The Core of an Economy with Production [1,2,3]

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I. INTRODUCTION

The theory of the core of exchange economies is well developed. For economies with production it has been an open problem as to how the production possibilities of groups of consumers should be represented and what the relationship between aggregate production possibilities and the production possibilities of coalitions should be. Since the core describes allocations which cannot be improved upon by any group of consumers, the definition of the bargaining power of each coalition with respect to their technical knowledge is the crucial point in a theory of the core with production. However, as the analysis indicates, the important features of such a definition are not only related to the specification of production possibilities *per se*, but also to the distribution of the total available productive knowledge and to the institutional and organizational framework in which coalitions are able to employ their respective parts.

Until recently, the existing results on the core of productive economies dealt only with some special cases, e.g. Debreu and Scarf [8], Champsaur [5], Hildenbrand [10], [11], and recently also Arrow and Hahn [1]. Debreu and Scarf assume that the total production possibility set is a convex cone and that it is available to each coalition. Both assumptions together imply additivity which is also the basic assumption made by Hildenbrand, and in a different form by Arrow and Hahn.

This paper attempts to supply a general description of an economy in which each group of consumers has at its disposal some production possibility set which it can use independently in case it is dissatisfied with a proposed allocation. Section II describes the general economic framework of the distribution of production possibilities. It contains the characterization by Debreu and Scarf and by Hildenbrand as special cases. The main theorem in Section IV indicates when the core will be non-empty. The proof of the theorem requires an extension of Scarf's theorem on balanced games which is given in Section III.

II. THE MODEL

The basic framework is an economy as described in [7]. The commodity space of the economy is the finite dimensional Euclidean space R^l. There is a finite set of consumers, $I = \{1, ..., i, ..., n\}$ who are characterized by their consumption sets $X_i \subset R^l$, their preference relation $\underset{i}{\succsim}$, and their endowment $e_i \in R^l$. Let 2^I denote the set of coalitions of consumers. With each $S \in 2^I$ is associated a non-empty production possibility set $Y^S \subset R^l$ with the

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convention that $Y\phi = \{0\}$. The set Y^S will reflect all features which are related to the ability of coalition S to make certain net output bundles available to the members of S through a joint action. Apart from purely technological determinants, which reflect the technical knowledge of the coalition S, Y^S will also be determined by organizational and institutional features inherent to coalition S. More specifically, since the pooling and the application of technical knowledge always requires some organizational form, Y^S depends on the managerial skill of the members of S as well as on their efficiency and ability to cooperate and to distribute the proceeds of their joint activities. This includes the case where there are costs of forming the coalition as well as costs of reaching a joint decision. If such costs occur, then, typically, $0 \notin Y^S$ will hold and the collection (Y^S) will not be super-additive.

To complete the description of the production characteristics of the economy, one has to define the total production possibility set Y. In general, for any two coalitions S_1 and S_2, the outcome of two separate decisions $y_1 \in Y^{S_1}$ and $y_2 \in Y^{S_2}$ will not be related in any specific way to a production decision by the coalition $S = S_1 \cup S_2$. Hence for the general case the technology will be given by a collection of non-empty subsets $((Y^S), Y)$ of the commodity space R^l. This somewhat abstract approach can be justified in the following way. Since it is not assumed that $Y = Y^I$, it is easily seen that there may exist feasible allocations for the economy as a whole which are not enforceable by the all consumer coalition I, i.e. for example, if the process of reaching joint decisions becomes more and more inefficient the larger the coalition grows. In this case there would exist a true incentive for decentralization which may result in Pareto superior allocations to any allocation which is enforceable by coalition I. On the other hand, the relationship of (Y^S) to Y may embody certain forms of external effects which are part of the institutional structure of the economy.

The definition of the core is now straightforward. A list of commodity vectors $x = (x_i)$, $i = 1, ..., n$, is an allocation if $x_i \in X_i$ for every $i \in I$. An allocation is feasible if there exists a $y \in Y$ such that $\sum_{i \in I} x_i = \sum_{i \in I} e_i + y$.

