Temporary Equilibria with Quantity Rationing

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1. INTRODUCTION

Quantity rationing as a means to determine actual net trades has recently received wide interest in the study of economic situations in which desired net trades of all agents are not compatible. The necessity of defining a list of additional constraints for all agents or of introducing explicit rationing schemes has become apparent particularly in the theory of temporary equilibrium analysis where it is assumed that prices are fixed in the short run. With rationing it is then possible to describe and analyse typical short run disequilibrium phenomena.[1]

Two basic models with quantity rationing have been analysed in the literature. Drèze (1975) provided a solution to the problem of existence of equilibria in an exchange economy where price flexibility was restricted to some a-priori given intervals, including the case of completely rigid prices. Quantity constraints on net trades are taken to be observed and respected by the agents concerned and taken into account in their determination of optimal consumption plans. At equilibrium each consumer maximizes his utility subject to his budget constraints *and* to the quantity constraints.[2] This model assumes that each individual agent has no influence on the outcome of the rationing. For some markets this assumption is not very reasonable. A typical example is the rationing sometimes used by banks in case of an oversubscription for notes of obligation where people are served according to their relative share of the total bid. Then an individual agent might also find it in his interest to manipulate the outcome in his favour by expressing a demand which he knows he will never realize. A similar phenomenon can be observed in labour markets. In either case such dependence leads directly to a revision of the equilibrium concept proposed by Drèze since an agent's maximization against a fixed perceived constraint is no longer his best possible action.

The second model was proposed by Benassy (1975). Although Benassy introduces rationing functions for which the actual rationing on each market depends explicitly on each agent's decision in the market, he assumes that for every agent the expressed net trade on each market is a best decision given an agent's subjective rationing function usually different from the actual rationing mechanism. At equilibrium an agent's expressed net trade may be neither a best strategy against the perceived and realized constraints nor may it be optimal exploiting his influence on his effective constraints.

These difficulties have lead us to a general reformulation of the equilibrium concept when a fixed rationing scheme is in operation. Apart from some general characteristics of the class of rationing schemes which we consider we do not supply a general theory of rationing schemes. The restrictions are the same as those introduced by Benassy (1975).

Our equilibrium concept corresponds to a best response strategy equilibrium of each individual agent, exhibiting a Nash property in desired net trades.[3] To us this concept represents best the notion of a short run equilibrium in a temporary equilibrium framework where each agent's beliefs about other agents' strategies are taken as given. It can clearly describe situations of Keynesian unemployment or other types of short run imbalances.

The paper is organized as follows. Section 2 presents the general model of a temporary economy with production and consumption. In Section 3, after defining the equilibrium concept, we discuss its relationship with respect to the temporary competitive equilibrium and to the equilibrium concepts proposed by Drèze and Benassy. Section 4 contains an example which displays the characteristic differences among the different concepts. An answer to the existence question—a *sine qua non* for any newly proposed equilibrium concept—has been deferred to the last section in order not to interrupt the discussion and the characterization of equilibria with rationing. Section 5 states and proves a general existence theorem.

2. A MODEL OF TEMPORARY EQUILIBRIUM WITH QUANTITY RATIONING

To represent the temporary nature of the economy we will follow the description given by Benassy (1975) and by Grandmont and Laroque (1976) as closely as possible. (See also Grandmont (1974) and Sondermann (1974)).

The economy consists of a finite number of agents, consumers and producers, indexed by $i = 1, ..., I$. We will also choose I to denote the set of all agents and write $i \in I$. No confusion will arise. Consider the economy in a given period. There are K consumption goods indexed by $k = 1, ..., K$ and H types of labour $h = 1, ..., H$. The set of all non-monetary commodities is $N = K \cup H = \{1, ..., n, ..., N\}$. Money is the $N+1$st commodity, which is the only store of value to be carried over from one period to the next.

For the typical consumer i, let $\underline{l}_i \in R^H_-$ denote the vector of maximum hours of labour which he is able to perform. Let $L_i = \{l \in R^H_- \mid \underline{l}_i \leqq l \leqq 0\}$. We choose as consumer i's consumption set X_i equal to $R^K_+ \times L_i$, a closed and convex set. The initial resources of consumer i are given by a vector $\omega_i \in R^N_+$ with $\omega_{ih} = 0$, $h \in H$, and by his initial stock of money holdings $m_i^0 \in R_+$.

Let $s = (p, w, 1)$ denote the vector of fixed prices in the given period where $p \in \text{int } R^K_+$, $w \in \text{int } R^H_+$ are commodity prices and wages respectively and where the price of money has been chosen to be equal to unity. The preferences of consumer i, given his expectations about future prices can then be represented by an expected utility index $u_i: R^K_+ \times L_i \times R_+ \to R$ associating with each consumption plan $x_i \in X_i$, each transfer of money to the next period m_i, a finite utility $u_i(a_i)$ where $a_i = (x_i, m_i)$. His actions are constrained to satisfy

$$s.a_i = s.(\omega_i, m_i^0).$$

For any action a_i let $z_i = x_i - \omega_i$ denote the associated net trade of consumer i.

The typical producer $i \in I$ is characterized by his production possibility set $\tilde{X}_i$ and his initial endowment (ω_i, m_i^0) where $\omega_i \in R^N_+$ is the vector of all produced and stocked commodities which are the result of production decisions of the previous periods. As for the consumer $\omega_{ih} = 0$ for $h \in H$, $m_i^0 \geqq 0$ represents the stock of money which the producer has at his disposal at the beginning of the period. For the production possibilities of every producer it is assumed that the length of any production process is at least one period implying that for all inputs he chooses in the current period outputs will accrue not earlier than the next period. Therefore all of his sales possibilities in the current period are completely determined by his initial resources and all of his purchases will be inputs to be used in production.

Let X_i denote the projection of $\tilde{X}_i$ into the space $R_+^K \times R_+^H$ which represents the set of possible input combinations to be chosen in the current period. We will assume that X_i is a closed and convex set containing the origin and the point ω_i of the particular producer where the latter assumption implies that he can at least store or destroy whatever he produced without further inputs. Following Sondermann (1974) we assume that for given current prices s the producer's preferences can be represented by an expected utility index u_i: $R_+^K \times R_+^H \times R_+ \rightarrow R$ which associates with any action $a_i = (y_i, m_i)$, $y_i \in X_i$, his expected utility $u_i(a_i)$ given current prices and his expectations on future prices. His actions are constrained to satisfy $s.a_i = s.(\omega_i, m_i^0)$. For any action a_i the producer's desired net trade is given by $z_i = y_i - \omega_i$.

Given any list of desired net trades (z_i), $i \in I$, a rationing scheme is a list of functions which determine for each agent i his actual net trade of every commodity such that all markets are cleared. The most general form of such schemes would clearly include other variables than the desired net trades of each agent. The case with price dependent rationing schemes was discussed by Drèze (1975) where it served to derive sufficient conditions for the existence of the particular equilibrium concept already discussed. For our purposes here such additional complexity does not seem to add further insight into the problem of defining equilibria at fixed prices. In principle, we will follow the formal structure of rationing schemes given by Benassy (1975).

For any given market $n \in N$, we assume that there exists a list of functions (F_{in}), $i \in I$, F_{in}: $R^I \rightarrow R$, such that for every list of desired net trades (z_{in}) on this market $F_{in}(z_{1n}, ..., z_{In})$ describes the actual net trade of i and such that $\Sigma_i F_{in}(z_{1n}, ..., z_{In}) = 0$. To simplify notation let $z = (z_1, ..., z_N)$ an element of $(R^N)^I$, $F_i = (F_{i1}, ..., F_{iN})$, for all i, $z_n = (z_{1n}, ..., z_{In})$ and $F_n = (F_{1n}, ..., F_{In})$. Thus we can write F_i as a function from $(R^N)^I$ to R^N, and the rationing scheme as a list of such functions, one for every i such that $\Sigma_i F_i(z) = 0$ for every $z \in (R^N)^I$. Finally, we adopt the notational convention of denoting by a subscript $\hat{i}$ any list of fixed desired net trades of agents other than i, for example $z_{\hat{i}} = (z_1, ..., z_{i-1}, z_{i+1}, ..., z_I)$ or, when considering a single market n, $z_{\hat{i}n} = (z_{1n}, ..., z_{i-1,n}, z_{i+1,n}, ..., z_{I,n})$. Then, by the obvious extensions, $F_i(z) = F_i(z_i, z_{\hat{i}})$ when F_i is considered as a function of z_i alone at fixed $z_{\hat{i}}$.

