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Rationing and Optimality in Overlapping Generations Models*

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I. Introduction

The theory of overlapping generations models provides examples of economies where the first welfare theorem of static finite economies fails to hold. Since the discovery of this result, several authors have examined the relationship between competitive equilibria and Pareto optimal allocations, although they have systematically ignored alternatives to the competitive mechanism as the only allocation rule. To our knowledge, the question of whether a different allocation mechanism may perform better than the competitive one has not been posed.

One alternative to the competitive allocation rule is provided by the theory of equilibria with rationing which has been developed extensively for static finite economies. Such models have been under constant criticism because they provide no or very little explanation as to why prices are fixed at non-Walrasian levels, although binding quantity constraints imply a clear signal as to how prices should adjust. Therefore, if the model of quantity rationing can be placed in a full dynamic framework where prices adjust from one period to the next, the criticism of assumed and unexplained price rigidity could be overcome. By adopting the approach of the law of supply and demand, i.e., demand rationing implies price increases and supply rationing implies price decreases, a well-defined class of price adjustment rules may be obtained which provides a first step toward a theory of price dynamics under nonmarket-clearing conditions.

Optimality properties of rationing equilibria in static finite economies have been studied by several authors. Balasko (1979, 1982), Böhm and Müller (1977) and Drèze and Müller (1982) examined the question of different rationing equilibria for the same set of prices, whereas Böhm (1984)

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and Silvestre (1982) addressed optimality questions by comparing different sets of non-Walrasian prices. In the context of both problems, however, the static model does not provide any clues regarding possibilities of dominating Walrasian equilibria, since the first welfare theorem holds in these cases. In this note, we present a positive answer to the question of how inefficient competitive equilibria can be dominated by equilibria with quantity constraints which follow a stable price adjustment process according to the law of supply and demand. The framework is a simple overlapping generations model. Rationing equilibria were first described in such models by Azariadis and Böhm (1982), who combined the concept of equilibria with quantity constraints, as proposed by Drèze (1975), with the notion of perfect foresight.

II. The Model

Consider a simple overlapping generations model. The producer-consumers live for two consecutive periods and one agent is born in each period. Time t extends from 1 to infinity. The agent born in period t is called agent t, he/she is called young at t and old at $t+1$. At $t=1$, there is an old agent who lives only during that period and is endowed with one unit of money. For $t \geqq 1$, preferences are described by a continuous utility function $u(c_{t+1}, l_t)$ over current work $l_t \geqq 0$ and future consumption $c_{t+1} \geqq 0$. We assume that $u(c_{t+1}, l_t)$ is strictly quasi-concave, strictly monotonically increasing in its first argument and strictly monotonically decreasing in its second. Each young agent is endowed with $\bar{l}$ units of leisure which he can use to produce a single perishable consumption good using the linear technology $y_t \leq l_t$.

For $t=1,\ldots,$ let p_t denote the current price of the consumption good in period t in terms of money and let m_t denote the amount of money held by agent t to finance his consumption when old. Then, formally, the producer-consumers solve the following optimization problems:

$$\text{(P0)} \quad \max u_0(c_1) \quad \text{s.t.} \quad p_1 c_1 \leq 1, \quad c_1 \geq 0.$$

$$\text{(P1)} \quad \max u(c_{t+1}, l_t) \qquad t \geqq 1$$

$$\text{s.t.} \quad m_t \leq p_t l_t$$

$$p_{t+1} c_{t+1} \leq m_t$$

$$c_{t+1} \geq 0, 0 \leq l_t \leq \bar{l}$$

$$l_t \geq y_t.$$

The unique solution to (P1) defines the demand and supply behavior of agent t, which can be described either by his supply function $l_t = y_t = s(p_t/p_{t+1})$ or equivalently by the associated offer curve. It is assumed that the offer curve can be written as a strictly increasing function $c_{t+1} = G(l_t)$ such that $G(l_t) < l_t$ for some $0 < l_t < \bar{l}$. The monotonicity of G implies that leisure and future consumption are normal goods and that income effects are not too strong. Moreover, $G(l)/l$ is strictly increasing with l, and from the budget equation and the definition of G, we have for all p_t and p_{t+1}:

$$\frac{G(s(p_t/p_{t+1}))}{s(p_t/p_{t+1})} = \frac{p_t}{p_{t+1}}.$$

Definition: A sequence $\{p_t\}_1^\infty$ of prices is a competitive equilibrium if for all t

(a) $c_t = y_t = s(p_t/p_{t+1})$

(b) $c_t = \dfrac{1}{p_t}$

(c) $m_t = 1$.

