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# Returns to Size vs Returns to Scale: The Core with Production Revisited

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This note reports and reemphasizes the importance of a balancedness condition for production sets which—if combined with otherwise standard assumptions—guarantees the non-emptiness of the core. Balancedness neither requires nor implies any property related to returns to input scale. It is argued that the emphasis laid by Scarf and others on a particular kind of increasing returns to scale is not essential for the existence result in general. It is shown, however, that Scarf's distributivity assumption implies balancedness. Finally, it is pointed out that earlier results indicate that equilibria in the core require a specific profit distribution to consumers rather than a particular behavioral rule of producers even in the presence of non-convexities. *Journal of Economic Literature* Classification Number: 021. 

## INTRODUCTION

In Chap. 21 of the "Contributions to Mathematical Economics: In Honor of Gérard Debreu" [9], H. Scarf makes available a set of unpublished notes on the core of a productive economy when there are increasing returns. The ideas and concepts of these notes seem to have had a substantial influence on researchers who were trying to develop an alternative to the competitive equilibrium concept when production sets exhibit increasing returns to scale (see, for example, the recent paper by Dehez and Drèze [6]). Scarf's article and the work of others (e.g., [8]) create the impression as if increasing returns to scale in production are a major cause for an empty core in a production economy. Starting with such a premise he and others approached the existence question of the core by searching for equilibrium concepts with increasing returns with the hope (1) that existence of such equilibria could be established for a large set of economies and (2) that the standard inclusion property holds.



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A number of publications of the mid-seventies, however, support quite a different point of view. Non-convexities of production sets are not the central issue for the existence of the core, and any specific behavioral rule of a non-convex producer does not yield the inclusion property. Specifically:

(1) Increasing returns to scale in production and the existence of the core in a productive economy have very little to do with each other on a general conceptual level. What matters for the existence of the core instead are the returns to coalition size relative to the feasible production set. Scarf's model treats the special case only where returns to scale of production in an enlarged space imply the right kind of returns to coalition size in the space of commodities with which agents are concerned. The distributivity assumption introduced by Scarf within his special structure implies the balancedness condition introduced in [3], so that existence of the core in his case follows from the 1974 theorem.

(2) The equilibrium concepts suggested in [11] and [1] independently, which stipulate a particular profit distribution, guarantee in a natural way the inclusion property and a possible limit theorem. However, it is also evident from these papers that returns to coalition size rather than returns to input scale are the important assumptions for the result. It remains an important but separate question which behavioral rule of the firm replaces profit maximization in a non-convex environment to obtain existence of market equilibria. However, [1] and [11] provide the answer to the question of how profits are to be distributed if one requires at the same time that these equilibria are in the core.

## Returns to Size vs Returns to Scale

From a conceptual viewpoint the general model presented in 1974 is an appropriate extension of an exchange economy to treat the problem of blocking and of the core in an economy with production. The abstract description of a production possibility set $Y^S$ associated with a coalition $S$ allows for a wide range of interpretations. From a modelling point of view of a cooperative theory, both of the following situations should be included in the formulation: larger coalitions may become more effective because of the pooling of their technological as well as their organizational know-how (increasing returns to size). On the other hand, a larger group may become very ineffective (decreasing returns to size) if, for example, it does not know how to pool its resources efficiently, or if there are costs to coalition formation. Existence of the core depends on the blocking power of any coalition $S$ relative to the overall feasible set; i.e., the size of all of the sets

$Y^S$ and their relationship matters and *not* whether these sets are convex. Hence, the essential property required for the general existence proof in [3] imposes that coalitions' productive power is not too large.

Let $\mathfrak{E} = \{I, (X_i, e_i, \succsim_i), ((Y^S), Y)\}$ denote a coalition production economy with commodity space $\mathbb{R}^l$. The interpretation of $((Y^S), Y)$ means that $Y^S \subset \mathbb{R}^l$ is what coalition $S$ can enforce through cooperation, and $Y$ is what is feasible for the economy. Therefore, a priori one need not have to have $Y^I \neq Y$.

DEFINITION. $((Y^S), Y)$ is called balanced if for every balanced family of coalitions $\mathfrak{S}$ and associated positive weights $\{d_S \mid S \in \mathfrak{S}\}$

$$\sum_{S \in \mathfrak{S}} d_S Y^S \subset Y.$$

Balancedness neither requires nor imposes properties like free disposal, $0 \in Y^S$, or non-increasing returns to scale. To see this, consider the following example. Let $l = 2$; choose a positive number $c$ and define

$$Y^S = \left\{ (y_1, y_2) \,\middle|\, y_1 + \frac{1}{|S|} y_2 \leqslant -c,\ y_1 \leqslant 0,\ y_2 \geqslant 0 \right\}.$$

Then $((Y^S), Y^I)$ is balanced.