The rationing scheme we defined is independent across markets. This assumption has been made partly for simplicity and partly for lack of a general theory which explains interdependent rationing schemes. Moreover, this assumption enables us to demonstrate the continuity of the correspondence of attainable trades for each agent from the simple continuity assumption of each F_{in}. This would not have been the case for an interdependent rationing scheme.

The following two assumptions are sufficiently intuitive and need no further justification. They were given in a slightly different form by Benassy. If they are not fulfilled one is typically led to other than decentralized market economies.

(R1) *Voluntary exchange*

For every i, for every $n \in N$ and for every $z_{\hat{i}n}$

(a) $|F_{in}(z_{in}, z_{\hat{i}n})| \leqq |z_{in}|$

(b) $F_{in}(z_{in}, z_{\hat{i}n}).z_{in} \geqq 0$

(a) and (b) express the fact that an agent i is never forced to accept a trade which is on the other side of the market for which he proposes a net trade and that he is never forced to accept any larger transaction than he is proposing himself. We shall say that the individual is not rationed on market n if $F_{in}(z_{in}, z_{\hat{i}n}) = z_{in}$, i.e. if he receives or is able to sell what he proposes.

(R2) *Rationing on the long side only*

For every market n, for every $z_n \in R^I$ and for every $i \in I$

(a) $(\sum_j z_{jn})(z_{in} - F_{in}(z_n)) \geqq 0$

(b) $\sum_j z_{jn} = 0 \Rightarrow z_{in} = F_{in}(z_n)$

(a) in conjunction with (R1) implies that whenever there is an excess demand (supply) in a market then no seller (buyer) is rationed.

The formal structure of our model allows now to associate in a straightforward manner with any vector z of proposed net trades the actual transactions $F_i(z)$, $i \in I$, and, furthermore, the utility of each agent associated with the resulting action. Some care must be taken, however, when making this final step since nothing in the assumptions made so far guarantees that the transactions $F_i(z)$ are feasible given the actual constraints of the agents in the economy. The rationing scheme was not linked in any way to the actual data of the given economy—and there was no theoretical reason to do it so far.

Since every agent may express any arbitrary quantity as his desired demand or supply we will, in general, take as agent i's strategy space Z_i all of R^N. Then, to complete the model at this stage we define each agent's utility as follows. Given any $z \in (R^N)^I$ we call $F_i(z)$ feasible for i if

$$(F_i(z) + \omega_i,\ m_i^0 - \textstyle\sum_n s_n F_{in}(z_n)) \in X_i \times R_+.$$

Then the utility of agent i for an I-tuple of strategies z is given by

$$U_i(z) = \begin{cases} u_i(F_i(z) + \omega_i,\ m_i^0 - \sum_n s_n F_{in}(z_n)) & \text{if } F_i(z) \text{ feasible} \\ -\infty & \text{otherwise.} \end{cases}$$

3. EQUILIBRIUM WITH RATIONING

The structure developed in the previous section suggests directly a Nash equilibrium as the appropriate equilibrium concept of this model since it can be viewed as a non-cooperative game in normal form. If $(z_1^*, \ldots, z_I^*)$ is a Nash equilibrium and if, in the short run, each agent i believes that the other agents will propose net trades $z_{\bar{i}}^*$, then his best strategy is simply one of choosing his best response net trade z_i^* given the net trades of all others and taking into account his influence on his constraints through the rationing scheme.

Let

$$\xi_i(z_{\bar{i}}) = \{z_i \in R^N \mid U_i(z_i, z_{\bar{i}}) \geqq U_i(z_i', z_{\bar{i}}) \quad \text{for all } z_i' \in Z_i\}$$

denote the set of best response net trades of agent i given the strategies of all other agents. Then:

Definition. A list of net trades $z^* = (z_1^*, \ldots, z_I^*)$ is an equilibrium with quantity rationing if $z_i^* \in \xi_i(z_{\bar{i}}^*)$ for every $i \in I$.

The discussion of the non-emptiness of all ξ_i and of the existence of equilibria will be deferred to the last section. However, some general properties of such equilibria can be given at this point.

With our assumptions on the initial resources and on the consumption and production sets any agent can always realize a feasible allocation for himself regardless of all other proposed net trades by choosing $z_i = 0$. Thus he can always achieve a finite utility level avoiding a realization of $-\infty$. Furthermore, if $F_i(z)$ is feasible for every $i \in I$, then the resulting allocation (a_i), $i \in I$, is obviously feasible.

Our next two results relate the equilibrium with rationing to the temporary competitive equilibrium, i.e. where each agent maximizes his utility on his budget set without any additional constraints, and to the equilibrium proposed by Drèze.

Before proceeding to these results, however, we will indicate in Lemma 1 some basic properties of the class of continuous rationing schemes. These properties describe precisely how far each agent can manipulate the outcome of the rationing process in his favour. Roughly speaking these properties enable any agent to decrease the amount exchanged continuously to the level of zero by reducing his own proposed net trade given the proposed net trades of all others. On the other hand any agent on the short side of a market in disequilibrium can also increase the amount exchanged.

Lemma 1. *Let* $F = (F_1, \ldots, F_I)$ *be a continuous rationing scheme for which* (R1) *and* (R2) *hold. Consider any market n and any list of desired net trades* z_n. *Then*

(a) *For every* $i \in I$ *and for every* α *such that* $0 \leqq \alpha \leqq |F_{in}(z_n)|$ *there exists* z'_{in} *such that*

$$|F_{in}(z'_{in}, z_{\bar{i}n})| = \alpha.$$

(b) *If* $\Sigma_j z_{jn} > 0$, *then for every i such that* $z_{in} \leqq 0$, *there exist* $\zeta_{in} < F_{in}(z_n)$ *such that for all* $z'_{in} \in]\zeta_{in}, F_{in}(z_n)]$, *one has* $F_{in}(z'_{in}, z_{\bar{i}n}) = z'_{in}$. *Symmetrically,* $\Sigma_j z_{jn} < 0$ *and* $z_{in} \geqq 0$ *implies the existence of* $\bar{\zeta}_{in} > F_{in}(z_n)$ *such that for all* $z'_{in} \in [F_{in}(z_n), \bar{\zeta}_{in}[$ *one has*

$$F_{in}(z'_{in}, z_{\bar{i}n}) = z'_{in}.$$

Proof. (a) follows immediately from (R1) and from the continuity of F by using the intermediate value theorem.
To show (b) one observes that (R1) (R2), and, for example, $\Sigma_j z_{jn} > 0$ with $z_{in} \leqq 0$ together imply

$$(\Sigma_j z_j)(z_{in} - F_{in}(z_{in}, z_{\bar{i}n})) = 0$$

or equivalently $z_{in} = F_{in}(z_{in}, z_{\bar{i}n})$. Hence there exists ζ_{in} such that, for every

$$z'_{in} \in]\zeta_{in}, F_{in}(z_n)], \ \Sigma_{j \neq i} z_{jn} + z'_{in} > 0.$$

Therefore, using (R1) and (R2),

$$(\textstyle\sum_{j \neq i} z_{jn} + z'_{in})(z'_{in} - F_{in}(z'_{in}, z_{\bar{i}n})) \geqq 0$$

implies $z'_{in} = F_{in}(z'_{in}, z_{\bar{i}n})$. ‖

Another way of stating property (b) of the Lemma is the following. Let

$$\Phi_{in}(z_{\bar{i}n}) = \{x_{in} \mid x_{in} = F_{in}(z_{in}, z_{\bar{i}n}), z_{in} \in R\}$$

denote the set of attainable exchanges of agent i in market n. If at z_n agent i is on the short side of the market, then $F_{in}(z_n) \in \text{int}\ \Phi_{in}(z_{\bar{i}n})$. Proposition 1 uses this property in a strong way.