Definition 1. A non-empty coalition S is said to block an allocation x if there exist $x'_i \in X_i$, $i \in S$, and $y^S \in Y^S$ such that

$$x'_i \succ_i x_i \quad \text{for all } i \in S \qquad ...(1)$$

$$\sum_{i \in S} x'_i = \sum_{i \in S} e_i + y^S. \qquad ...(2)$$

Then the core is defined as the set of feasible allocations which are blocked by no coalition.

III. A THEOREM ON BALANCED GAMES

The purpose of the remaining part of the paper is to find sufficient conditions for a non-empty core of the economy. For the case of an exchange economy Scarf gave a general result in [12] and in [13]. His procedure of representing the economy as a game and of applying a result on balanced games will also be followed here. However, the actual proof requires an extension of Scarf's theorem on balanced games since two of his assumptions will not necessarily hold for economies with production. This section intends to show (*a*) that Scarf's theorem holds essentially without one of his assumptions and (*b*) that this stronger version of the theorem is also applicable to a conceptual extension of a balanced game.

Consider a game without side payments given by the triple (I, v, H), where

$$I = \{1, ..., n\}$$

is the set of players, v is the characteristic function which assigns to each coalition $S \subset I$ a non-empty subset $v(S) \subset E^S$, the subspace of the Euclidean space E^n associated with the members of S. Let $H \subset E^n$ be the set of possible utility outcomes. In most treatments of games in characteristic function form it is assumed that $v(I) = H$. From a conceptual point of view, however, a distinction between what is enforceable by I and what is possible seems necessary. An exchange economy where blocking is costly provides an example. The core of the game (I, v, H) is the set of possible outcomes which cannot be blocked by any coalition $S \subset I$.

Definition 2. A family $\mathscr{S} = \{S\}$ of non-empty coalitions $S \subset I$ is called balanced iff for every $S \in \mathscr{S}$ there exist weights $d_s > 0$ such that for all $i \in I$

$$\sum_{\substack{S \in \mathscr{S} \\ S \ni i}} d_s = 1.$$

Definition 3. A game (I, v, H) is called balanced iff for all balanced families $\mathscr{S}$

$$\bigcap_{S \in \mathscr{S}} (v(S) \times E^{I \setminus S}) \subset H.$$

This definition coincides with the traditional definition in [2] and [12] for the special case where $H = v(I)$. The main result is the following theorem.

Theorem 1. *Let (I, v, H) be a balanced game. Assume for every $S \subset I$*

(1) *$v(S)$ is non-empty and closed*

(2) *$x \in v(S)$, $y \in E^S$, $y \leqq x$ implies $y \in v(S)$*

(3) *H is non-empty, closed, and bounded from above*

(4) *$x \in H$, $z \in E^n$, $z \leqq x$ implies $z \in H$.*

Then the core of (I, v, H) is non-empty.

This theorem will be proved in two steps which separate the two extensions of Scarf's theorem as indicated in the introduction. Since the proofs rely on his result a complete statement of the theorem ([12], Theorem 1, p. 54) is in order.

Theorem (Scarf). *Let (I, v, H) be a balanced game such that $v(I) = H$ and for every $S \subset I$*

(1) *$v(S)$ is non-empty and closed*

(2) *$x \in v(S)$, $y \in E^S$, $y \leqq x$ implies $y \in v(S)$*

(3) *$\{u^S \in v(S) \mid \forall i \in S \;\; u_i^S \geqq v(\{i\})\}$ is non-empty and bounded.*

Then (I, v, H) has a non-empty core.

Lemma 1. *Let (I, v, H) be a balanced game such that $v(I) = H$ is bounded from above and for every $S \subset I$*

(1) *$v(S)$ is non-empty and closed*

(2) *$x \in v(S)$, $y \in E^S$, $y \leqq x$ implies $y \in v(S)$.*

Then the core of (I, v, H) is non-empty.

Let (I, v) denote the game for which $v(I) = H$.

Proof. First one observes, since every partition $\mathscr{P}$ of I is a balanced family, that

$$\bigcap_{S \in \mathscr{P}} (v(S) \times E^{I \setminus S}) = \prod_{S \in \mathscr{P}} v(S) \subset v(I),$$

where Π denotes the Cartesian product. Hence every $v(S)$ is bounded from above. Let $U(I) = \prod_{i \in I} v(\{i\})$ and let $U(S)$ denote the projection of $U(I)$ into the subspace associated with coalition S.

