i-k}(p), k=1,\ldots, i-1, e_{\{i\}}\cdot h = f_i(p)\}$$

$$= \{h \mid h \in \tau_{i-1}(p), e_{\{i\}}\cdot h = f_i(p)\}.$$

Clearly, for all $i=2,\ldots, n$, f_i is continuous, $\tau_i(p)$ is non-empty, compact, of dimension at most equal to Max $\{0, n-i\}$, and τ_i is upper hemi-continuous. In particular, $\tau_n(p)$ will be the unique point $[f_1(p), \ldots, f_n(p)]$. Since τ_n is upper hemi-continuous the function $g{:}P \to R^n$ defined by $g(p) = [f_1(p),\ldots, f_n(p)]$ is continuous and for all $p \in P$, $g(p) \in \tau(p)$. Q. E. D.

Let $\bar{\eta}_j(p) = \{\bar{y}_j \in Y_j^1 \mid p\cdot\bar{y}_j = v_j(p)\}$. Under assumption ($P$ 2) $\bar{\eta}_j(p)$ is non-empty and $\bar{\eta}_j$ is upper hemi-continuous. For each $p \in P$ and $d \in \delta(p)$ define a supply correspondence

$$\eta(d, p) = \sum_{j\in\mathscr{J}} d_j\, \bar{\eta}_j(p)$$

Since the strictly positive components of d define a balanced set $J(d)$ it follows that $\eta(d, p) = \sum_{j\in J(d)} d_j\bar{\eta}_j(p)$, $d \in \delta(p)$. Now define as the aggregate supply correspondence

$$\eta(p) = \text{conv} \bigcup_{d\in\delta(p)} \eta(d, p)$$

where conv denotes convex hull.

Lemma 3: If Y_j^1 *is compact and non-empty, and if (P6) and (P7) hold, then* $\eta(p)$ *is non-empty,* η *is an upper hemi-continuous correspondence, and* $y \in \eta(p)$ *implies*

(1) $p.y = \Pi(p)$,

(2) there exists a set $J \subset \mathscr{J}$ *such that* $y \in \sum_{j\in J} Y_j$.

Proof: The non-emptiness follows from Lemma 1 and from the definition of $\eta(p)$.

Let $y \in \eta(d, p)$, i. e. $y = \sum_{j\in\mathscr{J}} d_j\bar{y}_j$ where $(d_j) \in \delta(p)$ and $\bar{y}_j \in \bar{\eta}_j(p)$. Then

$$p\cdot y = p\cdot \sum_{j\in\mathscr{J}} d_j\bar{y}_j = \sum_{j\in\mathscr{J}} d_j p\cdot\bar{y}_j = \sum_{j\in\mathscr{J}} d_j v_j(p) = \Pi(p)$$

which proves (1), since the same argument can be used for any finite convex combination of points in $\bigcup_{d\in\delta(d)} \eta(d, p)$.

To prove (2) one uses the fact that with each element in $\eta(d, p)$ is associated a balanced set $J(d)$. Let $y \in \eta(p)$. Then y can be written as a convex combination of at most $l+1$ vectors $y^k \in \eta(d^k, p)$, $k=1,\ldots, l+1$, i. e., $y=\sum_{k=1}^{l+1} \lambda^k y^k$ with $0 \leqq \lambda^k \leqq 1$ and $\sum_{k=1}^{l+1} \lambda^k = 1$. Let J^k be the balanced set associated with d^k and let $y_j{}^k \in \bar{\eta}_j(p)$ be such that $y^k = \sum_{j \in J^k} d_j{}^k y_j{}^k$. Then

$$y = \sum_{k=1}^{l+1} \lambda^k y^k = \sum_{k=1}^{l+1} \lambda^k \sum_{j \in J^k} d_j{}^k y_j{}^k = \sum_{k=1}^{l+1} \sum_{j \in J^k} \lambda^k d_j{}^k y_j{}^k.$$