This definition summarizes market-clearing conditions for the commodity and money markets. It is clear that the sequence $\{p_t\}_1^\infty$ is a competitive equilibrium if and only if the difference equation

$$\frac{1}{p_{t+1}} = G\left(\frac{1}{p_t}\right)$$

holds for all t.

If prices are not market clearing in a dynamic model, two separate issues arise. One concerns the type of feasible allocation in each period. We adopt the approach suggested by Drèze, implying a nonmanipulable rationing mechanism which preserves voluntary trade and the one-sidedness of binding constraints. The second issue is the relationship between actual price changes and the observed disequilibrium state in each period. We impose consistency with the law of supply and demand.

Suppose agent t faces prices (π_t, π_{t+1}) and a quantity constraint x_t on his supply of goods. Then his decision problem can be written in the following way:

(P2) $\max u(c_{t+1}, l_t)$

s.t. $m_t \leq \pi_t l_t$

$$\pi_{t+1}c_{t+1} \le m_t$$
$$l_t \le x_t$$
$$c_{t+1} \ge 0, 0 \le l_t \le \bar{l}$$
$$l_t \ge y_t.$$

If the constraint is binding, it must be the case that $x_t < s(\pi_t/\pi_{t+1})$. Then the optimum solutions will be $\tilde{l}_t = x_t$ and $\tilde{c}_{t+1} = \pi_t x_t/\pi_{t+1}$.

Definition: A sequence $\{\pi_t, x_t\}_1^\infty$ *of prices and supply constraints is a supply-constrained equilibrium if for all t*

(a) $x_t \le s(\pi_t/\pi_{t+1})$, *with strict inequality for some t*
(b) $\pi_t x_t = 1$.

Condition (a) says that for some t, prices are such that the constraint is strictly binding. Condition (b) stipulates that money is never rationed, which is equivalent to the fact that no demand rationing occurs.

The following lemma relates the supply-constrained price sequence to the offer curve.

Lemma: A price sequence $\{\pi_t\}_1^\infty$ *supports a supply-constrained equilibrium if and only if*

$$\frac{1}{\pi_{t+1}} \ge G\left(\frac{1}{\pi_t}\right)$$

holds for all t, with strict inequality for some t.

Proof: Assume that the sequence $\{\pi_t\}_1^\infty$ supports a supply-constrained equilibrium. There then exists a sequence of supply constraints $\{x_t\}_1^\infty$ such that $x_t \le s(\pi_t/\pi_{t+1})$. From the budget identity $\pi_t s(\pi_t/\pi_{t+1}) = \pi_{t+1} G(s(\pi_t/\pi_{t+1}))$, we obtain

$$\frac{1}{\pi_{t+1}} = \frac{G(s(\pi_t/\pi_{t+1}))}{\pi_t s(\pi_t/\pi_{t+1})} \geqq \frac{G(x_t)}{\pi_t x_t} = G\left(\frac{1}{\pi_t}\right)$$

since $G(l)/l$ is strictly increasing and $\pi_t x_t = 1$.

Conversely, suppose

$$\frac{1}{\pi_{t+1}} \geqq G\left(\frac{1}{\pi_t}\right)$$

holds for all t, i.e.,

$$\frac{G(s(\pi_t/\pi_{t+1}))}{s(\pi_t/\pi_{t+1})}=\frac{\pi_t}{\pi_{t+1}}\geq G\left(\frac{1}{\pi_t}\right)\Big/\frac{1}{\pi_t}\text{ for all } t.$$

For $x_t=1/\pi_t$ this implies $s(\pi_t/\pi_{t+1})\geqq x_t$, since $G(l)/l$ is strictly increasing.

Q.E.D.

As an immediate consequence of the lemma, we note that the set of time maps F which generate supply-constrained equilibria for a given offer curve G is very large. In order to demonstrate this, let $\bar{p}>0$ denote the lowest price which is consistent with the stationary competitive equilibrium, i.e., $G(1/\bar{p})=1/\bar{p}$. Then any continuous function $F:[0, 1/\bar{p}]\rightarrow[0, 1/\bar{p}]$ such that $F(s)\geq G(s)$ for all $s\in[0, 1/\bar{p}]$, with strict inequality for some s, supports a supply-constrained equilibrium. Since $F\geq G$, prices under rationing will generally be lower than in a competitive equilibrium if both have the same initial condition.

The class of price adjustment mechanisms has to be restricted since they should correspond to the associated sequence of quantity constraints. One appropriate way of imposing consistency is to require that prices adjust upward (downward) whenever there is a binding demand (supply) constraint which is a natural consistency requirement in a competitive framework. Therefore, a time map F is called consistent if it satisfies the law of supply and demand. Such an F associated with a supply-constrained equilibrium must generate a nonincreasing sequence of prices with strictly falling prices if constraints are binding. This implies that F must fulfill $1/\bar{p}\geqq F(s)\geqq s$ for all $s\in[0, 1/\bar{p}]$. It is easily seen that these price adjustment rules are globally stable.