Consider the game (I, w) where $w(S)$ is defined by $w(S) = v(S) \cup U(S)$. Clearly

$w(I) = v(I)$. Furthermore, $w(S)$ satisfies assumptions (1) and (2) of Scarf's theorem. since $v(S)$ and $U(S)$ satisfy (1) and (2). Also, for all $S \subset I$, $w(S)$ is individually rational, i.e. the set $\{u^S \in w(S) | u_i^S \geqq w(\{i\}),\ i \in S\}$ is non-empty. Since all $v(S)$ are bounded above the set is also bounded. Hence (I, w) satisfies assumptions (1)-(3) of Scarf's theorem.

Consider any balanced family $\mathscr{S}$. Then

$$\bigcap_{S \in \mathscr{S}} (w(S) \times E^{I \setminus S}) = \bigcap_{S \in \mathscr{S}} \{\{v(S) \cup U(S)\} \times E^{I \setminus S}\}$$

$$= \bigcap_{S \in \mathscr{S}} \{(v(S) \times E^{I \setminus S}) \cup (U(S) \times E^{I \setminus S})\}$$

$$= \bigcup_{\substack{\mathscr{S}', \mathscr{S}'' \subset \mathscr{S} \\ \mathscr{S}' \cap \mathscr{S}'' = \phi \\ \mathscr{S}' \cup \mathscr{S}'' = \mathscr{S}}} \{\{\bigcap_{S \in \mathscr{S}'} (v(S) \times E^{I \setminus S})\} \cap \{\bigcap_{T \in \mathscr{S}''} (U(T) \times E^{I \setminus T})\}\}$$

Clearly the members of the union for which $\mathscr{S}' = \mathscr{S}$ and for which $\mathscr{S}'' = \mathscr{S}$ are subsets of $v(I)$ and hence of $w(I)$. For any mixed element, i.e. $\mathscr{S}', \mathscr{S}'' \neq \phi$ define a new family of coalitions in the following way. Let d_T be the weight associated with $T \in \mathscr{S}''$ and consider the family of singletons $\mathscr{T}(T) = \{\{i\} | i \in T\}$ with associated weights $d_{\{i\}} = d_T$ for $i \in T$. It is easy to check that the collection of coalitions $\mathscr{C} = \{\{\mathscr{T}(T) | T \in \mathscr{S}''\}, \mathscr{S}'\}$ is a balanced family of coalitions. Moreover, for every $T \in \mathscr{S}''$

$$U(T) \times E^{I \setminus T} = \bigcap_{i \in T} (v(\{i\}) \times E^{I \setminus \{i\}}).$$

Hence for any $\mathscr{S}''$

$$\bigcap_{T \in \mathscr{S}''} (U(T) \times E^{I \setminus S}) = \bigcap_{T \in \mathscr{S}''} \{\bigcap_{i \in T} (v(\{i\}) \times E^{I \setminus \{i\}})\}$$

$$= \bigcap_{\substack{i \in T \\ T \in \mathscr{S}''}} (v(\{i\}) \times E^{I \setminus \{i\}})$$

which implies that for any $\mathscr{S}'$ and $\mathscr{S}''$

$$\{\bigcap_{S \in \mathscr{S}'} (v(S) \times E^{I \setminus S})\} \cap \{\bigcap_{T \in \mathscr{S}''} (U(T) \times E^{I \setminus T})\} = \bigcap_{S \in \mathscr{C}} (v(S) \times E^{I \setminus S}) \subset v(I) = w(I).$$

Hence (I, w) is a balanced game. According to Scarf's theorem it has a non-empty core.

Let x be in the core of (I, w). x is feasible for the game (I, v) since $v(I) = w(I)$. Moreover, $v(S) \subset w(S)$ implies that x cannot be blocked by any S in the game (I, v). Hence x belongs to the core of (I, v).

Q.E.D.