First one observes that $J = \bigcup_{k=1}^{l+1} J^k$ is a balanced set which is defined by the positive components of the associated vector of weights $\gamma = \sum_{k=1}^{l+1} \lambda^k d^k$. Clearly, $\gamma \in \delta(p)$ according to Lemma 2. Furthermore,

$$\sum_{k=1}^{l+1} \sum_{j \in J^k} \lambda^k d_j{}^k y_j{}^k = \sum_{j \in J} \sum_{\substack{k \\ J^k \ni j}} \lambda^k d_j{}^k y_j{}^k$$

$$= \sum_{j \in J} \sum_{\substack{k \\ J^k \ni j}} \Big(\sum_{\substack{k \\ J^k \ni j}} \lambda^k d_j{}^k \Big) \frac{\lambda^k d_j{}^k}{\sum_{\substack{k \\ J^k \ni j}} \lambda^k d_j{}^k} y_j{}^k$$

$$= \sum_{j \ni J} \gamma_j \sum_{\substack{k \\ J^k \ni j}} \frac{\lambda^k d_j{}^k}{\gamma_j} y_j{}^k \in \sum_{j \in J} \gamma_j \operatorname{conv} Y_j$$

$$= \sum_{j \in J} \operatorname{conv} \gamma_j Y_j = \operatorname{conv} \sum_{j \in J} \gamma_j Y_j \subset \sum_{j \in J} Y_j$$

where the last inclusion follows from (P6) and (P7). Hence $y \in \sum_{j \in J} Y_j$ which proves (2).

The upper hemi-continuity of η will be shown in two steps. First, it will be demonstrated that $\tilde{\eta}(p) = \bigcup_{d \in \delta(p)} \eta(d, p)$ is upper hemi-continuous.

Since for all p and all $d \in \delta(p)$, $\eta(d, p)$ is bounded it suffices to show that $\bigcup_{d \in \delta(p)} \eta(d, p)$ has a closed graph. Consider sequences $y^n \to y$, $p^n \to p$ such that $y^n \in \bigcup_{d \in \delta(p^n)} \eta(d, p^n)$. Then there exist sequences $d^n \to d$ and $y^n{}_j \to y_j$ for every $j \in \mathscr{J}$ such that $d^n \in \delta(p^n)$ and $y^n{}_j \in \bar{\eta}_j(p^n)$, $j \in \mathscr{J}$. Since for every $j \in \mathscr{J}$, $\bar{\eta}_j$ and δ have a closed graph, it follows that $y_j \in \bar{\eta}_j(p)$, $j \in \mathscr{J}$ and $d \in \delta(p)$. Hence

$$y = \sum_{j \in \mathscr{J}} d_j y_j \in \eta(d, p) \subset \bigcup_{d \in \delta(p)} \eta(d, p).$$

It remains to be shown that $\eta(p)$ has a closed graph. Let μ be any correspondence $\mu : P \to Y$, $Y \subset R^l$ and Y compact, and assume that μ has a closed graph. Consider sequences $z^n \to z$, $p^n \to p$, $z^n \in \text{conv}\, \mu(p^n)$. Then there exist sequences $z_k{}^n \to z_k$, $\lambda_k{}^n \to \lambda_k$ for $k=1,\ldots,l+1$ with $0 \leqq \lambda_k{}^n \leqq 1$ and $\sum_{k=1}^{l+1} \lambda_k{}^n = 1$ such that $z^n = \sum_{k=1}^{l+1} \lambda_k{}^n z_k{}^n$ and $z_k{}^n \in \mu(p^n)$. Since μ has a closed graph it follows that $z_k \in \mu(p)$ for every $k=1,\ldots,l+1$ and clearly $\sum_{k=1}^{l+1} \lambda_k = 1$ with $0 \leqq \lambda_k \leqq 1$ for $k=1,\ldots, l+1$. Hence $\sum_{k=1}^{l+1} \lambda_k z_k \in \text{conv}\, \mu(p)$, which completes the proof of Lemma 3.