Definition. For any arbitrary strategy set Z_i and any $z_{\bar{i}}$, let

$$\Phi_i(z_{\bar{i}}) = \{x_i \in R^N \mid x_i = F_i(z_i, z_{\bar{i}}), \text{ some } z_i \in Z_i\}.$$

Proposition 1. *Let* (z_i^*) *be an equilibrium for the rationing scheme* F. *If for every agent i the utility function* u_i *is semi strictly quasi-concave and if* $F_i(z^*) \in \text{int}\ \Phi_i(z_{\bar{i}}^*)$, *then* $x_i^* = F_i(z^*)$ *is a temporary competitive equilibrium, i.e. for every* $i \in I$, $(\omega_i + x_i^*, m_i^0 - \Sigma_{n \in N} s_n x_{in}^*)$ *maximises* $u_i(\omega_i + x_i, m_i)$ *on the budget set* $s.(x_i, m_i) = m_i^0$.

The assumption that, for every i, $F_i(z^*) \in \text{int}\ \Phi_i(z_{\bar{i}}^*)$ is very strong. The class of proportional rationing schemes (see for example Shapley (1976)) provides an example where this property holds for all agents and for all proposed net trades. One can easily construct non-pathological examples where it does not hold. However, it seems impossible to replace it by a weaker one to obtain the same result since global optimality of each agent's decision cannot be shown to hold unless some local perturbation of his final transactions in both directions in all markets is possible. The result indicates the importance of the interiority property to obtain a global unconstrained maximum for each consumer if prices are fixed correctly. A rationing scheme which does not enable each agent to manipulate his final outcome locally may fail to generate a temporary equilibrium even if prices were set correctly.

Proof. Suppose the result were false. Then there exist an agent i and a feasible net trade x_i in his budget set, such that

$$u_i(\omega_i+x_i,\ m_i)>u_i(\omega_i+x_i^*,\ m_i^0-\sum_{n\in N} s_n x_{in}^*),$$

where $m_i = m_i^0-\sum_{n\in N}s_n x_{in}$. Since u_i is semi strictly quasi-concave,

$$u_i(\omega_i+\lambda x_i+(1-\lambda)x_i^*,\ \lambda m_i+(1-\lambda)m_i^*)>u_i(\omega_i+x_i^*,\ m_i^*)$$

for every λ between zero and one.

Since $F_i(z^*)\in \operatorname{int}\Phi_i(z_i^*)$ for λ small enough there exist z'_{in}, $n\in N$, such that

$$\lambda x_{in}+(1-\lambda)x_{in}^* = F_{in}(z'_{in},\ z_{in}^*).$$

As a consequence one obtains

$$\lambda m_i+(1-\lambda)m_i^* = m_i^0-\sum_{n\in N} s_n F_{in}(z'_{in},\ z_{in}^*),$$

which contradicts the fact that (z_i^*) is an equilibrium for the rationing scheme. ||

To compare this with the equilibrium proposed by Drèze we first give a complete definition of the concept. For every $i\in I$, let $(\underline{\zeta}_i, \bar{\zeta}_i)$, $\underline{\zeta}_i\in R_-^N$ and $\bar{\zeta}\in R_+^N$ denote a pair of lower and upper constraints on net trades perceived by agent i.

Let

$$\beta_i(\underline{\zeta}_i,\ \bar{\zeta}_i) = \{x_i\in R^N \mid \underline{\zeta}_i \leqq x_i \leqq \bar{\zeta}_i,\ s.(\omega_i+x_i,\ m_i) = m_i^0,\ (\omega_i+x_i,\ m_i)\in X_i\times R_+\}$$

denote the associated budget set. Define the set of net trades which maximize utility on the budget set, while respecting the constraints, by

$$\xi_i(\underline{\zeta}_i,\ \bar{\zeta}_i) = \{x_i\in\beta_i(\underline{\zeta}_i,\ \bar{\zeta}_i) \mid u_i(\omega_i+x_i,\ m_i) \geqq u_i(\omega_i+x'_i,\ m'_i) \quad \text{for all } x'_i\in\beta_i(\underline{\zeta}_i,\ \bar{\zeta}_i)\}.$$

Definition. A list (x_i), $i\in I$, describes an equilibrium with rationing in the sense of Drèze for perceived constraints $(\underline{\zeta}_i, \bar{\zeta}_i)$, $i\in I$, if

(E.1) $\Sigma_{i\in I}\, x_i = 0$;

(E.2) $x_i\in\xi_i(\underline{\zeta}_i, \bar{\zeta}_i)$ for every $i\in I$;

(E.3) for every $n\in N$

(i) $x_{in} = \bar{\zeta}_{in}$ for some $i\in I$ implies $\underline{\zeta}_{in}<x_{in}$ for all $i\in I$;

(ii) $x_{in} = \underline{\zeta}_{in}$ for some $i\in I$ implies $x_{in}<\bar{\zeta}_{in}$ for all $i\in I$.

(E.1) and (E.2) are self-explanatory. (E.3) avoids rationing of supply and demand at the same time on the same market. The strict inequalities in (i) and (ii) require that at the equilibrium actual optimal trades for any agent are strictly below the perceived constraints on the short sides of all markets. This indicates a very close relationship to the interiority condition and supplies the justification for assumption (b) in Proposition 2.

For a thorough understanding of the relation between an equilibrium relative to a rationing scheme and the equilibrium proposed by Drèze a few more remarks are in order. It is well known that for a fixed price system the set of equilibria in the sense of Drèze may be very large. This is partly due to the fact that apart from condition (E.3) the constraints have to satisfy no further requirement. In particular no specific distribution is required. On the other hand a particular rationing scheme F in the present context eliminates a great deal of this ambiguity by describing a specific allocation rule for each agent in each market. One would therefore expect that the set of equilibrium allocations is smaller than under the Drèze definition. It is therefore natural to ask the question whether an equilibrium allocation relative to a specific rationing scheme F could also be sustained as a Drèze equilibrium, i.e. whether one can find a distribution of fixed perceived constraints such that non-strategic behaviour would lead to the same allocation as the strategic behaviour relative to F. The answer to this question is in the affirmative as long as all markets are in disequilibrium without any further requirement on the rationing scheme. This clearly indicates the qualitative strength of property (E.3) as a necessary condition for an equilibrium with rationing, whether non-strategic or strategic behaviour is allowed. If some markets are balanced at an equilibrium relative to F, the allocation may not be an equilibrium in the sense of Drèze unless some interiority conditions hold at the equilibrium. These results are stated in Proposition 2.

Proposition 2. *Let* F *be a continuous rationing scheme which satisfies* (R1) *and* (R2). *Let* (z_i^*) *be an equilibrium relative to* F. *Then, there exists a list of perceived constraints* $(\underline{\zeta}_i, \bar{\zeta}_i)$, $i \in I$ *with* $\underline{\zeta}_i \leqq 0 \leqq \bar{\zeta}_i$ *such that* $F_i(z^*) \in \xi_i(\underline{\zeta}_i, \bar{\zeta}_i)$ *for every* $i \in I$.