Proof of Theorem 1. First one observes that the balancedness implies $v(I) \subset H$ since $\{I\}$ is a balanced family. Consider the enlarged game (I, w) defined by $w(S) = v(S)$ for $S \neq I$ and $w(I) = H$. Clearly (I, w) is a game satisfying assumptions (1) and (2) of the Lemma. Since $H = w(I)$ one has for every balanced family not including the all-player coalition

$$\bigcap_{S \in \mathscr{S}} (w(S) \times E^{I \setminus S}) = \bigcap_{S \in \mathscr{S}} (v(S) \times E^{I \setminus S}) \subset w(I).$$

On the other hand, if $\mathscr{S}$ contains I the inclusion is obvious. Hence (I, w) is a balanced game which has a non-empty core. Let x be in the core of (I, w). Clearly x cannot be blocked by any $S \neq I$ in the game (I, v, H). Furthermore, since $v(I) \subset w(I)$ x is also unblocked by I for the game (I, v, H). Hence (I, v, H) has a non-empty core.

Q.E.D.

IV. MAIN RESULT

The crucial step for the application of Theorem 1 to an economy with production will be to demonstrate that it is representable as a balanced game. This requires some specification of the distribution of the technology. The following definition describes a relationship which will in general yield a non-empty core.

Definition 4. Let $((Y^S), Y)$ be the technology distribution of the economy. $((Y^S), Y)$ is called balanced if and only if for every balanced family $\mathscr{S}$ and associated weights (d_S)

$$\sum_{S \in \mathscr{S}} d_S Y^S \subset Y.$$

The following examples describe balanced technologies. Consider an economy with three consumers, i.e. $I = \{1, 2, 3\}$ and let $Y^S = \{0\}$ for all S not equal to the grand coalition. Then, if $Y = Y^{\{1,2,3\}}$ is "starshaped" as depicted in Figure 1, $((Y^S), Y)$ is balanced. The significance of this example is that for a balanced technology any individual set as well as the aggregate set may be non-convex.

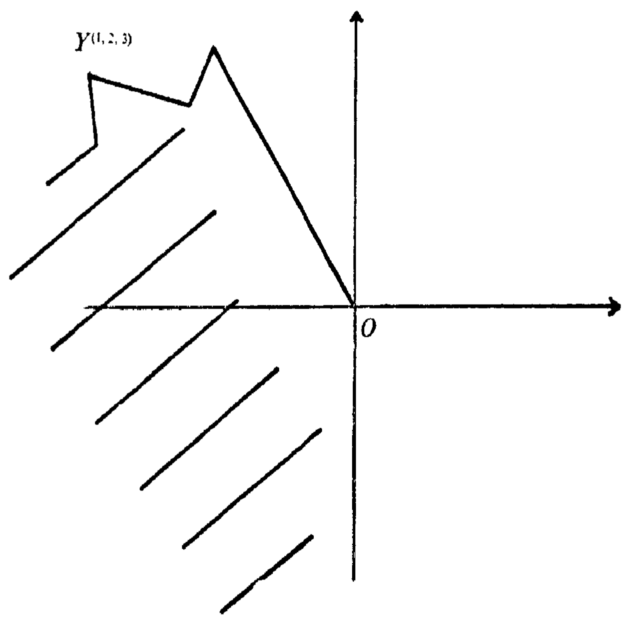

FIGURE 1

Consider an economy of the same size but with two types of production sets, type A and type B. Let Y^A be given by the two line segments $\{(OA), (AB)\}$ and Y^B by $\{(OB)\}$ (see Figure 2). Now let $Y^{\{1,2,3\}}$ be the zero production point, $Y^{\{1\}} = Y^{\{3\}} = Y^{\{1,3\}} = Y^B$ and $Y^{\{2\}} = Y^{\{1,2\}} = Y^{\{2,3\}} = Y^A$. Clearly the sets $Y^A + Y^B$, $Y^A + Y^A + Y^B$, $2Y^B + Y^A$ are convex. If

$$Y = \bigcup_{\mathscr{S} \subset 2^I} \sum_{S \in \mathscr{S}} Y^S$$

then it is easy to check that this technology is balanced. Lemma 2 describes a general class of balanced technologies.

