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BALANCED LOCAL RINGS WITH COMMUTATIVE RESIDUE FIELDS

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1. Let R be a ring with unity. An R-module M is said to be balanced or to have the double centralizer property, if the natural homomorphism from R to the double centralizer of M is surjective. If all left and right R-modules are balanced, R is called *balanced.* It is well known that every artinian uniserial ring is balanced. In [5], J. P. Jans conjectured that those were the only (artinian) balanced rings. Jans' conjecture has been shown to be false in [3] by constructing a class of balanced nonuniserial rings. In the present paper, we show that the rings of [3] together with the (local) uniserial rings are the only balanced rings which are local (i.e. have a unique one-sided maximal ideal) and whose residue division ring R/W is commutative (here, as well as in what follows, W denotes always the radical of R).

THEOREM. *Let R be a local ring with the radical W, and let $Q = R/W$ be commutative. Then R is balanced if and only if either*

(a) *R is an artinian uniserial ring, or*

(b) $W^2 = 0$ *and* $\dim({}_QW) \cdot \dim(W_Q) = 2$.

In view of [2] combined with [4], an arbitrary (not necessarily local) balanced ring is a finite direct sum of full matrix rings over balanced local rings, and thus we get the following:

COROLLARY. *Let R be a ring with the radical W such that R/W is commutative. Then R is balanced if and only if R is a finite direct sum of local rings of type* (a) *or* (b) *of Theorem.*

The proof of Theorem is based on the structure theorems of [2] and a result of V. P. Camillo and K. R. Fuller [1] on the index of a certain division subring of R/W. More specifically, the following two theorems of [2] will be used in our arguments (in [2], the formulations of these theorems are more general).

THEOREM A* OF [2]. *A balanced ring is artinian.*

THEOREM B* OF [2]. *Let R be a balanced local ring with the radical W; put $Q = R/W$. Then either*

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(i) *R is uniserial; or*

(ii) $W^2 = 0$, $\dim({}_QW) \leqq 2$, $\dim(W_Q) \leqq 2$ *and, for any two nonzero elements* x, y *of* W, $Rx + yR = W$; *or*

(iii) $W^3 = 0$, $\dim({}_QW^2) = \dim(W^2_Q) = 1$ *and* R/W^2 *is a ring described in* (ii), *i.e.* R *is a quasi-Frobenius ring of length* 4.

2. Let us start with the following lemma due to V. P. Camillo and K. R. Fuller [1].

Lemma. *Let* R *be a left QF-1 (i.e. every faithful left R-module is balanced) local ring with the radical* W *and let* s *be a nonzero element of the intersection* S *of the left and the right socles of* R. *Let*

$$\Phi = \{\varphi \in R \mid s\varphi \in Rs\} \subseteq R.$$

Then the radical of the ring Φ *is* W, $D = \Phi/W$ *is a division subring of* R/W *and* $\dim(R/W)_D \leqq 2$.

Proof. The result holds trivially if $\Phi = R$. Thus, let $\Phi \neq R$. As an immediate consequence, $S \neq Rs$ and, furthermore, Rs is not a two-sided ideal. Thus the left R-module $M = R/Rs$ is faithful. Note that $\overline{\Phi} = \Phi/Rs$ is canonically isomorphic to $\mathrm{End}({}_RM)$.

Evidently, $\Phi \supseteq W$, W is the radical of Φ and $D = \Phi/W$ is a division ring. Observe that both R/W and S/Rs are nonzero (right) vector spaces over D and that $\dim(R/W)_D \geqq 2$. In order to prove that $\dim(R/W)_D = 2$, it is obviously sufficient to show that every nonzero $\overline{\Phi}$-homomorphism $\psi: R/W \to S/Rs$ such that $\psi(\Phi/W) = 0$ has property that

$$\mathrm{Ker}\,\psi = \Phi/W.$$

Thus, take such a Φ-homomorphism ψ and let $r + W \in \mathrm{Ker}\,\psi$. Now, the $\overline{\Phi}$-homomorphism

$$M \xrightarrow{\varepsilon} R/W \xrightarrow{\psi} S/Rs \xrightarrow{\iota} M,$$

with the canonic epimorphism ε and embedding ι, belongs to the double centralizer of M and is therefore induced by the ring multiplication, say, by the element $\rho \in R$. Hence,

$$\rho \neq 0, \qquad \rho\Phi \subseteq Rs \quad \text{and} \quad \rho r \in Rs.$$

Consequently, since $1 \in \Phi$, $R\rho = Rs$. Thus, $sr \in Rsr = R\rho r \subseteq Rs$. This means that $r \in \Phi$, as required.