Furthermore, if

(a) $\Sigma_{i \in I} z_{in}^* \neq 0$, *for all markets n*

or if

(b) $\Sigma_{i \in I} z_{in}^* = 0$ *implies*

$$F_{in}(z_n^*) \in \operatorname{int} \Phi_{in}(z_{in}^*) \quad \textit{for all i such that } z_{in}^* < 0,$$

then the list of actual net trades (x_i), $i \in I$, *with* $x_i = F_i(z^*)$ *satisfies condition* (E.3) *as well.*

Proof. For every $i \in I$ and every $n \in N$, $\Phi_{in}(z_{in}^*)$ is a bounded interval which contains zero, since F is continuous and because of (R1). For every $i \in I$ define the constraints $(\underline{\zeta}_i, \bar{\zeta}_i)$ by

$$\underline{\zeta}_{in} = \inf \Phi_{in}(z_{in}^*)$$

$$\bar{\zeta}_{in} = \sup \Phi_{in}(z_{in}^*).$$

Since (z_i^*), $i \in I$, is an equilibrium relative to F one clearly has

$$\max_{z_i} u_i(\omega_i + F_i(z_i, z_i^*), m_i^0 - \textstyle\sum_n s_n F_{in}(z_{in}, z_{in}^*))$$

$$= \max \{u_i(\omega_i + x_i', m_i') \mid x_i' \in \Phi_i(z_i^*), s.(\omega_i + x_i', m_i') = m_i^0\}$$

$$= u_i(\omega_i + x_i, m_i)$$

where $x_i = F_i(z^*)$ and $m_i = m_i^0 - \Sigma_n s_n F_{in}(z_n^*)$.
Hence $x_i \in \xi_i(\underline{\zeta}_i, \bar{\zeta}_i)$ for every $i \in I$.

Let n be such that $\sum_{j \in I} z_{jn}^* \neq 0$. Then, according to Lemma 1,

$$\textstyle\sum_{j \in I} z_{jn}^* > 0 \text{ and } z_{in}^* \leqq 0 \quad \text{implies } \underline{\zeta}_{in} < x_{in}$$

and

$$\textstyle\sum_{j \in I} z_{jn}^* < 0 \text{ and } z_{in}^* \geqq 0 \quad \text{implies } x_{in} < \bar{\zeta}_{in}.$$

Therefore, for these markets (E.3) holds. On the other hand, if $\Sigma_{j \in I} z_{jn}^* = 0$, we have by hypothesis:

$$x_{in} \in \operatorname{int} \Phi_{in}(z_{in}^*) \quad \text{if } z_{in}^* < 0$$

which implies $\underline{\zeta}_{in} < x_{in}$. Hence (E.3) is satisfied for all $n \in N$. ||

It is also true that, for any Drèze equilibrium, there exists a rationing scheme F which has an equilibrium with the same actual net trades as the Drèze equilibrium. The proof of this result is straightforward. We therefore only state the result as Proposition 2′.

Proposition 2′. *Let (x_i), $i \in I$, denote an equilibrium in the sense of Drèze for given constraints $(\underline{\zeta}_i, \bar{\zeta}_i)$, $i \in I$. Then there exists a rationing scheme* F *satisfying* (R1) *and* (R2) *and strategies (z_i^*), $i \in I$ such that, for every $i \in I$, z_i^* maximizes*

$$u_i(\omega_i + F_i(z_i, z_{\bar{i}}^*), m_i^0 - \Sigma_n s_n F_{in}(z_{in}, z_{\bar{i}n}^*))$$

and $x_i = F_i(z^*)$.

Finally, we would like to indicate a relationship between our equilibrium concept and the one proposed by Benassy (1975, Section 4)

It is well known that at an equilibrium in the sense of Benassy agents maximize utility given the rationing constraints provided the utility functions are strictly quasi-concave. However no strategic element of the behaviour of an agent is involved. Proposition 3 shows that all equilibria in the sense of Benassy can also be sustained as equilibria under strategic behaviour for the class of independent rationing schemes (for a precise definition see below) which confirm the individual perceptions of the rationing mechanism at equilibrium.

Let (F_i), $i \in I$, denote the rationing scheme and let $(\underline{G}_i, \bar{G}_i)$, $i \in I$, denote the list of the two functions determining perceived lower and upper constraints on all markets for each $i \in I$. For given effective demands[4] $\tilde{z}$, $\underline{G}_{in}(\tilde{z}_{in}, \tilde{z}_{\bar{i}n})$ represents the lower constraint perceived by i on market n if $\tilde{z}_{in} \leqq 0$ and $\bar{G}_{in}(\tilde{z}_{in}, \tilde{z}_{\bar{i}n})$ represents the upper constraint perceived by i on market n if $\tilde{z}_{in} \geqq 0$.

The following definition of an independent rationing scheme describes the class of functions for which each rationed agent i has no influence on his realized trades by overstating his desired demands or supplies.

Definition. A rationing scheme (F_i) $i \in I$, is called independent for all $i \in I$, if for every market n, there exists for every $i \in I$ a pair of functions $(\underline{\phi}_{in}, \bar{\phi}_{in})$ defined on R^{I-1} such that $\underline{\phi}_{in} \leqq 0 \leqq \bar{\phi}_{in}$ and

$$z_{in} \geqq 0 \quad \text{implies } F_{in}(z_{in}, z_{\bar{i}n}) = \min\{z_{in}, \bar{\phi}_{in}(z_{\bar{i}n})\}$$

and

$$z_{in} \leqq 0 \quad \text{implies } F_{in}(z_{in}, z_{\bar{i}n}) = \max\{z_{in}, \underline{\phi}_{in}(z_{\bar{i}n})\}$$

Proposition 3. *Let* F *be an independent rationing scheme and let $(\underline{G}_i, \bar{G}_i)$, $i \in I$, denote the list of functions describing perceived constraints which satisfy the conditions (α) and (β) of Benassy* (1975, *p.* 506) *and assume that*

(i) $\underline{G}_{in}(\tilde{z}_n) = F_{in}(\tilde{z}_n)$ *implies* $F_{in}(\tilde{z}_n) = \underline{\phi}_{in}(\tilde{z}_{\bar{i}n})$

and

(ii) $\bar{G}_{in}(\tilde{z}_n) = F_{in}(\tilde{z}_n)$ *implies* $F_{in}(\tilde{z}_n) = \bar{\phi}_{in}(\tilde{z}_{\bar{i}n})$.

Then every equilibrium in the sense of Benassy for $(\underline{G}_i, \bar{G}_i)$ is an equilibrium relative to F, *if u_i is strictly quasi-concave for every $i \in I$.*

Conditions (i) and (ii) require that at the equilibrium actual constraints and perceived constraints coincide, whenever they are binding. The requirement that the rationing mechanism is independent is a natural consequence of the property of the perceived constraints function to coincide with the actual rationing level at an equilibrium in the sense of Benassy. Moreover the description of the perceived constraints functions by Benassy seems to imply the general idea of independent rationing schemes (see also Benassy (1976)). If the rationing scheme F is not flat at an equilibrium allocation but perceived constraints are binding, then strategic behaviour of some agent will cause a change of the allocation. In this case a result as stated in Proposition 3 will not hold in general.

Proof. If $(\tilde{z}_i)$, $i \in I$, is an equilibrium of Benassy and if u_i is strictly quasi-concave for every $i \in I$, then (see Grandmont (1977, Lemma 4, p. 562))

$$x_i = F_i(\tilde{z}) \quad \text{maximizes } u_i(x_i' + \omega_i, m_i')$$

subject to

$$(x_i'+\omega_i, m_i') \quad \text{is feasible for } i$$

and

$$\underline{G}_i(\tilde{z}) \leqq x_i' \leqq \bar{G}_i(\tilde{z}).$$

Since (F_i), $i \in I$ is independent one has

$$\Phi_i(\tilde{z}_i) = [\underline{\phi}_i(\tilde{z}_i), \bar{\phi}_i(\tilde{z}_i)].$$

We will show that

$$x_i = F_i(\tilde{z}) \quad \text{maximizes } u_i(x_i'+\omega_i, m_i')$$

subject to

$$(x_i'+\omega_i, m_i') \quad \text{is feasible for } i$$

and

$$x_i' \in \Phi_i(\tilde{z}_i).$$

Four possible cases may occur for each market n

(a) $F_{in}(\tilde{z}_n) \in \text{int}\,[\underline{G}_{in}(\tilde{z}_n), \bar{G}_{in}(\tilde{z}_n)]$.