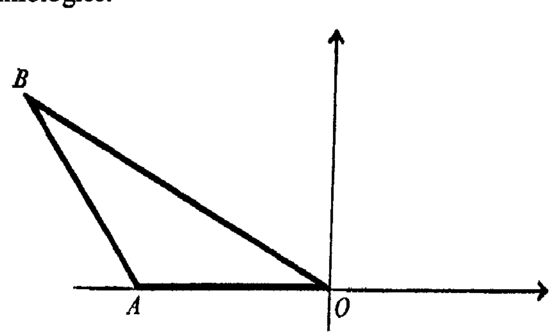

FIGURE 2

Lemma 2. *Let $0 \in Y^S$ for all $S \subset I$ and let Y be convex and defined by*

$$Y = \bigcup_{\mathscr{S} \subset 2^I} \sum_{S \in \mathscr{S}} Y^S.$$

Then $((Y^S), Y)$ is balanced.

Proof. First observe that $0 \in Y^S$ for all S implies $Y = \sum_{S \subset I} Y^S$, since for any collection $\mathscr{S}$, $\sum_{S \subset I} Y^S \supset \sum_{S \in \mathscr{S}} Y^S$. Hence,

$$\sum_{S \in \mathscr{S}} d_S Y^S = \sum_{S \in \mathscr{S}} (d_S Y^S + (1-d_S)\{0\}) \subset \sum_{S \in \mathscr{S}} \text{conv}\, Y^S = \text{conv} \sum_{S \in \mathscr{S}} Y^S \subset \text{conv} \sum_{S \in 2^I} Y^S$$

where conv Y^S denotes the convex hull of Y^S. Q.E.D.

To give another example of a class of balanced technologies, let (Y^S) be any collection of sets in R^l and let Y be the smallest convex cone at zero which contains every Y^S. Then $((Y^S), Y)$ is balanced. The examples indicate that a balanced technology describes a wide range of collections of production possibility sets where the individual sets as well as the aggregate set Y may embody elements of increasing returns and/or indivisibilities.

Theorem 2. *Let the economy* $\mathscr{E} = \{I, (X_i), (e_i), (\underset{i}{\succsim}), ((Y^S), Y)\}$ *be such that for every* $i \in I$

(1) $X_i \subset R^l_+$ *is closed, convex, and bounded below and* $e_i \in X_i$

(2) $\underset{i}{\succsim}$ *is a complete, transitive, continuous preordering on* X_i *such that for any* x'_i *and* x''_i *with* $x'_i \underset{i}{\succsim} x''_i$ *and for all* λ, $0 \leqq \lambda \leqq 1$ $\lambda x'_i + (1-\lambda)x''_i \underset{i}{\succsim} x''_i$

(3) $0 \in Y^{\{i\}}$

(4) *for every* $S \subset I$ Y^S *is closed*

(5) Y *closed and* $AY \cap R^l_+ = \{0\}$ *where* AY *denotes the asymptotic cone of* Y [1]

(6) $((Y^S), Y)$ *is balanced.*

Then $\mathscr{E}$ *has a non-empty core.*

Proof. First it will be shown that $\mathscr{E}$ is representable as a game of the form (I, v, H) satisfying conditions (1)-(4) of Theorem 1.