III. Optimality

Given a competitive equilibrium $\{p_t\}_1^\infty$, the welfare of each agent t is given by his indirect utility function. Taking into account the equilibrium condition $l_t=1/p_t$, we obtain $v_t(1/p_t)=u(G(l_t),l_t)=u(G(1/p_t),1/p_t)$. Hence, for any agent $t\geq 1$, the utility in equilibrium is an increasing function of $1/p_t$. For a supply-constrained equilibrium $\{\pi_t, x_t\}_t^\infty$, the utility of agent t becomes

$$\tilde{v}\left(\frac{1}{\pi_{t+1}},\frac{1}{\pi_t}\right)=u\left(\frac{1}{\pi_{t+1}},\frac{1}{\pi_t}\right).$$

It follows from the monotonicity of G that any inefficient competitive equilibrium $\{p_t\}_1^\infty$ must have $p_1>\bar{p}$, where $G(1/\bar{p})=1/\bar{p}$ with $\bar{p}$ as the stationary price at the golden rule state. Moreover, the sequence of equilibrium prices diverges monotonically to infinity. The allocation converges to the no-trade equilibrium with monotonically decreasing utility. Then it is immediately seen that any such equilibrium can be Pareto dominated by a supply-constrained equilibrium simply by choosing a stationary price level $\tilde{\pi} = \pi_t$ for all t with $\bar{p} \leqq \tilde{\pi} < p_1$. However, such a stationary price sequence is not consistent since it permanently ignores supply pressures to lower prices implied by the law of supply and demand. Hence, if the notion of equilibria under quantity rationing is used as a paradigm to search for Pareto-improving allocations, binding quantity constraints imply nonstationary prices. The following theorem states that there exists a globally stable price adjustment rule, consistent with the law of supply and demand, which supports a Pareto-dominating supply-constrained equilibrium that is improving for all agents. Moreover, the price adjustment can be chosen such that all agents $t \geqq 1$ receive a utility level higher than at the golden rule state.

Theorem: Let $\bar{p} > 0$ denote the stationary price of the golden rule state. There exists a consistent time map F supporting a Pareto-dominating supply-constrained equilibrium for any inefficient competitive equilibrium such that

$$u\left(\frac{1}{\pi_{t+1}}, \frac{1}{\pi_t}\right) > u\left(\frac{1}{\bar{p}}, \frac{1}{\bar{p}}\right) \quad \textit{all} \quad t \geqq 1.$$

Proof: Let $\bar{y} > 0$ be such that $u(\bar{y}, 0) = u(1/\bar{p}, 1/\bar{p})$ and define the price adjustment mechanism by

$$\frac{1}{\pi_{t+1}} = \bar{y} + (1 - \bar{p}\bar{y})\frac{1}{\pi_t} = F\left(\frac{1}{\pi_t}\right),$$

where $\pi_1 = p_1 - \varepsilon > \bar{p}$ for some $\varepsilon > 0$. Clearly, $\{\pi_t\}_1^\infty$ is a decreasing sequence which converges to $\bar{p}$, since $\bar{y}\bar{p} < 1$. Moreover, $s \leqq 1/\bar{p}$ implies $F(s) \geqq s$, so that F supports a supply-constrained equilibrium. $\pi_1 < p_1$ improves agent zero. For $t \geqq 1$, we obtain

$$u\left(\frac{1}{\pi_{t+1}}, \frac{1}{\pi_t}\right) = u\left(\bar{y} + \frac{\bar{p}}{\pi_t}\left(\frac{1}{\bar{p}} - \bar{y}\right), \frac{\bar{p}}{\pi_t} \cdot \frac{1}{\bar{p}}\right) > u(\bar{y}, 0) = u\left(\frac{1}{\bar{p}}, \frac{1}{\bar{p}}\right),$$

since $0 < (\bar{p}/\pi_t) < 1$ and u is strictly quasi-concave.

Q.E.D.

The proof and theorem are illustrated in Figure 1. The price adjustment rule used in the proof corresponds to the linear time map F, defined by the straight line connecting the two points $(\bar{y}, 0)$ and $(1/\bar{p}, 1/\bar{p})$. Clearly, many other time maps can be constructed which yield the same result.