(b) $\underline{G}_{in}(\tilde{z}_n) < F_{in}(\tilde{z}_n) = \bar{G}_{in}(\tilde{z}_n)$
which implies $F_{in}(\tilde{z}_n) = \bar{\phi}_{in}(\tilde{z}_{in})$.

(c) $F_{in}(\tilde{z}_n) = \underline{G}_{in}(\tilde{z}_n) < \bar{G}_{in}(\tilde{z}_n)$
which implies $F_{in}(\tilde{z}_n) = \underline{\phi}_{in}(\tilde{z}_{in})$.

(d) $F_{in}(\tilde{z}_n) = \underline{G}_{in}(\tilde{z}_n) = \bar{G}_{in}(\tilde{z}_n)$
which implies $\{F_{in}(\tilde{z}_n)\} = \Phi_{in}(\tilde{z}_{in})$.

Suppose that $\tilde{z}_i$ does not maximize i's utility against $\tilde{z}_i$. There exists a feasible $x_i' \in \Phi_i(\tilde{z}_i)$ such that

$$u_i(\omega_i+x_i', m_i^0-\textstyle\sum_n s_n x_{in}') > u_i(\omega_i+F_i(\tilde{z}), m_i^0-\textstyle\sum_n s_n F_{in}(\tilde{z}_n)).$$

Since u_i is strictly quasi-concave, for all λ, $0<\lambda<1$, one has

$$u_i(\omega_i+\lambda F_i(\tilde{z})+(1-\lambda)x_i'), m_i^0-\textstyle\sum_n s_n(\lambda F_{in}(\tilde{z}_n)+(1-\lambda)x_{in}') > u_i(\omega_i+F_i(\tilde{z}), m_i^0-\textstyle\sum_n s_n F_{in}(\tilde{z}_n)).$$

For λ sufficiently close to one, we have

$$\lambda F_{in}(\tilde{z}_n)+(1-\lambda)x_{in}' \in [\underline{G}_{in}(\tilde{z}_n), \bar{G}_{in}(\tilde{z}_n)]$$

in case (a). Likewise for (b) since

$$\lambda F_{in}(\tilde{z}_n)+(1-\lambda)x_{in}' \leqq \bar{\phi}_{in}(\tilde{z}_n) = \bar{G}_{in}(\tilde{z}_n)$$

and for (c) since

$$\lambda F_{in}(\tilde{z}_n)+(1-\lambda)x_{in}' \geqq \underline{\phi}_{in}(\tilde{z}_n) = \underline{G}_{in}(\tilde{z}_n).$$

For case (d) the inclusion holds trivially. Hence, one obtains a contradiction. ||

4. AN EXAMPLE

This section contains a comparison of the three different equilibrium concepts discussed in Section 3 for a particular economy. The example has been chosen in such a way that for all three equilibrium concepts there exist equilibria with unemployment and with overproduction.

Consider an economy with two consumers $i = 1, 2$ and one producer $i = 3$. There is only one consumption good x, one type of labour l and money m.

For a consumer $i = 1, 2$, let $x_i \geqq 0$ denote his consumption, $|l_i|$ denote the amount of labour supplied satisfying $-1 \leqq l_i \leqq 0$ and $m_i \geqq 0$ his stock of money at the end of the period. Both consumers have identical temporal utility functions defined by

$$u_i(x_i, l_i, m_i) = x_i^2(l_i+1)m_i.$$

The initial resources of each consumer consist only of money transferred from the previous period. Let $m_i^0 \geqq 0$, $i = 1, 2$, denote the initial stock.

The producer sells the quantity $x_3 \geqq 0$ of the consumption good constrained by his initial resources $\omega_3 \geqq 0$. Since labour produces output one period later, labour l_3 does not constrain x_3. For the current price p^1, the wage rate w and an expected price p^2, the producer's utility function is defined by

$$u_3(x_3, l_3) = p^2\sqrt{l_3}-wl_3+p^1x_3.$$

A feasible state of the economy is given by a list (x_i, l_i, m_i), $i = 1, 2$, and (x_3, l_3) such that

$$x_1+x_2 = x_3$$

$$l_1+l_2+l_3 = 0,$$

and for $i = 1, 2$

$$x_i \geqq 0,\ 0 \geqq l_i \geqq \underline{l}$$

$$m_i = m_i^0-p^1x_i-wl_i \geqq 0$$

and

$$l_3 \geqq 0,\ 0 \leqq x_3 \leqq \omega_3.$$

For the parameters of the system we take the following values:

$$m_1^0 = 2w,\ m_2^0 = 0$$

$$m_3^0 \geqq -w\underline{l}$$

$$p^1 = 4\sqrt{\tfrac{2}{3}}w$$

$$p^2 = \frac{2}{\sqrt{6}}\left(1-\frac{\underline{l}}{2}\right)w$$

$$\omega_3 = \frac{1}{\sqrt{6}}(1-\underline{l})$$

where $\underline{l} = -2(5-2\sqrt{6})$ which is greater than $-\frac{1}{4}$. With these assumptions one clearly has $p^1>w$ and $p^2<p^1$, where the latter inequality implies pessimistic expectations of the producer.

Let z_{il} and z_{ix}, $i = 1, 2, 3$ denote desired net trades of agent i for labour and the consumption good respectively. Since we will be concerned with excess supply equilibria only it suffices to describe the rationing scheme on the goods market for the producer and on the labour market for the consumers. On the goods market the rationing F_{3x} follows the simple rule

$$F_{3x}(z_{1x}, z_{2x}, z_{3x}) = -\min\{-z_{3x}, (z_{1x}+z_{2x})\}.$$

For the labour market we have for $i = 1, 2$

$$F_{il}(z_{1l}, z_{2l}, z_{3l}) = \begin{cases} -\max\{z_{3l}+z_{\bar{\imath}l}, 0\} & \text{if } z_{il}>z_{\bar{\imath}l} \\ -\frac{1}{2}\min\{z_{3l}, -(z_{1l}+z_{2l})\} & \text{if } z_{il} = z_{\bar{\imath}l} \\ -\min\{z_{3l}, -z_{il}\} & \text{if } z_{il}<z_{\bar{\imath}l}. \end{cases}$$

If $z_{il}<\underline{l}$, $i = 1, 2$, then $F_{il}(z_{il}, z_{\bar{\imath}l}, z_{3l}) = F_{il}(\underline{l}, z_{\bar{\imath}l}, z_{3l})$. This rationing scheme displays the special feature that the consumer who is offering to work the larger amount will always be served first.

4.1. *Equilibrium relative to* F

An excess supply equilibrium relative to F is a triple (z_1^*, z_2^*, z_3^*) such that

$$\textstyle\sum_{i=1}^{3} z_{il}^* < 0 \quad \text{and} \quad \sum_{i=1}^{3} z_{ix}^* < 0.$$

Since in this case the producer is not rationed on the labour market, one finds immediately that $z_{3l}^* = -l$. Best response strategies for the consumers are then calculated in a straightforward manner to come to

$$z_{il}^* = \underline{l} \quad i = 1, 2$$

$$z_{1x}^* = \frac{1}{\sqrt{6}}\left(1 - \frac{l}{4}\right) \quad \text{and} \quad z_{2x}^* = -\frac{1}{4\sqrt{6}}\, l.$$

Hence $F_{3x}(z_{1x}^*, z_{2x}^*, z_{3x}) = -(1/\sqrt{6})(1-(l/2))$ for $z_{3x} \leqq -(1/\sqrt{6})(1-(l/2))$. Since $\omega_3 = (1/\sqrt{6})(1-l) > (1/\sqrt{6})(1-(l/2))$ and since u_3 is linear in x_3 we have z_{3x}^* optimal if $-\omega_3 \leqq z_{3x}^* \leqq -(1/\sqrt{6})(1-(l/2))$.