Let $g_i(p)$, $i=1,\ldots, n$ be the i-th component of the continuous selection $g(p) \in \tau(p)$, i. e., consumer i's profit payment. Then his budget correspondence β_i can be defined as

$$\beta_i(p) = \{x_i \in X_i \cap K_1 \mid p \cdot x_i \leqq p \cdot e_i + g_i(p)\}.$$

Lemma 4: If (C1), (C4), and (P1) hold and if $Y_j{}^1$ is compact, then β_i is lower hemi-continuous at every p and has a closed graph.

Proof: (C4), (P1), and Lemma 2 imply that, for all $p \in P$, $\beta_i(p)$ is non-empty. Let $x_i{}^n \to x_i$, $p^n \to p$, and for all n, $x_i{}^n \in \beta_i(p^n)$. Hence, $p^n \cdot x_i{}^n \leqq p^n \cdot e_i + g_i(p^n)$ and the continuity on both sides imply $p \cdot x_i \leqq p \cdot e_i + g_i(p)$, i. e. $x_i \in \beta_i(p)$.

Let $x_i \in \beta_i(p)$ and $p^n \to p$. According to (C4) and since $g_i(p) \geqq 0$, $p \cdot x_i{}^0 < p \cdot e_i + g_i(p)$. Consider the straight line L passing through $x_i{}^0$ and x_i and let $a^n \in L$ be such that $p^n \cdot a^n = p^n \cdot e_i + g_i(p^n)$. Define

$$x_i{}^n = \begin{cases} a^n \text{ if } p^n \cdot a^n < p^n \cdot x_i \\ x_i, \text{ otherwise} \end{cases}$$

Clearly, $x_i{}^n \to x_i$ and, also, $x_i{}^n \in \beta_i(p^n)$ for all n. Hence, β_i is lower hemi-continuous. Q. E. D.

Let the demand correspondence ξ_i of each consumer i be defined by $\xi_i(p) = \{x_i \in \beta_i(p) \mid x_i \underset{i}{\succsim} z_i \text{ for all } z_i \in \beta_i(p)\}$.

Lemma 5: Let (C1)—(C4) be satisfied. Then $\xi_i(p)$ is non-empty and convex, and the correspondence ξ_i is upper hemi-continuous.

Proof: Since $X_i \cap K_1$ is compact, $\beta_i(p)$ is compact. According to Lemma 4, β_i is a continuous correspondence. Since $\underset{i}{\succsim}$ is a complete preorder there exists a maximal element in $\beta_i(p)$, hence $\xi_i(p)$ is non-empty.

Let $x_i' \in \xi_i(p)$ and $x_i'' \in \xi_i(p)$. For any $0 \leqq \lambda \leqq 1$, $\lambda x_i' + (1-\lambda) x_i'' \in \beta_i(p)$. Furthermore, by the convexity of $\underset{i}{\gtrsim}$, $\lambda x_i' + (1-\lambda) x_i'' \underset{i}{\gtrsim} x_i'$ which implies $\lambda x_i' + (1-\lambda) x_i'' \in \xi_i(p)$.

For ξ_i to be upper hemi-continuous, it suffices to show that ξ_i has a closed graph. Let $x_i^n \to x_i$, $p^n \to p$, and $x_i^n \in \xi_i(p^n)$. Clearly, $x_i \in \beta_i(p)$. Since β_i is lower hemi-continuous, for every $z \in \beta_i(p)$ there exists a sequence $z^n \to z$ and $z^n \in \beta_i(p^n)$. Hence, $x_i^n \underset{i}{\gtrsim} z^n$ for all n and by the continuity of $\underset{i}{\gtrsim}$, $x_i \underset{i}{\gtrsim} z$ for all $z \in \beta_i(p)$, implying that ξ_i is upper hemi-continuous. Q. E.D.

Let $\xi(p) = \sum_{i \in I} \xi_i(p)$ and define the excess demand correspondence ζ as

$$\zeta(p) = \xi(p) - \sum_{i \in I} e_i - \eta(p)$$

which is non-empty, convex, and upper hemi-continuous. ζ maps P into some compact subset Z of R^l.