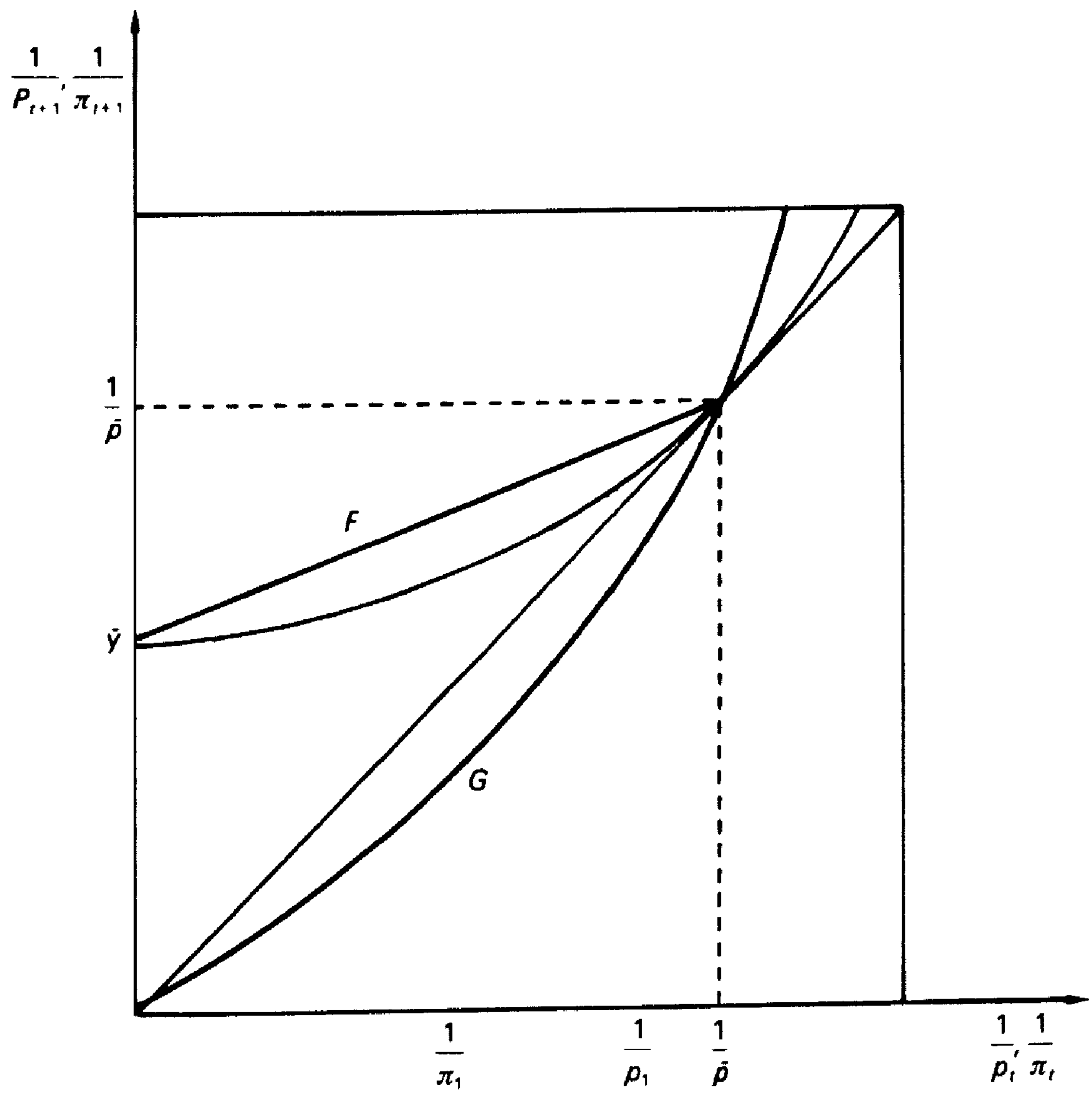

Fig. 1.

IV. Concluding Remarks

This note represents a first attempt to analyze welfare questions in an overlapping generations model, using the notion of equilibria with quantity rationing in a consistent dynamic framework. The model analyzed is very special and the techniques rely heavily on the two-dimensional format in which the model can be represented and on the assumption of a monotonic offer curve. It is well known that when the offer curve is backward bending, the set of competitive equilibria may be very complex. However, the theorem remains true if the words "Pareto dominating" are eliminated. Thus, there always exist globally stable supply-constrained equilibria with a utility above the golden rule state for all but the initial agent.

It is obvious that the above results are true for the general exchange economy case with one good and one agent per period. To our knowledge extensions have not been made to more than one consumer and/or more than one commodity. Some tentative experimentation seems to indicate that Pareto dominance of inefficient competitive equilibria with quantity rationing can be achieved in many, but not all, cases.

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