It can easily be checked that the realized net demands are unique for all (z^*). The optimal labour supplies $z_{il}^* = \underline{l}$ reflect directly the type of rationing on the labour market.

4.2 *Equilibria in the sense of Drèze*

Since the producer is not rationed on the labour market we have again $z_{3l}^* = -l$. Since the consumers will be rationed on the labour market the bounds $(\underline{\zeta}_{il})$, $i = 1, 2$, satisfy

$$-\tfrac{1}{4} \leqq \underline{\zeta}_{1l} \leqq 0,\ -\tfrac{3}{4} \leqq \underline{\zeta}_{2l} \leqq 0 \quad \text{with } \underline{\zeta}_{1l} + \underline{\zeta}_{2l} = l.$$

Optimal consumption for i is given by

$$z_{1x}^* = \frac{1}{\sqrt{6}}\left(1 - \frac{\underline{\zeta}_{1l}}{2}\right) \qquad \text{with } \underline{\zeta}_{1l} + \underline{\zeta}_{2l} = l$$

$$z_{2x}^* = -\frac{1}{\sqrt{6}}\,\frac{\underline{\zeta}_{2l}}{2}$$

so that

$$z_{3x}^* = -\frac{1}{\sqrt{6}}\left(1 - \frac{l}{2}\right).$$

4.3. *Equilibrium in the sense of Benassy*

At an excess supply equilibrium $(\tilde{z}_i)$, $i = 1, 2, 3$, one has for the producer:

$$\tilde{z}_{3,l} = -l \quad \text{and} \quad \tilde{z}_{3x} = -\omega_3.$$

Let $\bar{G}_i$, $i = 1, 2$, denote the functions describing perceived constraints for the consumers on the goods market with the properties

$$\bar{G}_1(z_x) \geqq \frac{9}{8\sqrt{6}}$$

$$\bar{G}_2(z_x) \geqq \frac{3}{8\sqrt{6}}$$

if

$$\textstyle\sum_{i=1}^{3} z_{ix} \leqq \dfrac{1}{\sqrt{6}}\,\dfrac{l}{2}.$$

Then effective supplies of labour for both consumers are equal to their optimal supplies without constraints, i.e.

$$\tilde{z}_{1l} = -\tfrac{1}{4},\ \tilde{z}_{2l} = -\tfrac{3}{4}.$$

The rationing on the labour market implies:

$$F_{1l}(\tilde{z}_l) = 0, \quad F_{2l}(\tilde{z}_l) = l$$

and effective demands in consumption:

$$\tilde{z}_{1x} = \frac{1}{\sqrt{6}} \quad \text{and} \quad \tilde{z}_{2x} = \frac{5-2\sqrt{6}}{\sqrt{6}}.$$

Therefore, $\tilde{z}$ is the only excess supply equilibrium in the sense of Benassy.

This result shows clearly that the first consumer would be better off if he had used the information on the rationing scheme (as in the case of an equilibrium relative to F).

5. EXISTENCE OF TEMPORARY EQUILIBRIA WITH QUANTITY RATIONING

The assumptions (R1) and (R2) together imply that the list of strategies $z_i = 0$, $i \in I$ is an equilibrium relative to F. Although this represents a trivial equilibrium formally, it may very well picture a situation in which, due to anticipations about future prices, every producer prefers to store his output from the previous period and not employ any labour, whereas the consumer may decide to consume his initial resources and not offer any labour. From this it is clear that without any further assumption existence of non-trivial equilibria cannot be guaranteed.

Our assumptions divide into three categories. The first contains the assumptions which guarantee the necessary continuity and convexity properties of all best response strategies given any list of appropriately chosen compact strategy sets Z_i for every agent i. Then, applying the standard fixed point argument for a non-cooperative game one obtains existence of an equilibrium for any economy with a closed set of attainable states. The second category consists of an assumption for one particular consumer which avoids having a unique trivial equilibrium. The third category describes a property of the rationing scheme which keeps best response strategies bounded if $Z_i = R^N$ for every $i \in I$. It is obvious that such an assumption must in some way exclude the strictly proportional rationing schemes since for these existence of equilibria may easily fail even if the set of feasible states of the economy is bounded. Our assumption intuitively describes the property of a rationing scheme for which an agent's influence to manipulate the final outcome, if he is on the long side, disappears if he increases his desired net trade beyond any limit.

For the economy we require the following three assumptions:

(A1) For every agent $i \in I$ the utility function u_i is continuous and strictly quasi-concave.

(A2) The set of feasible states of the economy is compact, i.e. the set

$$\{(a_i) \mid a_i \in X_i \times R_+ \quad \text{and} \quad \textstyle\sum_{i \in I} (a_i - (\omega_i, m_i^0)) = 0\}$$

is compact.

(A3) There exists a consumer j, a type of labour h' and a constant $\bar{l}_{jh'} < 0$, such that, for all feasible consumption plans $((y_j, l_j) + \omega_j, m_j)$ with $l_{jh'} \leqq \bar{l}_{jh'}$,

$$u_j((y_j, l_j) + \omega_j, m_j) \geqq u_j((y_j, l'_j) + \omega_j, m'_j)$$

for

$$l'_j = (l_{j1}, \ldots, l_{jh'-1}, 0, l_{jh'+1}, \ldots, l_{jH})$$

and

$$m'_j = m_i^0 - \textstyle\sum_{k \in K} s_k y_{jk} - \sum_{h \in H} s_h l'_{jh} \geqq 0.$$

Assumptions (A1) and (A2) need no comment. (A3) states that for some consumer and some type of labour which he can perform he is always willing to offer some non-zero amount of that type of labour up to the bound of his consumption set in exchange for money. This implies that he holds a positive preference for future consumption. The purpose of this assumption is to rule out the case where the equilibrium with zero proposed net trades is the only one. A similar assumption for at least one producer who would always prefer to sell some non-zero amount of one of his products rather than storing it or using it as an input would have served the same goal.

For the rationing scheme we impose three more assumptions:

(R3) F_i is a continuous function for every $i \in I$.

(R4) For every i, for every z_i and for every market n, $F_{in}(., z_{in})$ is non-decreasing in z_{in}.

It was pointed out at the beginning of this section that one crucial problem for proving existence is to eliminate those cases for which the rationing scheme implies an unbounded incentive to overstate desired net trades. This phenomenon may occur if the rationing scheme is proportional. Therefore, the condition one has to impose will involve properties so that an agent cannot always gain, i.e. increase his actual transaction by further overstating his desired net trade beyond some large but finite proposal. For an economy with a compact set of feasible states and a one-sided rationing scheme unbounded proposals can only occur on the long side of a market. Therefore the restriction on the class of rationing schemes which we are going to impose in assumption (R5) takes this explicitly into account. More specifically, we assume that unbounded proposals will lead to realizations which could have been obtained by some finite proposals if e.g. the proposals of agents on the short side remain bounded. Moreover, we require that this limitation on manipulation of the outcome depends in a continuous way on the bounded proposals of the other agents.