Following standard arguments of equilibrium analysis, define a correspondence μ by $\mu(z) = \{p \in P \mid p \cdot z = \text{Max}\, P \cdot z\}$. Clearly, $\mu(z)$ is non-empty and convex, and μ is upper hemi-continuous. Now let ψ be the correspondence defined by $\psi(z, p) = \zeta(p) \times \mu(z)$. ψ is a map from $Z \times P$ into itself. Furthermore, ψ is upper hemi-continuous and $\psi(p)$ is non-empty and convex. Applying Kakutani's Fixed Point Theorem, there exists a (z^1, p^1) such that $(z^1, p^1) \in \psi(z^1, p^1)$, i. e. $z^1 \in \zeta(p^1)$ and $p^1 \in \mu(z^1)$. It remains to be shown that $z^1 \leqq 0$. For any $p \in P$ and $z \in \zeta(p)$, i. e. $x \in \xi(p)$ and $y \in \eta(p)$, $z = x - \sum_{i \in I} e_i - y$, $p \cdot z = p \cdot x - p \cdot \sum_{i \in I} e_i - p \cdot y \leqq g(p) \cdot \mathbf{1} - \Pi(p) = 0$. Hence in particular, $p^1 \cdot z^1 \leqq 0$. Since $p^1 \in \mu(z^1)$, $p \cdot z^1 \leqq p^1 \cdot z^1 \leqq 0$ for all $p \in P$ implies $z^1 \leqq 0$. Since each consumer is locally not satiated, $p^1 \cdot x_i^1 = p^1 \cdot e_i + g_i(p^1)$ which implies $p^1 \cdot z^1 = 0$. Hence it has been shown that there exists a list $[(x_i^1), J^1, (y_j^1), (t_i^1), p^1]$ where J^1 is determined according to Lemma 3, and $t_i^1 = g_i(p^1)$, $i \in I$. By construction $[J^1, (y_j^1), (t_i^1)]$ is stable relative to p^1. Furthermore, for each $i \in I$, x_i^1 is a best element in the restricted budget set, and market excess demand is non-positive.

Now consider an increasing sequence $(K_q)_{q=1,\dots}$ of closed cubes in R^l with center at the origin and whose diameters tend to infinity. With each K_q associate the truncated economy $\mathscr{E}_q$. Thus, for every $q = 1, \dots$, there exists a list $[(x_i^q), (y_j^q), p^q, (t_i^q), d^q]$ such that

(1) d^q determines the set of firms J^q,

(2) for every $i \in I$
$x_i \in X_i \cap K_q$ and $x_i \underset{i}{\succ} x_i^q$ implies $p^q \cdot x_i > p^q \cdot e_i + t_i^q$,

(3) $[J^q, (y_j^q), (t_i^q)]$ is a firm structure relative to p^q, i. e.
$\sum_{i \in I} t_i^q = \sum_{j \in J^q} p^q \cdot y_j^q$,

(4) for every $S_j \subset I$
$\sum_{i \in S_j} t_i^q \geqq \text{Max}\,\{p^q \cdot y \mid y \in Y_j \cap K_q\}$,

(5) $\sum_{i \in I} (x_i^q - e_i) - \sum_{j \in J^q} y_j^q \leqq 0$.

By the choice of K_1 and since $K_1 \subset K_q$ for all $q = 1, \ldots$, one knows, that for all $i \in I$, $x_i^q \in \text{int}\, K_1$, $t_i^q \geqq 0$, $\sum_{j \in J^q} y_j^q \in \text{int}\, K_1$, $0 \leqq d^q \leqq 1$. Hence the sequences $(x_i^q)_{q=1,\ldots}$ and $(y_j^q)_{q=1,\ldots}$ are bounded as well as (t_i^q) since $(p^q)_{q=1,\ldots}$ is bounded and $t_i^q = g_i(p^q)$. Thus, there exists a converging subsequence with limit point $[(\bar{x}_i), (\bar{y}_j), \bar{p}, (\bar{t}_i), \bar{d}]$. Clearly, $\bar{x}_i \in X_i$, $\bar{y}_j \in Y_j$, $\bar{t}_i \geqq 0$, $\bar{p} \in P$, and $0 \leqq \bar{d} \leqq 1$. Furthermore,

$$\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j \leqq 0 \text{ and } \sum_{i \in I} \bar{t}_i = \sum_{j \in \bar{J}} \bar{p} \cdot \bar{y}_j$$

where $\bar{J}$ is the set of firms determined by $\bar{d}$.