Let $x^S = (x_i^S)$, $i \in S$ and define $X^S = \{(x_i^S) \mid x_i^S \in X_i, \sum_{i \in S} x_i^S \in Y^S + \{\sum_{i \in S} e_i\}\}$. X^S is the set of allocations enforceable by coalition S. Let $\mathscr{T}$ denote the set of coalitions for which $X^S \neq \phi$. Clearly assumptions (1) and (3) imply that $\{i\} \in \mathscr{T}$ for all $i \in I$. For any $S \subset I$ the collection $\{S, \{i\} \mid i \in I \setminus S\}$ is a balanced family with all weights equal to one. Hence $Y^S + \sum_{i \in I \setminus S} Y^{\{i\}} \subset Y$ in conjunction with (3) implies $Y^S \subset Y$ for all $S \subset I$. $Y^S \subset Y$ implies $AY^S \subset AY$, which yields $AY^S \cap R^l_+ = \{0\}$ for all $S \subset I$ using assumption (5). Hence standard results on asymptotic cones show that the set of feasible allocations for the economy as well as the sets X^S, $S \in \mathscr{T}$ are compact. Since for every $i \in I$ the preference relation is continuous there exist continuous representations $u_i(x_i)$, $i \in I$. Furthermore, for each $S \in \mathscr{T}$, there exists a characteristic function $\bar{v}$ from $\mathscr{T}$ into the utility space E^n representing the enforceable utility vectors for each $S \in \mathscr{T}$. Without loss of generality one can normalize $\bar{v}(\cdot)$ such that $\bar{v}(\{i\}) = \max\{u_i(x_i) \mid x_i \in Y^{\{i\}} + \{e_i\}\} = 0$ and one can extend $\bar{v}$ to v by defining $v(S) = \bar{v}(S) + E^S_-$, where E^S_- denotes the negative orthant of the utility subspace associated with coalition $S \in \mathscr{T}$. For $S \notin \mathscr{T}$ define $v(s) = \prod_{i \in S} v(\{i\})$. Hence $v(\cdot)$ is a characteristic function in the sense of Theorem 1. Assumption (6) implies that $0 \in Y$, hence there exists a non-empty set $\bar{H} \subset E^n$ of possible utility allocations, which is closed and bounded from above and which can be extended to $H = \bar{H} + E^n_-$ without any loss of generality. Hence $\mathscr{E}$ is representable as a game (I, v, H) in characteristic function form satisfying all assumptions of Theorem 1. For the theorem to be applicable it remains to be shown that it is a balanced game.

[1] For basic results on asymptotic cones, see Debreu [7, p. 22].

Let $\mathscr{S}$ be a balanced family of non-empty coalitions, $u \in E^n$, and for all $S \in \mathscr{S}$, $u^S \in v(S)$ with associated $x^S \in X^S$, $S \in \mathscr{T}$ and $(x_i^S) \in \prod_{\substack{i \in S \\ S \notin \mathscr{T}}} X^{(i)}$. Define, for each $i \in I$, $x_i = \sum_{\substack{S \in \mathscr{S} \\ S \ni i}} d_S x_i^S$.

It will be shown that (x_i) is a feasible allocation for the economy.

$$\sum_{i \in I} x_i = \sum_{i \in I} \sum_{\substack{S \in \mathscr{S} \\ S \ni i}} d_S x_i^S = \sum_{S \in \mathscr{S}} \sum_{i \in S} d_S x_i^S = \sum_{S \in \mathscr{S}} d_S \sum_{i \in S} x_i^S$$

$$= \sum_{S \in \mathscr{S}} d_S(y^S + \sum_{i \in S} e_i) = \sum_{S \in \mathscr{S}} d_S y^S + \sum_{i \in I} e_i \sum_{\substack{S \in \mathscr{S} \\ S \ni i}} d_S$$

$$= \sum_{S \in \mathscr{S}} d_S y^S + \sum_{i \in I} e_i.$$

Since $((Y^S), Y)$ is balanced, $\sum_{S \in \mathscr{S}} d_S y^S$ is a feasible production plan. Hence (x_i) is a feasible allocation. Q.E.D.

V. REMARKS

One of the outcomes of the general formulation of the technology distribution is that, in general, one cannot expect every allocation in the core to be Pareto optimal for the economy as a whole, i.e. relative to Y. This is evident, since $Y^I \subset Y$ and $Y^I \neq Y$ in general, e.g. if there are sufficient decreasing returns to cooperation. Hence, there may exist feasible allocations which are not Pareto optimal, but which are unblocked. However, in such a case, the core will always contain a subset of the set of Pareto optimal allocations. Let x be in the core. If x is not Pareto optimal, then there exists an allocation z such that $z_i \underset{i}{\succsim} x_i$ for all $i \in I$, $z_i \underset{i}{\succ} x_i$ some i, and $\sum_{i \in I} (z_i - e_i) \in Y$. Hence z must be in the core.

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