Let (z_n^r), $r = 1, \ldots$ denote a sequence of desired net trades in market n. Define

$$I_n = \{i \in I \mid \; |z_{in}^r| \to +\infty\}$$

and

$$\Phi_{in}^r(z_{in}^r) = \{x_{in}^r \mid \exists \, |z_{in}| \leqq r\colon x_{in}^r = F_{in}(z_{in}, z_{in}^r)\}$$

(R5) For every market n and for every sequence (z_n^r) satisfying

(a) $\| z_n^r \| \to +\infty$,

(b) for every $i \in I$,

$$z_{in}^r(\textstyle\sum_{j \in I} z_{jn}^r) \leqq 0 \quad \text{for every } r = 1, \ldots$$

or

$$z_{in}^r(\textstyle\sum_{j \in I} z_{jn}^r) \geqq 0 \quad \text{for every } r = 1, \ldots,$$

(c) $z_{in}^r(\Sigma_{j \in I} z_{jn}^r) \leqq 0$ implies $i \in I \backslash I_n$,

there exist bounded continuous functions

$$f_{in}\colon R^{I \backslash I_n} \to R \backslash \{0\} \quad i \in I_n$$

such that, for the sequence

$$\bar{z}_{in}^r = \begin{cases} f_{in}(z_{I \backslash I_n}^r) & i \in I_n \\ z_{in}^r & \text{otherwise} \end{cases}$$

one has for r sufficiently large

(i) $F_{in}(\bar{z}_n^r) = F_{in}(z_n^r)$ for all $i \in I$

and

(ii) $\Phi_{in}^r(\bar{z}_{in}^r) \subseteq \Phi_{in}^r(z_{in}^r)$.

Conditions (b) and (c) specify those sequences for which the boundedness assumptions (i) and (ii) hold. In particular condition (c) states that agents on the short side quote only bounded proposals. Property (i) indicates that every agent realizes the same actual transaction for the bounded sequence $\bar{z}_n^r$. Property (ii) strengthens (i) in the sense that the set of feasible transactions for each agent does not contain points which were not available for the unbounded sequence.

It should be clear that the class of rationing functions satisfying (R1)–(R5) is non-empty since any continuous one-sided rationing function with a fixed ranking over agents to serve them belongs to this class. In the diagram below, Figure 1, we illustrate property (R5) for an economy with one agent supplying a quantity y and two agents demanding z_1 and z_2 with $z_1+z_2 \geqq y$. The lines drawn in the diagram are the isodistribution curves for the two agents demanding which correspond to the distribution of y for

$$z_1+z_2 = x_1+x_2 = y.$$

The continuity requirement of (c) in (R5) requires that the isodistribution curves change continuously for alternative values of y so that the flat sections for large z_1 and z_2 do not disappear.

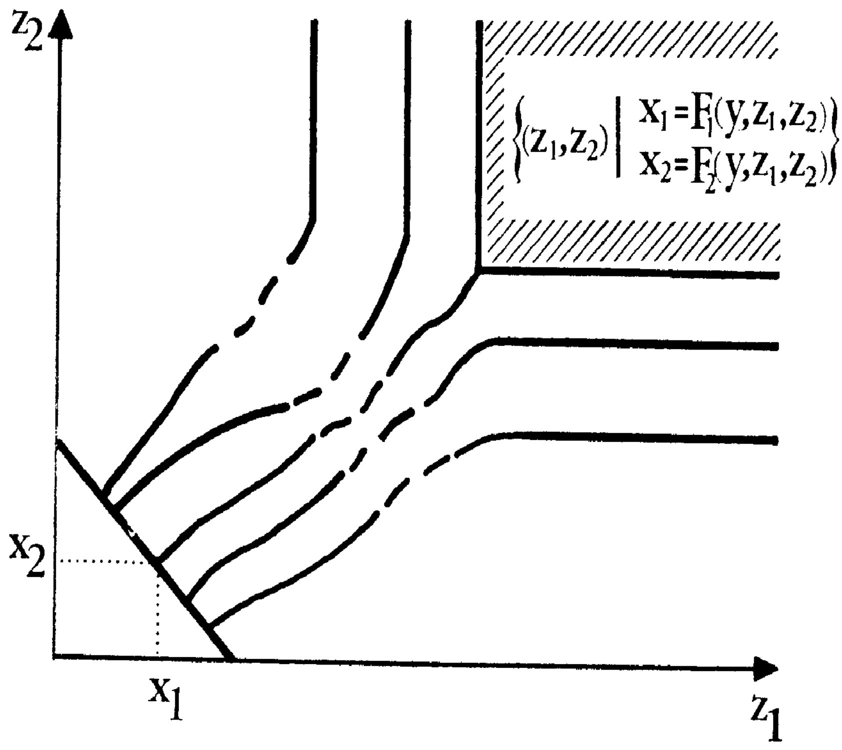

FIGURE 1

Lemma 2. *Assume that for all agents $i \in I$, except for consumer j with the characteristics described in* (A3), *the set of possible strategies $Z_i \subset R^N$ is a compact cube C of R^N containing the origin and* $(0, L_i)$ *for every consumer. For consumer j, choose $Z_j = C \cap \{z \mid z_{h'} \leqq l_{jh'}\}$. If the assumptions* (A1)–(A3) *and* (R1)–(R4) *hold, then there exists a non-trivial Nash equilibrium $\bar{z}$ for the game $((Z_i)_{i\in I}, (U_i)_{i\in I})$ such that $F_i(\bar{z})$ is feasible for every $i \in I$.*

Let $\xi_i: Z_l \to Z_i$ denote the correspondence defined by

$$\xi_i(z_l) = \{z_i \in Z_i \mid U_i(\omega_i+F_i(z_i, z_l), m_i^0-\textstyle\sum_n s_n F_{in}(z_{in}, z_{ln}))$$
$$\geqq U_i(\omega_i+F_i(z_i', z_l), m_i^0-\textstyle\sum_n s_n F_{in}(z_{in}', z_{ln})) \text{ for all } z_i' \in Z_i\}.$$

We will show that the correspondences (ξ_i), $i \in I$, are non-empty and convex-valued, and upper semi-continuous. Then, standard fixed point arguments imply the lemma.

For any producer i

$$0 \in Z_i \text{ and } F_i(0, z_i) = 0 \quad \text{imply } 0 \in \Phi_i(z_i)$$

for every z_i. On the other hand, for any consumer, $l_i \in L_i$ and $(0, l_i) \in Z_i$ implies $(F_i(0, l_i), z_i) \in X_i$ for every z_i. Since the rationing functions F_i are continuous and independent across markets, $\Phi_i(z_i)$ is a cartesian product of intervals $\Phi_{in}(z_{in})$, i.e.

$$\Phi_i(z_i) = \Pi_{n \in N} \Phi_{in}(z_{in}).$$

The end points of $\Phi_{in}(z_{in})$ are the maximum and minimum respectively of $F_{in}(z_{in}, z_{in})$ taken over a fixed compact interval Z_{in} which is the projection of Z_i onto the nth component. Therefore, according to the standard maximum theorem both end points move continuously with z_{in}. Therefore, Φ_i is both upper hemi-continuous and lower hemi-continuous as the cartesian product of intervals whose end points are varying continuously. Let

$$\mathscr{T}_i = \{x_i \in R^N \mid \omega_i + x_i \in X_i, m_i^0 - \sum_n s_n x_{in} \geqq 0\}.$$

$\mathscr{T}_i$ is a closed, convex set containing zero for $i \neq j$, and $\eta_j = (0, ..., l_{jh'}, ..., 0)$ for j.

Lemma 3. *The correspondence ψ_i defined by*

$$\psi_i(z_i) = \mathscr{T}_i \cap \Phi_i(z_i)$$

is continuous.

Proof of Lemma 3. ψ_i is uhc because Φ_i is uhc and $\mathscr{T}_i$ is a fixed closed set.