Now, we are ready to give

Proof of Theorem. A ring of type (a) is well known to be balanced. And, every ring of type (b) is balanced according to [3].

In order to prove the converse, assume first that R is a balanced non-uniserial local ring such that $W^2 = 0$. Consequently, in view of Theorem B*,

$$\dim({}_Q W) \leqq 2, \qquad \dim(W_Q) \leqq 2$$

and, for any two nonzero $x, y \in W$, $Rx + yR = W$. In particular, W is the unique minimal two-sided ideal.

To prove our statement, we need to exclude the possibility of the case $\dim({}_Q W) = 2 = \dim(W_Q)$. To this end, assume $\dim({}_Q W) = 2$ and take a nonzero element $s \in W$ and $\tau \in R$ such that $s\tau \notin Rs$, i.e. such that

$$\tau \notin \Phi = \{\varphi \in R \mid s\varphi \in Rs\}.$$

Hence,

$$W = Rs \oplus Rt \quad \text{with } t = s\tau.$$

Now, by Lemma, $\dim(Q_{\Phi/W}) \leqq 2$ and since Q is commutative,

$$Q = \Phi/W \oplus \tau\Phi/W = \Phi/W \oplus \Phi/W\tau.^2$$

Thus, if $\sigma \in R$, then

$$\sigma + W = \varphi_1 + \varphi_2\tau + W \quad \text{with } \varphi_1, \varphi_2 \in \Phi,$$

and therefore

$$s\sigma = s\varphi_1 + s\varphi_2\sigma.$$

Since, obviously, $s\varphi_1 \in Rs \cap sR$ and $s\varphi_2\tau \in Rt \cap sR$, we have

$$W = Rt + sR = Rt + (Rs \cap sR) + (Rt \cap sR) = Rt + (Rs \cap sR).$$

However, $W = Rt \oplus Rs$, and hence $Rs = Rs \cap sR$, i.e. $Rs \subseteq sR$. Consequently, sR is a two-sided ideal and therefore $sR = W$, i.e. $\dim(W_Q) = 1$, as required.

In order to complete the proof of our Theorem, consider the case when R is not uniserial and $W^2 \neq 0$. Then, again by Theorem B*, $W^3 = 0$ and R is a quasi-Frobenius ring. But then, in view of the preceding part of the proof (applied to R/W^2), R is either left or right uniserial and therefore uniserial, in contradiction to our hypothesis.

The proof of Theorem is completed.

At the end, let us remark that, making use of the structure theorems of [2] and of Lemma B of [1], an alternative proof of Theorem can be given along the lines of the proof of Lemma C of [1].

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[2] It is only here that the commutativity of Q is used. In fact, it is enough here to assume that every division subring of Q of right index 2 has also left index 2 (in Q).

ADDED IN PROOF (June, 1972). A full characterization of balanced rings is given in Lecture Notes in Math., no. 246, Springer-Verlag, 1972.

REFERENCES

1. V. P. Camillo and K. R. Fuller, *Balanced and* QF-1 *algebras* (to appear).
2. V. Dlab and C. M. Ringel, *Rings with the double centralizer property*, J. Algebra (to appear).
3. ———, *A class of balanced non-uniserial rings*, Math. Ann. **195** (1972), 279–291.
4. K. R. Fuller, *Primary rings and double centralizers*, Pacific J. Math. **34** (1970), 379–383. MR **42** #6040.
5. J. P. Jans, *On the double centralizer condition*, Math. Ann. **188** (1970), 85–89. MR **42** #7698.

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