In [4] a characterization of balanced technologies was given which shows that a balanced distribution of productive knowledge can be generated from productive factors specific to each agent and coalition and a common larger technology. Assume that there are $k$ productive factors in addition to the $l$ commodities. Agents possess endowments $\omega_i \in \mathbb{R}^k_+$ $(i \in I)$. The aggregate technology available to all coalitions is a set $\bar{Y} \subset R^l \times R^k_-$ and the productive power of coalitions is defined by

$$Y^S = \left\{ y \in \mathbb{R}^l \,\middle|\, \left( y, -\sum_{i \in S} \omega_i \right) \in \bar{Y} \right\}.$$

Reference [4] contains the following two lemmas.

LEMMA 1. *If $\bar{Y}$ is a convex cone, then $((Y^S), Y^I)$ is totally balanced.*

LEMMA 2. *Let $((Y^S), Y^I)$ be totally balanced. Then there exists a set*

$$\bar{Y} \subset \mathbb{R}^l \times \mathbb{R}^k_- \text{ with } k = |I| = n, \text{ for all } i \in I,$$

$$\omega_i = (\omega_{1i}, ..., \omega_{ni}),\ \omega_{ii} = 1, \text{ and } \omega_{ij} = 0 \text{ if } i \neq j,$$

*such that*

$$Y^S = \left\{ y \in \mathbb{R}^l \,\middle|\, \left( y, -\sum_{i \in S} \omega_i \right) \in \bar{Y} \right\}.$$

Scarf confines himself exclusively to the particular model used in both lemmas. Technological knowledge is the same for all agents, whereas group specific productive power is defined by group specific productive resources. With his additional assumption of free disposal this excludes decreasing returns to size, coalition formation costs etc. Moreover, his assumption of distributivity on $\bar{Y}$ implies a totally balanced technology in the space of all other commodities.

DEFINITION. $\bar{Y} \subset \mathbb{R}^l \times \mathbb{R}^k_-$ is called distributive if for any finite number of points $y_i \in \bar{Y}$, $y_i = (y_i^l, y_i^k)$, and any non-negative $\alpha_i$, the point $y = (y^l, y^k) = \sum \alpha_i y_i \in \bar{Y}$, if $y_i^k - y^k \leqslant 0$.

It is straightforward to show that a distributive $\bar{Y}$ generates a totally balanced technology $((Y^S), Y^I)$. Consider $\mathfrak{S}$ balanced with $\{d_S \mid S \in \mathfrak{S}\}$. Then, $y^S \in Y^S$, $S \in \mathfrak{S}$ implies

$$\sum_{S \in \mathfrak{S}} d_S \left( y^S, -\sum_{i \in S} \omega_i \right) = \left( \sum_{S \in \mathfrak{S}} d_S Y^S, -\sum_{i \in I} \omega_i \right) \in \bar{Y}$$

since for all $S$, $\sum_{i \in S} \omega_i \leqslant \sum_{i \in I} \omega_i$. It is apparent from this that the increasing returns to scale in $\mathbb{R}^{l+k}$ imply a balanced form of increasing returns to size which is all that matters for the existence of the core. A consequence of this observation is that the existence theorems in [9] and [8] become a special case of Theorem 2 in [3], restated here in a slightly different version.

THEOREM. *Let* $\mathfrak{E} = \{I, (X_i, e_i, \succsim_i), ((Y^S), Y)\}$ *be such that for every* $i \in I$

(1) $X_i \subset \mathbb{R}^l$ *is closed convex, and bounded below.*

(2) $\succsim_i$ *is a complete, transitive, continuous, and convex preference ordering on* $X_i$.

(3) $X_i \cap \{Y^{\{i\}} + \{e_i\}\}$ *is non-empty.*

*Moreover,*

(4) $Y^S$ *is closed for all* $S \subset I$.

(5) $Y$ *is closed and* $AY \cap \mathbb{R}^l_+ = \{0\}$, *where* $AY$ *is the asymptotic cone of* $Y$.

(6) $((Y^S), Y)$ *is balanced.*

*Then* $\mathfrak{E}$ *has a non-empty core.*

## MARKET EQUILIBRIA IN THE CORE

The essential feature of the equilibrium concept proposed in [1, 5, 11] is that profits are distributed to consumers in such a way that the distribution belongs to the core of an associated side-payment game. This property drives the inclusion theorem. No particular behavioral assumption as to profit maximization is required by definition. However, increasing returns to size may imply this additional feature in equilibrium. In [1] a very general concept of equilibrium with firms is proposed which does not prescribe a specific behavioral rule vis-à-vis prices and markets. This shows once again that the requirements, to describe market equilibria which belong to the core, may be distinctly different from the behavioral rules which the increasing returns literature seeks in order to decentralize production decision under increasing returns. Profit distribution and its relationship to the coalition specific productive factors are more important than the behavioral rule of each firm. It would be interesting to establish the relationship between the equilibrium concepts used by Scarf [9], by Quinzii [8], and by Sharkey [10] to the one with the appropriate profit distribution.

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