To show lower hemi-continuity, consider a sequence (z_i^r), $r = 1, ...,$ converging to z_i and $x_i \in \psi_i(z_i)$. We need to show that there exists a sequence y_i^r converging to x_i such that $y_i^r \in \psi_i(z_i^r)$. Let $y_i(z_i) = 0$ if $i \neq j$ where j is the consumer whose characteristics satisfy (A3), and let $y_j(z_j) = F_j(\eta_j, z_j)$ if $i = j$. Then, for every t, $0 \leqq t \leqq 1$,

$$tx_i + (1-t)y_i(z_i) \in \mathscr{T}_i \cap \Phi_i(z_i) \quad \text{and} \quad y_i(z_i) \in \Phi_i(z_i^r)$$

for all $r = 1, ...$ Let

$$t(z_i^r) = \max\{t \mid 0 \leqq t \leqq 1, tx_i + (1-t)y_i(z_i) \in \Phi_i(z_i^r)\}.$$

Clearly this maximum exists. Since $\Phi_i(z_i)$ is a rectangle and since Φ_i is a continuous correspondence, $t(.)$ is a continuous function. Thus, $\lim t(z_i^r) = 1$. Define

$$y_i^r = t(z_i^r)x_i + (1 - t(z_i^r))y_i(z_i).$$

Clearly, $y_i^r \in \Phi_i(z_i^r) \cap \mathscr{T}_i$ for all r and $y_i^r \to x_i$. Therefore, ψ_i is a continuous correspondence.

Proof of Lemma 2. It can easily be seen now that, for every $i \in I$, $z_i \in \xi_i(z_i)$ if and only if $x_i = F_i(z_i, z_i)$ and x_i maximises $u_i(x_i + \omega_i, m_i^0 - \Sigma_n s_n x_{in})$ subject to $x_i \in \psi_i(z_i)$. Let $v_i(z_i)$ denote this maximum. v_i is a continuous function since u_i is continuous on $\mathscr{T}_i$ and ψ_i is a continuous correspondence. Furthermore

$$\gamma_i(z_i) = \{x_i \in \psi_i(z_i) \mid u_i(x_i + \omega_i, m_i^0 - \sum_n s_n x_{in}) = v_i(z_i)\}$$

is upper hemi-continuous. Finally, to show upper hemi-continuity of ξ_i, consider a sequence (z_i^r), $r = 1, ...$ which converges to z_i and a sequence of best response strategies (z_i^r), $r = 1, ..., z_i^r \in \xi_i(z_i^r)$, which converges to z_i. Then for the sequence (x_i^r) defined by $x_i^r = F_i(z_i^r, z_i^r)$ it follows that $u_i(x_i^r + \omega_i, m_i^0 - \Sigma_n s_n x_{in}^r) = v_i(z_i^r)$ for all r. Since u_i, v_i are continuous and since F_i is continuous one has

$$u_i(\omega_i + F_i(z_i, z_i), m_i^0 - \sum_n s_n F_{in}(z_{in}, z_{in})) = v_i(z_i).$$

Hence $z_i \in \xi_i(z_i)$.

It remains to be shown that $\xi_i(z_i)$ is convex for every z. First observe that $\gamma_i(z_i)$

is a function since u_i is strictly quasi-concave. Hence, for any two z_i^1 and z_i^2 which belong to $\xi_i(z_1)$, it follows that

$$F_i(z_i^1, z_1) = F_i(z_i^2, z_1).$$

Therefore, for every $0 \leqq \lambda \leqq 1$,

$$F_i(\lambda z_i^1+(1-\lambda)z_i^2, z_1) = F_i(z_i^1, z_1) = F_i(z_i^2, z_1)$$

since F_i is monotonic. This completes the proof of the lemma. Notice that the resulting equilibrium is non-trivial because j cannot announce a trivial strategy.

Theorem. *Assume that, for every $i \in I$, $Z_i = R^N$ and that the assumptions* (A1)–(A3) *and* (R1)–(R5) *are satisfied. Then there exists a non-trivial equilibrium (z_i), $i \in I$, relative to* F *such that $F_i(z)$ is feasible for every $i \in I$.*

Proof. Consider the sequence of economies with strategy sets

$$Z_i^r = \{z_i \in R^N \mid \ |z_{in}| \leqq r,\ r = 1, \ldots;\ n \in N\} \quad \text{for } i \neq j,$$

and

$$Z_j^r = \{z_j \in R^N \mid \ |z_{jn}| \leqq r,\ r = 1, \ldots;\ n \in N,\ z_{jh'} \leqq l_{jh'}\}.$$

According to Lemma 2 there exists a non-trivial equilibrium (z_i^r), $i \in I$, relative to F for every r. It is clear that if (z^r) is bounded, then the limit of any converging subsequence is a non-trivial equilibrium relative to F.

Suppose the sequence (z^r), $r = 1, \ldots$ is unbounded. For every $n \in N$ there exists a subsequence, also denoted by (z_n^r), $r = 1, \ldots$, such that, for every $i \in I$

(1) $z_{in}^r(\sum_{j \in I} z_{jn}^r) \leqq 0$ for every r

or

(2) $z_{in}^r(\sum_{j \in I} z_{jn}^r) > 0$ for every r.

Since the set of feasible states of the economy is compact, (R1) and (R2) imply that $x_i^r = F_i(z^r)$ is bounded for every $i \in I$. All agents on the short side are not rationed so that their proposed net trades z_{in}^r stay bounded. Therefore, condition (1) will hold for all agents on the short side.

Let (z^r) be the sequence of strategies associated with a converging subsequence of (x_i^r), $i \in I$, and use (R5) to define for every $i \in I$ and every $n \in N$

$$\bar{z}_{in}^r = \begin{cases} f_{in}(z_{I\setminus I_n}^r) & i \in I_n \\ z_{in}^r & \text{otherwise.} \end{cases}$$

Then, according to (R5), for r sufficiently large $F_{in}(\bar{z}_n^r) = F_{in}(z_n^r) = x_{in}^r$ and $\Phi_{in}^r(\bar{z}_{in}^r) \subseteqq \Phi_{in}^r(z_{in}^r)$. Since the functions f_{in}, $i \in I_n$, are bounded and continuous, there exists a subsequence which will also be denoted by $(\bar{z}^r)$, $r = 1, \ldots$, converging to $\bar{z} \neq 0$. It remains to be shown that $\bar{z}$ is an equilibrium for F. Suppose this were false. Then there would exist an agent i and a realization $\bar{\bar{x}}_i \in \Phi_i(\bar{z}_i)$ such that, if $\bar{x}_i = F_i(\bar{z})$,

$$u_i(\omega_i+\bar{\bar{x}}_i,\ m_i^0-\textstyle\sum_n s_n\bar{\bar{x}}_{in}) > u_i(\omega_i+\bar{x}_i,\ m_i^0-\sum_n s_n\bar{x}_{in}).$$

Let $i \neq j$. Since Φ_i^r is lower hemi-continuous there exists a sequence $\bar{\bar{x}}_i^r$ converging to $\bar{\bar{x}}_i$ such that $\bar{\bar{x}}_i^r \in \Phi_i^r(\bar{z}_i^r)$. For r sufficiently large $\Phi_i^r(\bar{z}_i^r) \subseteqq \Phi_i^r(z_i^r)$ so that $\bar{\bar{x}}_i^r \in \Phi_i^r(z_i^r)$. Since u_i is continuous, for r sufficiently large

$$u_i(\omega_i+\bar{\bar{x}}_i^r,\ m_i^0-\textstyle\sum_n s_n\bar{\bar{x}}_{in}^r) > u_i(\omega_i+x_i^r,\ m_i^0-\sum_n s_n x_{in}^r)$$

which contradicts the fact that x_i^r is a realization for i at an equilibrium relative to F for Z_i^r. If $i = j$, the same argument in conjunction with (A3) yields that $\bar{z}_j$ is optimal for j in R^N. ‖

Existence of a non-zero equilibrium could have been shown by restricting (R3)–(R5) to market h'. However, the more uniform presentation here allows one to show existence of an equilibrium with non-zero proposed net trades in any market which satisfies an assumption like (A3), without requiring changes in the proof.

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NOTES

1. For a survey on temporary equilibria with rationing see Grandmont (1977).
2. This same notion of equilibrium was used by Grandmont and Laroque (1976).
3. The same notion of equilibrium was introduced by Shapley (1976). After completion of this paper, the work of Heller and Starr (1977), who address the same problem, was brought to our attention.
4. For a precise definition of effective demand see Benassy (1975).

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