Suppose the firm structure $[\bar{J}, (\bar{y}_j), \bar{t}_i]$ were not stable relative to $\bar{p}$. Then there exists a coalition S_j and a bundle $y' \in Y_j$ such that $\bar{p} \cdot y' > \sum_{i \in S_j} \bar{t}_i$. Clearly, for q large enough $y' \in Y_j \cap K_q$ and

$$p^q \cdot y' > \sum_{i \in S_j} t_i^q \geqq \text{Max}\,\{p^q \cdot y \mid y \in Y_j \cap K_q\}$$

which contradicts (4).

Let $z_i \in X_i$, $z_i \neq \bar{x}_i$ such that $\bar{p} \cdot z_i \leqq \bar{p} \cdot e_i + \bar{t}_i$. There exists a sequence $(z_i^q)_{q=1,\ldots}$ converging to z_i such that $p^q \cdot z_i^q \leqq p^q \cdot e_i + t_i^q$ and $z_i^q \in X_i \cap K_q$. Since for all $q = 1, \ldots$ $x_i^q \underset{i}{\succsim} z_i^q$ the continuity of $\underset{i}{\succsim}$ implies $\bar{x}_i \underset{i}{\succsim} z_i$. Hence, $\bar{x}_i$ is a best element in the unrestricted budget set.

It remains to be shown that $\bar{p}$ supports an equilibrium with zero excess demand. Since each consumer is locally not satiated, we have for all $i \in I$, $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot e_i + \bar{t}_i$, which implies $\bar{p}.\ (\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j) = 0$. If $\bar{p} \gg 0$, then $p \cdot [\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j] \leqq \bar{p} \cdot [\sum_{i \in I} (\bar{x}_i - e_i) - \sum_{j \in \bar{J}} \bar{y}_j] = 0$ and

$\sum_{i\in I} (\bar{x}_i - e_i) - \sum_{j\in J} \bar{y}_j \leqq 0$ implies that excess demand is equal to zero. If $\bar{p}$ contains some zero component, then the assumption of free disposal guarantees that there exist bundles $\bar{y}_j'$ such that $\bar{p} \cdot \sum_{j\in J} \bar{y}_j' = p \cdot \sum_{j\in J} \bar{y}_j$ and $\sum_{i\in I} (\bar{x}_i - e_i) - \sum_{j\in J} \bar{y}_j' = 0$. This completes the proof of the theorem.

Q. E. D.

References

[1] K. J. Arrow and F. Hahn: Competitive Equilibrium Analysis, San Francisco, 1971.

[2] V. Boehm: The Core of an Economy with Production (Forthcoming).

[3] P. Champsaur: Modèle de Jeu pour une Economic Concurrentielle avec Production, ENSAE, Paris; paper presented at the conference of the Econometric Society, Cambridge, September 1970.

[4] G. Debreu: New Concepts and Techniques for Equilibrium Analysis, International Economic Review *3* (1962), pp. 257—273.

[5] G. Debreu: On a Theorem of Scarf, Review of Economic Studies *30* (1963), pp. 177—180.

[6] G. Debreu: Theory of Value, New York, 1959.

[7] G. Debreu and H. Scarf: A Limit Theorem on the Core of an Economy, International Economic Review *4* (1963), pp. 235—246.

[8] W. Hildenbrand: The Core of an Economy with a Measure Space of Economic Agents, Review of Economic Studies 35 (1968), pp. 443—452.

[9] W. Hildenbrand: Existence of Equilibria for Economies with Production and a Measure Space of Consumers, Econometrica *38* (1970), pp. 608—623.

Address of author: Volker Boehm, Institut für Gesellschafts- und Wirtschaftswissenschaften der Universität Bonn, Wirtschaftstheoretische Abteilung I, Adenauerallee 24—42, D-5300 Bonn.